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THE LATTICE *f*-DERIVATIONS OF LATTICES

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Abstract

In this paper, we introduce the notion of a lattice *f*-derivation for a lattice and investigate some related properties. Moreover, we study the fixed set $Fix_d(L)$, ker d and the higher order derivation of a lattice *f*-derivation for a lattice.

1. Introduction

Let R be a ring. An additive mapping $D : R \to R$ is called a *derivation* of $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. Several authors [1, 2, 4] and [6] studied derivations in rings and near rings. Szasz [7] introduced the concept of derivation for lattices and investigated some of its properties. In 2008, Xin et al. [8] studied derivation of lattice and investigated some of its properties. In 2011, Harmaitree and Leerawat [3] studied *f*-derivation of

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lattice and investigated some of its properties. In this paper, we introduced a new concept called *lattice f-derivation* on a lattice and then we investigate some related properties.

2. Preliminaries

First, we shall give some basic definitions and results used throughout the entire paper. Details and proofs can be found in [3, 5] and [8].

Definition 2.1 [5]**.** An (algebraic) lattice $(L, \wedge, \vee)$ is a nonempty set L with two binary operations "$\wedge$" and "$\vee$" (read "*meet*" and "*join*", respectively) on L which satisfy the following conditions for all $x, y, z \in L$:

(i) $x \wedge x = x, \quad x \vee x = x$;

(ii) $x \wedge y = y \wedge x, \quad x \vee y = y \vee x$;

(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad x \vee (y \vee z) = (x \vee y) \vee z$;

(iv) $x = x \wedge (x \vee y), \quad x = x \vee (x \wedge y)$.

Definition 2.2 [5]**.** A poset $(L, \leq)$ is a *lattice ordered* if and only if for every pair x, y of elements of L both the $\sup\{x, y\}$ and the $\inf\{x, y\}$ exist.

Definition 2.3 [8]**.** Let $(L, \wedge, \vee)$ be a lattice. A binary operation "$\leq$" is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

Lemma 2.4 [8]**.** *Let* $(L, \wedge, \vee)$ *be a lattice. Define the binary operation* "$\leq$" *as Definition* 2.3. *Then* $(L, \leq)$ *is a poset and for any* $x, y \in L$, $x \wedge y$ *is the* $\sup\{x, y\}$ *and* $x \vee y$ *is the* $\inf\{x, y\}$.

Theorem 2.5 [5]**.** (i) *Let* $(L, \leq)$ *be a lattice ordered set. If we define* $x \wedge y = \inf\{x, y\}$, $x \vee y = \sup\{x, y\}$, *then* $(L, \wedge, \vee)$ *is an algebraic lattice.*

(ii) *Let* $(L, \wedge, \vee)$ *be an algebraic lattice. If we define* $x \leq y$ *if and only if* $x \wedge y = x$ *(or* $x \leq y$ *if and only if* $x \vee y = y$*), then* $(L, \leq)$ *is a lattice ordered set.*

It can be verified that Theorem 2.5 yields a one-to-one relationship between lattice ordered sets and algebraic lattices. Therefore, we shall use the term lattice for both concepts.

Theorem 2.6 [5]**.** (i) *Every ordered set is lattice ordered*.

(ii) *In a lattice ordered set* $(L, \leq)$ *the following statements are equivalent for all* $x, y \in L$:

(a) $x \leq y$; (b) $\sup\{x, y\} = y$; *and* (c) $\inf\{x, y\} = x$.

Definition 2.7 [5]**.** If a lattice L contains a least (greatest) element with respect to ≤, then this uniquely determined element is called the *zero element* (*one element*) denoted by 0 (by 1).

Lemma 2.8. *Let L be a lattice*. *Then* $x \wedge y = x$ *if and only if* $x \vee y = y$ *for all* $x, y \in L$.

Proof. Let $x, y \in L$ and assume $x \wedge y = x$. Then $x \vee y = (x \wedge y) \vee y = y$. Conversely, let $x \vee y = y$. So $x \wedge y = x \wedge (x \vee y) = x$.

Lemma 2.9 [5]**.** *Let L be a lattice*. *If* $y \leq z$, *then* $x \wedge y \leq x \wedge z$ *and* $x \vee y \leq x \vee z$ *for all* $x, y, z \in L$.

Definition 2.10 [5]**.** A nonempty subset S of a lattice L is called *sublattice* of L if S is a lattice with respect to the restriction of $\wedge$ and $\vee$ of L onto S.

Definition 2.11 [5]**.** A lattice L is called *modular* if for any $x, y, z \in L$ if $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Definition 2.12 [5]**.** A lattice L is called *distributive* if either of the following conditions hold for all x, y, z in L: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ or $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Corollary 2.13 [5]**.** *Every distributive lattice is a modular lattice*.

Definition 2.14 [5]**.** Let $f : L \to M$ be a function from a lattice L to a lattice M.

(i) f is called a *join-homomorphism* if $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in L$.

(ii) f is called a *meet-homomorphism* if $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in L$.

(iii) f is called a *lattice-homomorphism* if f are both a join-homomorphism and a meet-homomorphism.

(iv) f is called an *order-preserving* if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in L$.

Lemma 2.15 [5]**.** *Let $f : L \to M$ be a function from a lattice L to a lattice M. If f is a join-homomorphism* (*or a meet-homomorphism or a lattice-homomorphism*), *then f is an order-preserving*.

Definition 2.16 [5]**.** An ideal is a nonempty subset I of a lattice L with the properties:

(i) if $x \leq y$ and $y \in I$, then $x \in I$ for all x, y in L,

(ii) $x \vee y \in I$ for all $x, y \in I$.

Definition 2.17 [3]**.** Let L be a lattice and $f : L \to L$ be a function. Then a function $d : L \to L$ is called an *f-derivation* on L if for any $x, y \in L$, $d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y))$.

Proposition 2.18 [3]**.** *Let L be a lattice and d be an f-derivation on L, where $f : L \to L$ is a function. Then the following conditions hold*: *for any element $x, y \in L$*:

(1) $dx \leq f(x)$;

(2) $dx \wedge dy \leq d(x \wedge y) \leq dx \vee dy$.

3. The Lattice *f*-derivations on Lattices

The following definition introduces a notion of a derivation for lattice:

Definition 3.1. Let L be a lattice, $d : L \to L$ and $f : L \to L$ be functions. We call d a *lattice f-derivation* on L if for any $x, y \in L$, $d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y))$ and $d(x \vee y) = d(x) \vee d(y)$.

Remark. If d is a lattice *f*-derivation on L, then d is an *f*-derivation on L.

We often abbreviate $d(x)$ to dx.

Now we give some examples and some properties for the lattice *f*-derivation on lattices.

Example 3.2. Consider the lattice as shown in Figure 1:

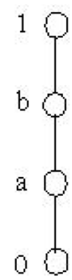

Figure 1

Define, respectively, a function d and a function f by

$$dx = \begin{cases} 0 & \text{if } x = 0, \\ a & \text{if } x = 1, a, b, \end{cases} \qquad f(x) = \begin{cases} 0 & \text{if } x = 0, \\ a & \text{if } x = a, b, \\ b & \text{if } x = 1. \end{cases}$$

Then it is easy to check that d is a lattice *f*-derivation.

Example 3.3. Consider the lattice as shown in Figure 1.

Define, respectively, a function d and a function f by

$$dx = \begin{cases} 0 & \text{if } x = 0, \\ b & \text{if } x = a, b, \\ a & \text{if } x = 1, \end{cases} \qquad f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x = a, b, 1. \end{cases}$$

Then it is easy to check that d is an *f*-derivation on L but it is not a lattice *f*-derivation, since $a = d(a \vee 1) \neq da \vee d1 = b$.

Proposition 3.4. *Let L be a lattice and d be a lattice f-derivation on L where $f : L \to L$ is a function. Then d is an order-preserving.*

Proof. Suppose that $x, y \in L$ such that $x \leq y$. Then $dy = d(x \vee y) = dx \vee dy$, it follows that $dx \leq dy$. So d is an order-preserving.

Proposition 3.5. *Let L be a lattice and d be a lattice f-derivation on L where $f : L \to L$ is a function. Then d is a lattice-homomorphism.*

Proof. Let $x, y \in L$. Then $d(x \vee y) = dx \vee dy$. Next, we will show that $d(x \wedge y) = dx \wedge dy$. By Proposition 2.18(2), we have $dx \wedge dy \leq d(x \wedge y)$. On the other hand, we get $d(x \wedge y) \leq dx$ and $d(x \wedge y) \leq dy$ since d is an order-preserving. So $d(x \wedge y) \leq dx \wedge dy$. Thus $d(x \wedge y) = dx \wedge dy$. Hence, d is a lattice-homomorphism.

Corollary 3.6. *Let L be a lattice and $d : L \to L$ be a function. Then d is a lattice-homomorphism if and only if d is a lattice d-derivation on L.*

Proof. Suppose that d is a lattice-homomorphism. Then we get d are both a join-homomorphism and a meet-homomorphism. So $d(x \vee y) = dx \vee dy$ and $d(x \wedge y) = (dx \wedge dy) = (dx \wedge dy) \vee (dx \wedge dy)$ for all $x, y \in L$. Thus, d is a lattice d-derivation. Conversely, it is obvious by Proposition 3.5.

Proposition 3.7. *Let L be a lattice with the greatest element* 1 *and d be a lattice f-derivation on L where $f : L \to L$ is a function such that $f(1) = 1$. Then $dx = f(x) \wedge d1$ for all $x \in L$.*

Proof. Let $x \in L$. By Proposition 2.18(1), we have $dx \leq f(x)$. By Proposition 3.4, we get $dx \leq d1$. So $dx \leq d1 \wedge f(x)$. Note that

$$\begin{aligned} dx &= d(x \wedge 1) = (dx \wedge f(1)) \vee (f(x) \wedge d1) = (dx \wedge 1) \vee (f(x) \wedge d1) \\ &= dx \vee (f(x) \wedge d1) = f(x) \wedge d1, \end{aligned}$$

completing the proof.

The following results are immediate from Proposition 3.7:

Corollary 3.8. *Let L be a lattice with the greatest element* 1 *and d be a lattice f-derivation on L where* $f : L \to L$ *is a function such that* $f(1) = 1$. *Then we have, for all* $x \in L$:

(1) $d1 \leq f(x)$ *if and only if* $dx = d1$;

(2) $f(x) \leq d1$ *if and only if* $dx = f(x)$;

(3) $d1 = 1$ *if and only if* $dx = f(x)$.

Corollary 3.9. *Let L be a lattice with the greatest element* 1 *and d be a lattice f-derivation on L where* $f : L \to L$ *is a function. Then* $d1 = 1$ *if and only if* $f(1) = 1$ *and* $dx = f(x)$ *for all* $x \in L$.

Proposition 3.10. *Let L be a lattice and d be a lattice f-derivation on L where* $f : L \to L$ *is a function. Then* $dx = dx \vee (f(x) \wedge dy)$ *for all* $x, y \in L$.

Proof. Let $x, y \in L$. Then we have $x = (x \wedge y) \vee x$. So

$$dx = d((x \wedge y) \vee x) = (d(x \wedge y) \vee dx) = ((dx \wedge f(y)) \vee (f(x) \wedge dy)) \vee dx$$

$$= ((dx \wedge f(y)) \vee dx) \vee (f(x) \wedge dy) = dx \vee (f(x) \wedge dy).$$

Proposition 3.11. *Let L be a lattice and d be a lattice f-derivation on L where* $f : L \to L$ *is an order-preserving. Then* $dx = d(x \vee y) \wedge f(x)$ *for all* $x, y \in L$.

Proof. Let $x, y \in L$. By Proposition 2.18(1) and Proposition 3.4, we get $dx \leq f(x) \leq f(x \vee y)$ and $dx \leq d(x \vee y)$. So $dx \leq d(x \vee y) \wedge f(x)$. Thus $dx = d((x \vee y) \wedge x) = (d(x \vee y) \wedge f(x)) \vee (f(x \vee y) \wedge dx) = (d(x \vee y) \wedge f(x)) \vee dx = d(x \vee y) \wedge f(x)$.

Proposition 3.12. *Let L be a lattice and d be a lattice f-derivation on L where* $f : L \to L$ *is an order-preserving. Then* $d(x \wedge y) = f(x) \wedge dy$ *for all* $x, y \in L$.

Proof. Let $x, y \in L.$ From Proposition 3.11, we have $dx = d(x \vee y) \wedge f(x).$ By Proposition 3.4, we know that d is a lattice-homomorphism. Then $d(x \wedge y) = dx \wedge dy = (d(x \vee y) \wedge f(x)) \wedge dy = f(x) \wedge (d(x \vee y) \wedge dy) = f(x) \wedge ((dx \vee dy) \wedge dy) = f(x) \wedge dy.$

Proposition 3.13. *Let L be a modular lattice and d be a lattice f-derivation on L where $f : L \to L$ is a function. Then $dx = f(x) \wedge d(x \vee y)$ for all $x, y \in L.$*

Proof. Let $x, y \in L.$ From Proposition 3.10, we have $dx = dx \vee (f(x) \wedge dy).$ Since L is a modular lattice and $dx \leq f(x),$ $dx = f(x) \wedge (dx \vee dy) = f(x) \wedge d(x \vee y).$

Proposition 3.14. *Let L be a modular lattice and d be a lattice f-derivation on L where $f : L \to L$ is a function. Then $d(x \wedge y) = f(x) \wedge dy$ for all $x, y \in L.$*

Proof. Let $x, y \in L.$ From Proposition 3.13, we have $dx = f(x) \wedge d(x \vee y).$ Then $d(x \wedge y) = dx \wedge dy = (f(x) \wedge d(x \vee y)) \wedge dy = f(x) \wedge (d(x \vee y) \wedge dy) = f(x) \wedge ((dx \vee dy) \wedge dy) = f(x) \wedge dy.$

Corollary 3.15. *Let L be a distributive lattice and d be a lattice f-derivation on L where $f : L \to L$ is a function. Then $dx = f(x) \wedge d(x \vee y)$ and $d(x \wedge y) = f(x) \wedge dy$ for all $x, y \in L.$*

Let L be a lattice and d be a lattice f-derivation on L where $f : L \to L$ is a function. Denote $Fix_d(L) = \{x \in L \mid dx = f(x)\}.$

In the following results, we assume that $Fix_d(L)$ is a nonempty proper subset of L.

Proposition 3.16. *Let L be a lattice and d be a lattice f-derivation on L where $f : L \to L$ is a meet-homomorphism. Let $x, y \in L$ be such that $y \leq x.$ If $x \in Fix_d(L),$ then $y \in Fix_d(L).$*

Proof. Let $x, y \in L$ be such that $y \leq x$. Then $dy = d(x \wedge y) = (dx \wedge f(y)) \vee (f(x) \wedge dy) = (f(x) \wedge f(y)) \vee (dx \wedge dy) = f(x \wedge y) \vee d(x \wedge y) = f(y) \vee dy = f(y)$.

Theorem 3.17. *Let L be a lattice and d be a lattice f-derivation on L where* $f : L \to L$ *is a lattice-homomorphism. Then* $Fix_d(L)$ *is a sublattice of L.*

Proof. Let $x, y \in Fix_d(L)$. Since $x, y \in Fix_d(L)$, $dx = f(x)$ and $dy = f(y)$. Since d and f are lattice-homomorphisms and by Proposition 3.5, $f(x \wedge y) = f(x) \wedge f(y) = dx \wedge dy = d(x \wedge y)$. So $x \wedge y \in Fix_d(L)$. Next, we will show that $x \vee y \in Fix_d(L)$. By Definition 3.1, $f(x \vee y) = f(x) \vee f(y) = dx \vee dy = d(x \vee y)$. Thus $x \vee y \in Fix_d(L)$. Hence $Fix_d(L)$ is a sublattice of L.

Theorem 3.18. *Let L be a lattice and d be a lattice f-derivation on L where* $f : L \to L$ *is a lattice-homomorphism. Then* $Fix_d(L)$ *is an ideal of L.*

Proof. The proof is by Proposition 3.16 and Theorem 3.17.

Let L be a lattice with a least element 0 and d be a lattice *f*-derivation on L where $f : L \to L$ is a function. Define $\ker d = \{x \in L \mid dx = 0\}$.

In the following results, we assume that $\ker d$ is a nonempty proper subset of L.

Theorem 3.19. *Let L be a lattice with a least element* 0 *and d be a lattice f-derivation on L where* $f : L \to L$ *is a function. Then* $\ker d$ *is a sublattice of L.*

Proof. Let $x, y \in \ker d$. Then $dx = 0 = dy$. By Proposition 3.5, we get $d(x \wedge y) = dx \wedge dy = 0$, it follow that $x \wedge y \in \ker d$. By Definition 3.1, we have $d(x \vee y) = dx \vee dy = 0 \vee 0 = 0$. So $x \vee y \in \ker d$. Hence $\ker d$ is a sublattice of L.

Theorem 3.20. *Let L be a lattice with a least element* 0 *and d be a lattice f-derivation on L where* $f : L \to L$ *is a function. Then* $\ker d$ *is an ideal of* L.

Proof. Let $x, y \in L$ such that $x \leq y$ and $y \in \ker d$. By Proposition 3.5, we get $dx = d(x \wedge y) = dx \wedge dy = dx \wedge 0 = 0,$ it follow that $x \in \ker d$. By Theorem 3.19, we know that $\ker d$ is a sublattice of L, and so $x \vee y \in \ker d$ for all $x, y \in \ker d$. Hence $\ker d$ is an ideal of L.

Definition 3.21. Let L be a lattice and $f : L \to L$ be a function. Then a nonempty subset I of L is said to be an *f-invariant* if $f(I) \subseteq I$, where $f(I) = \{y \in L \mid y = f(x) \text{ for some } x \in I\}$.

Theorem 3.22. *Let L be a lattice and d be a lattice f-derivation on L where* $f : L \to L$ *is a function. Let I be an ideal of L such that I is an f-invariant. Then I is a d-invariant.*

Proof. Assume that I is an ideal of L such that I is an f-invariant and let $y \in dI$. Then there exists $x \in I$ such that $y = dx$. Since I is an f-invariant, $f(x) \in I$. By Proposition 2.18(1), we have $y = dx \leq f(x)$. By Definition 2.16, we get $y \in I$. Thus $dI \subseteq I$.

Let L be a lattice, $d_1 : L \to L$ and $d_2 : L \to L$ be functions. Define a function $d_1 \circ d_2 : L \to L$ by $d_1 \circ d_2 = d_1(d_2 x)$ for all $x \in L$.

Theorem 3.23. *Let L be a lattice,* d_1 *and* d_2 *be lattice f-derivation on L where* $f_1 : L \to L$ *and* $f_2 : L \to L$ *are functions, respectively. Then* $d_1 \circ d_2$ *is a lattice* $f_1 \circ f_2$*-derivation on L.*

Proof. Let $x, y \in L$. Then

$$\begin{aligned} d_1 \circ d_2(x \wedge y) &= d_1(d_2(x \wedge y)) \\ &= d_1((d_2 x \wedge f_2(y)) \vee (f_2(x) \wedge d_2 y)) \\ &= d_1(d_2 x \wedge f_2(y)) \vee d_1(f_2(x) \wedge d_2 y) \end{aligned}$$

$$= (d_1(d_2x) \wedge f_1(f_2(y))) \vee (f_1(d_2x) \wedge d_1(f_2(y)))$$

$$\vee (d_1(f_2(x)) \wedge f_1(d_2y)) \vee (f_1(f_2(x)) \wedge d_1(d_2y))$$

$$= [(d_1(d_2x) \wedge f_1(f_2(y))) \vee (f_1(f_2(x)) \wedge d_1(d_2y))]$$

$$\vee (f_1(d_2x) \wedge d_1(f_2(y))) \vee (d_1(f_2(x)) \wedge f_1(d_2y))$$

$$\geq (d_1(d_2x) \wedge f_1(f_2(y))) \vee (f_1(f_2(x)) \wedge d_1(d_2y))$$

$$= (d_1 \circ d_2(x) \wedge f_1 \circ f_2(y)) \vee (f_1 \circ f_2(x) \wedge d_1 \circ d_2(y)).$$

On the other hand, we have $d_1(f_2(x)) \leq f_1(f_2(x))$ and $d_1(f_2(y)) \leq f_1(f_2(y))$. Then

$$d_1 \circ d_2(x \wedge y) = d_1(d_2(x \wedge y))$$

$$= d_1((d_2x \wedge f_2(y)) \vee (f_2(x) \wedge d_2y))$$

$$= d_1(d_2x \wedge f_2(y)) \vee d_1(f_2(x) \wedge d_2y)$$

$$= (d_1(d_2x) \wedge d_1(f_2(y))) \vee (d_1(d_2x) \wedge d_1(f_2(y)))$$

$$\leq (d_1(d_2x) \wedge d_1(f_2(y))) \vee (f_1(d_2x) \wedge d_1(f_2(y)))$$

$$\leq (d_1(d_2x) \wedge f_1(f_2(y))) \vee (f_1(d_2x) \wedge d_1(f_2(y)))$$

$$= (d_1 \circ d_2(x) \wedge f_1 \circ f_2(y)) \vee (f_1 \circ f_2(x) \wedge d_1 \circ d_2(y)).$$

So $d_1 \circ d_2(x \wedge y) = (d_1 \circ d_2(x) \wedge f_1 \circ f_2(y)) \vee (f_1 \circ f_2(x) \wedge d_1 \circ d_2(y))$. Moreover, we get $d_1 \circ d_2(x \vee y) = d_1(d_2(x \vee y)) = d_1(d_2x) \vee d_1(d_2y) = d_1 \circ d_2(x) \vee d_1 \circ d_2(y)$, that is, $d_1 \circ d_2$ is a lattice $f_1 \circ f_2$-derivation on L.

Theorem 3.24. *Let L be a lattice, d_i be a lattice f-derivation on L where $f_i : L \to L$ is a function for $i = 1, 2, 3, \ldots, n, \ldots$. Then $d_1 \circ d_2 \circ \cdots \circ d_n$ is a lattice $f_1 \circ f_2 \circ \cdots \circ f_n$-derivation on L.*

Proof. When $n = 2$. By Theorem 3.23, we get $d_1 \circ d_2$ is a lattice $f_1 \circ f_2$-derivation on L. Let $n \in N$ for $n \geq 3$ and assume that $d_1 \circ d_2 \circ$

$\cdots \circ d_n$ is a lattice $f_1 \circ f_2 \circ \cdots \circ f_n$-derivation on L. Since d^{n+1} is a lattice f_{n+1}-derivation on L and by Theorem 3.23, $d_1 \circ d_2 \circ \cdots \circ d_n \circ d_{n+1}$ is a lattice $f_1 \circ f_2 \circ \cdots \circ f_n \circ f_{n+1}$-derivation on L.

Definition 3.25. Let L be a lattice, $x \in L$ and $d : L \to L$ be a function. Denote $d^n(x) = \underbrace{d \circ d \circ d \circ \cdots \circ d}_{n}(x) = \underbrace{d(d(\cdots(d}_{n}(x))))$. A $d^n(x)$ is said to be a *lattice f-derivation order n* of x if d is a lattice f-derivation on L.

Theorem 3.26. *Let L be a lattice and d be a lattice f-derivation on L where* $f : L \to L$ *is an order-preserving. Then* $d^2x = f(d(x)) \wedge d(f(x))$ *for all* $x \in L$.

Proof. Let $x \in L$. Then we have $dx \leq f(x)$ and $d^2x = d(dx) \leq f(dx)$. Since f is an order-preserving, $f(dx) \leq f(fx)$. So $d^2x \leq f(dx) \leq f(f(x))$. Then $d^2x = d(dx \wedge f(x)) = (d^2x \wedge f(f(x))) \vee (f(dx) \wedge d(f(x))) = d^2x \vee (f(dx) \wedge d(f(x)))$. So $d^2x \geq f(dx) \wedge d(f(x))$. On the other hand, $d^2x \leq f(dx)$. By Lemma 2.9, $d^2x \wedge d(f(x)) \leq f(dx) \wedge d(f(x))$. By Proposition 3.4, we know that d is an order-preserving, then $d^2x \leq d(f(x))$, and so $d^2x \leq f(dx) \wedge d(f(x))$. Hence $d^2x = f(dx) \wedge d(f(x))$.

Theorem 3.27. *Let L be a lattice and d be a lattice f-derivation on L, where* $f : L \to L$ *is an order-preserving. Then* $d^nx = f(d^{n-1}x) \wedge d(f(d^{n-2}x))$ *for integer* $n \geq 2$.

Proof. For $n = 2$. By Theorem 3.26, we get $d^2x = f(dx) \wedge d(f(x))$. Let $n \in N$ for $n \geq 3$ and assume that $d^nx = f(d^{n-1}x) \wedge d(f(d^{n-2}x))$. Then

$$\begin{aligned} d^{n+1}x = d^n(dx) &= f(d^{n-1}(dx)) \wedge d(f(d^{n-2}(dx))) \\ &= f(d^n(x)) \wedge d(f(d^{n-1}(x))). \end{aligned}$$

Theorem 3.28. *Let L be a lattice and d be a lattice f-derivation on L where $f : L \to L$ is an order-preserving such that $d(f(x)) = f(dx)$ for all $x \in L$. Then $d^2x = d(f(x))$ for all $x \in L$.*

Proof. Let $x \in L$. By Theorem 3.26, we have $d^2x = f(dx) \wedge d(f(x))$. Since $d(f(x)) = f(dx)$, $d^2x = d(f(x))$.

Theorem 3.29. *Let L be a lattice and d be a lattice f-derivation on L where $f : L \to L$ is an order-preserving such that $d(f(x)) = f(dx)$ for all $x \in L$. Then $d^nx = d^{n-1}(f(x))$ for integer $n \geq 2$.*

Proof. For $n = 2$. By Theorem 3.28, we get $d^2x = d(f(x))$. Let $n \in N$ for $n \geq 3$ and assume that $d^nx = d^{n-1}(f(x))$. Then $d^{n+1}x = d^n(dx) = d^{n-1}(f(dx)) = d^{n-1}(d(f(x))) = d^n(f(x))$.

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