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A SHORT NOTE OF *F*-INDISTINGUISHABILITY OPERATORS

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Abstract

In this paper, we give equivalent conditions of an F-indistinguishability operator and investigate some related properties.

1. Introduction

The concept of fuzzy sets was proposed by Zadeh [7]. He generalized the idea of the characteristic function of a subset of a set X by defining a fuzzy subset of X as a map X into the unit interval $[0, 1];$. Several researchers have applied fuzzy sets to various branches of mathematics. The results of fuzzy relations and F-preorders were developed in [1-4, 6] among several others, Murali [1] defined the fuzzy equivalence relation on a set and proved that there exists a correspondence between fuzzy equivalence relations and certain classes of fuzzy subsets. Ounalli and Jaoua [2] defined the fuzzy difunctional relation on a set and studied some properties on such relation.

Seo et al. [4] proved that there exists a relationship between fuzzy

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equivalence relations and fuzzy difunctional relations. Valverde [6] proved that any F-indistinguishability operator on a set is generated by a family of fuzzy subsets of X and explored the links between F-indistinguishability operators and a kind of generalized metrics in the unit interval. Ovchinnikov [3] investigated numerical representations of fuzzy transitive relations. In this paper, we give equivalent conditions of an F-indistinguishability operator and investigate some related properties.

2. Preliminaries

In this section, we explain some basic definitions and result from [4] for reference purposes. Let X be a set, called the *universe of discourse*. Let $[0, 1]$ be the set of all real numbers α with $0 \leq \alpha \leq 1.$ A scalar is denoted by Greek letters such as $\alpha, \beta, \gamma, \ldots$ and so on, possibly with subscripts. The supremum and infimum of a set $\{\alpha_1, \alpha_2, \ldots, \alpha_m, \ldots\}$ of scalar are denoted by $\vee\{\alpha_1, \alpha_2, \ldots, \alpha_m, \ldots\}$ and $\wedge\{\alpha_1, \alpha_2, \ldots, \alpha_m, \ldots\}$, respectively. A fuzzy binary relation R on the universe X is a function $R : X \times X \to [0, 1].$ For $x, y \in X$, the value $R(x, y) = \alpha$ is called the *grade of membership* of (x, y) in R and means how far x and y are related under R. Without loss of generality, we define all fuzzy relations on a fixed universe X. For a family of relations $R_1, R_2, \ldots, R_n,$ we define the union $\sqcup_i R_i$ and intersection $\sqcap_i R_i$ as follows:

$$(\sqcup_i R_i)(x, y) = \vee_i R_i(x, y),$$

$$(\sqcap_i R_i)(x, y) = \wedge_i R_i(x, y).$$

As usual, symbols $\wedge$ and $\vee$ denote operations min and max, respectively, although occasionally, we adhere to the notations min and max.

Definition 2.1. A fuzzy relation R is a *fuzzy difunctional* if and only if it satisfies condition $R \circ R^{-1} \circ R \subseteq R$, which is equivalent to $R \circ R^{-1} \circ R = R$. $R \circ R^{-1} \circ R \supseteq R$ holds for any fuzzy relation R.

Definition 2.2. Let X be a nonempty set and let R be a fuzzy relation on X. R is called a *fuzzy equivalence relation* on X if and only if

(1) R is *reflexive*, i.e. $R(x, x) = 1, \forall x \in X$.

(2) R is *symmetric*, i.e. $R^{-1} = R$.

(3) R is *transitive*, i.e. $R \circ R \subseteq R$.

Definition 2.3. A t-norm F is a function $F : [0, 1]^2 \to [0, 1]$ satisfying the following conditions:

(i) $F(x, 1) = x$,

(ii) $F(x, y) = F(y, x)$,

(iii) $F(x, y) \leq F(u, v)$ for all $x \leq u, y \leq v$,

(iv) $F(x, F(y, z)) = F(F(x, y), z)$ for all $x, y, z \in [0, 1]$.

Definition 2.4. A t-norm F is *idempotent* defined by

$$F(x, x) = x \text{ for all } x \in X.$$

Definition 2.5. A fuzzy relation R on X is said to be F-*transitive* if

$$F(R(x, y), R(y, z)) \leq R(x, z)$$

for all x, y, z in X.

Definition 2.6. A map R from $X \times X$ into $[0, 1]$ is termed an F-*indistinguishability operator* if the following properties hold for any x, y and z in X:

(1) $R(x, x) = 1$ (reflexivity),

(2) $R(x, y) = R(y, x)$ (symmetry),

(3) $F(R(x, y), R(y, z)) \leq R(x, z)$ (F-transitivity).

3. Main Results

In what follows, X stands for a non-empty set and F for a continuous t-norm.

Theorem 3.1. *Let T be a t-norm on I. Then the following are equivalent*:

(1) $T(x, y) = x \wedge y$ *for all* $x, y \in I$.

(2) *T is idempotent*.

Proof. Assume that $T(x, y) = x \wedge y$ for all x, y in I, and let $x \in I$ be any given. Then $T(x, x) = x \wedge x = x,$ which yields T is idempotent. Conversely, assume that T is idempotent, then we show that $T(x, y) = x \wedge y$ for all $x, y \in I$. First, since $T(x, y) \leq x \wedge y,$ it suffices to show that $x \wedge y \leq T(x, y)$. Now, without loss of generality, we may assume that $x \leq y$. Then $x \leq y = x$ and $x = T(x, x) \wedge T(x, y)$. This leads to $x \wedge y \leq T(x, y)$.

Theorem 3.2. *Let a t-norm F be idempotent. Then P is an F-indistinguishability operator if and only if P is a fuzzy equivalence relation*.

Proof. Assume P is an F-indistinguishability operator. To show that P is an equivalence relation. It suffices to show that P is transitive. We show that P is transitive:

$$\begin{aligned}(P \circ P)(x, y) &= \vee_{z \in X} (P(x, z) \wedge P(z, y)) \\ &= \vee_{z \in X} F(P(x, z), P(z, y)) \\ &\leq \vee_{z \in X} (P(x, y)) \\ &= P(x, y) \text{ for all } x, y, z \in X.\end{aligned}$$

Conversely, assume that P is fuzzy equivalence relation. To show that P is an F-indistinguishability operator it suffices to show that P is transitive under F. Indeed,

$$\begin{aligned} F(P(x, y), P(y, z)) &= P(x, y) \wedge P(y, z) \\ &\leq \vee_{t \in X} (P(x, t) \wedge P(t, z)) \\ &= (P \circ P)(x, z) \\ &\leq P(x, z) \text{ for all } x, y, z \in X \end{aligned}$$

which yields P is transitive under F.

Theorem 3.3. *Let a t-norm F be idempotent, and let a fuzzy relation R on X be symmetric and transitive under F. If $S(x, y) = R(x, y)$ for all $x \neq y$ in X and $S(x, x) = 1$ for all x in X, then S is fuzzy difunctional.*

Proof. It suffices to show that S is symmetric and transitive under $*$. Now, for $x \neq y$ in X, we have $S(x, y) = R(x, y) = R(y, x) = S(y, x)$. This means S is symmetric. To show that S is transitive under F, we show that $F(S(x, y), S(y, z)) \leq S(x, z)$ for all x, y, z in X. If $x = z$, since $S(x, z) = 1$, then the inequality is trivial, hence, without loss of generality, we may assume that $x \neq z$. For $z = y$, we have $F(S(x, y), S(y, z)) = F(R(x, y), 1) = R(x, y) = R(x, z) = S(x, z)$, for $z \neq y$, we have $F(S(x, y), S(y, z)) = F(R(x, y), R(y, z)) \leq R(x, z) = S(x, z)$ which yields S is transitive under F. This completes the proof.

Theorem 3.4. *If R is strict F-indistinguishability operator on X, then $R/\sim$ is a strict F-indistinguishability operator on $X/\sim$.*

Proof. We show that $R/\sim$ is well-defined on $X/\sim$. For $x, x' \in u$ and $y, y' \in v$, we have

$$\begin{aligned} R(x, y) &= F(R(x, x'), R(x, y)) \\ &= F(R(x', x), R(x, y)) \\ &\leq R(x', y) \\ &= F(R(x', y), R(y, y')) \\ &\leq R(x', y'). \end{aligned}$$

Similarly, we get $R(x', y') \le R(x, y)$. Hence we have $R(x, y) = R(x', y')$. This implies that the definition of $(R/\sim)(u, v)$ does not depend on the choice of $x \in u$, $y \in v$. Next, we show that $R/\sim$ is strict. For any $u, v \in X/\sim$, let $(R/\sim)(u, v) = 1$ and $(R/\sim)(v, u) = 1$. This means there exist $x, y \in X$ such that $x \in u$ and $y \in v$. This leads to $R(x, y) = 1$ and $R(y, x) = 1$. Thus $x \sim y$, hence we have $u = v$. Therefore, $R/\sim$ is strict. Last, we show that $R/\sim$ is an F-indistinguishability operator on $X/\sim$. For reflexivity of $R/\sim$, now let $u \in X/\sim$ be any given, then there exist $x \in X$ such that $x \in u$. Hence we have $(R/\sim)(u, u) = R(x, x) = 1$. Hence $R/\sim$ is reflexive. For symmetry of $R/\sim$, let $u, v \in X/\sim$. Then there exist $x \in u$ and $y \in v$, hence

$$(R/\sim)(u, v) = R(x, y) = R(y, x) = (R/\sim)(v, u)$$

which yields $R/\sim$ is symmetric. To show that $R/\sim$ is transitive under F, let $u, v, w \in X/\sim$ be any given, then there exist $x, y, z \in X$ such that $x \in u$, $y \in v$ and $z \in w$, since

$$\begin{aligned} F(R/\sim(u, v), R/\sim(v, w)) &= F(R(x, y), R(y, z)) \\ &\le R(x, z) \\ &= R/\sim(u, w) \end{aligned}$$

which yields $R/\sim$ is transitive under F. This completes the proof.

Theorem 3.5 (cf. Seo et al. [4])**.** *Let a fuzzy relation R be reflexive. Then R is a fuzzy equivalence relation if and only if R is fuzzy difunctional.*

Theorem 3.6. *Let a t-norm F be idempotent, and let a fuzzy relation P be reflexive. Then P is fuzzy difunctional if and only if P is an F-indistinguishability operator on X.*

Proof. It follows from Theorem 3.2 and Theorem 3.5.

Theorem 3.7. *Let a t-norm F be idempotent. Then R is a fuzzy equivalence relation if and only if R is an F-indistinguishability operator.*

Proof. Assume that R is a fuzzy equivalence relation. To show that R is an F-indistinguishability operator, it suffices to show R is F-transitive

$$\begin{aligned} F(R(x, z), R(z, y)) &= R(x, z) \wedge R(z, y) \\ &\leq \vee_{z \in X}(R(x, z), R(z, y)) \\ &= (R \circ R)(x, y) \\ &\leq R(x, y) \text{ for all } x, y, z \in X \end{aligned}$$

which yields R is F-transitive.

Conversely, suppose that R is an F-indistinguishability operator. To show that R is a fuzzy equivalence, it suffices to show R is transitive, we show that R is transitive

$$\begin{aligned} (R \circ R)(x, y) &= \vee_{z \in X}(R(x, z) \wedge R(z, y)) \\ &= \vee_{z \in X} F(R(x, z), R(z, y)) \\ &\leq \vee_{z \in X} R(x, y) \\ &= R(x, y) \text{ for all } x, y \in X \end{aligned}$$

which yields, $R \circ R \subseteq R.$ This completes the proof.

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