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NOTE ON SPECTRAL SEMISTAR OPERATIONS, III

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Abstract

We prove that an integral domain D admits only spectral semistar operations if and only if D is a discrete valuation domain.

1. Introduction

This is a continuation of [6]. Let D be an integral domain with quotient field K. Let $\overline{\mathrm{F}}(D)$ be the set of non-zero D-submodules of K, let $\mathrm{F}(D)$ be the set of non-zero fractional ideals of D, i.e., $E \in \mathrm{F}(D)$ if $E \in \overline{\mathrm{F}}(D)$ and there is an element $d \in D \backslash \{0\}$ such that $dE \subseteq D.$ A semistar operation on D is a mapping $\star : \overline{\mathrm{F}}(D) \to \overline{\mathrm{F}}(D),\ E \mapsto E^{\star},$ such that for every $x \in K \backslash \{0\}$ and for every $E, F \in \overline{\mathrm{F}}(D),$ the following conditions hold: $(xE)^{\star} = xE^{\star};$ $E \subseteq F$ implies $E^{\star} \subseteq F^{\star};$ $E \subseteq E^{\star},$ and $(E^{\star})^{\star} = E^{\star}.$ For every $\Delta \subseteq \mathrm{Spec}(D),$ the mapping $E \mapsto \bigcap_{P \in \Delta} ED_P$ is a semistar operation on D, and is denoted by $\star_{\Delta}.$ A semistar operation $\star$ on D is called *spectral* if there is a subset $\Delta \subseteq \mathrm{Spec}(D)$ such that $\star = \star_{\Delta}.$ Picozza [7] posed the following

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question: When are all semistar operations on an integral domain spectral? We study the question, and prove the following:

Theorem. *D admits only spectral semistar operations if and only if D is a discrete valuation domain.*

A valuation domain with value group Γ is called *discrete* if, for every pair H_1, H_2 of adjacent convex subgroups of Γ (say, $H_1 \subsetneqq H_2$), the ordered factor group $\dfrac{H_2}{H_1}$ is order isomorphic with $\boldsymbol{Z}$.

Picozza [7, Corollary 4.13] showed the following: Let D be a local domain, i.e., D has only one maximal ideal. Then every semistar operation on D is spectral if and only if D is a discrete valuation domain. In [5, Theorem 1.1], we proved the following: Let D be a finite dimensional domain. Then every semistar operation on D is spectral if and only if D is a discrete valuation domain.

The paper consists of three sections. Section 2 contains lemmas. Section 3 contains the proof for Theorem.

2. Lemmas

Let D be a domain with quotient field K. A star operation on D is a mapping $\star : \mathrm{F}(D) \to \mathrm{F}(D)$, $E \mapsto E^{\star}$, such that for every $x \in K \backslash \{0\}$ and for every $E, F \in \mathrm{F}(D)$, the following conditions hold: $D^{\star} = D$; $(xE)^{\star} = xE^{\star}$; $E \subseteq F$ implies $E^{\star} \subseteq F^{\star}$; $E \subseteq E^{\star}$, and $(E^{\star})^{\star} = E^{\star}$.

For a Prüfer domain D, call two maximal ideals M and N of D *dependent* if $M \cap N$ contains a non-zero prime ideal of D. This defines an equivalence relation on the set of maximal ideals of D, and we denote by $\{A_{\lambda} \mid \lambda \in \Lambda\}$ the corresponding partition. For every $\lambda \in \Lambda$, denote by P_{λ} the largest prime ideal of D contained in $\bigcap_{M \in A_{\lambda}} M$, and let $S_{\lambda} = \bigcap_{M \in A_{\lambda}} D_M$. Recently, Houston proved the following:

Theorem 1 [2]**.** *An integrally closed domain D admits only finitely many star operations if and only if D is a Prüfer domain which satisfies the following conditions*: (1) *Every non-zero element of D is contained in only finitely many maximal ideals*, (2) $|A_\lambda| = 1$ *for almost all* $\lambda \in \Lambda$, (3) $\left|\mathrm{Spec}\left(\frac{D}{P_\lambda}\right)\right| < \infty$ *for every* $\lambda \in \Lambda$, *and* (4) *D has only finitely many non-invertible maximal ideals*.

Moreover, under these conditions, if we denote by $\mathrm{Star}(D)$ the set of star operations on *D*, then $|\mathrm{Star}(D)| = \prod_{\lambda \in \Lambda} |\mathrm{Star}(S_\lambda)|$.

For semistar operations, we have the following:

Theorem 2 [4, Section 5, (5.2), Theorem]**.** *An integrally closed domain D admits only finitely many semistar operations if and only if D is a finite dimensional Prüfer domain which has only finitely many maximal ideals*.

Throughout the rest of the section, *let D be a Prüfer domain with exactly two maximal ideals M and N*. Set $V = D_M$ and set $W = D_N$.

Lemma 1. *Assume that M and N are independent*. *Then we have* $V \notin \mathrm{F}(D)$ *and* $W \notin \mathrm{F}(D)$.

Proof. Suppose that $V \in \mathrm{F}(D)$. Then $aV \subseteq D$ for some $a \in D \setminus \{0\}$, hence $aVW \subseteq W$. It follows that $K = W$; a contradiction.

Lemma 2. (1) *Set* $\Delta = \{M, N\}$, *and set* $\star = \star_\Delta$. *Then* $I^\star = I$ *for every* $I \in \mathrm{F}(D)$.

(2) *Let P and Q be incomparable prime ideals of D*. *Set* $\Delta = \{P, Q\}$, *and set* $\star = \star_\Delta$. *If we set* $T = D_P \cap D_Q$, *then* $I^\star = IT$ *for every* $I \in \mathrm{F}(D)$.

Proof. (1) Let $x \in I^\star$. Since $x \in ID_M \cap ID_N$, we have $x = \frac{e_1}{s} = \frac{e_2}{t}$, where $e_1, e_2 \in I$, $s \in D \setminus M$ and $t \in D \setminus N$. Then $(s, t) = D$, hence $1 = sd_1$

$+ td_2$ with $d_1, d_2 \in D$. Then $x = sxd_1 + txd_2 = e_1d_1 + e_2d_2 \in I$, hence $I^\star = I$.

(2) We may assume that $P \subseteq M$ and $Q \subseteq N$. Set $T = D_P \bigcap D_Q$, set $M_1 = PD_P \bigcap T$, and set $N_1 = QD_Q \bigcap T$. T is a Prüfer domain with exactly two maximal ideals M_1 and N_1. Moreover, $T_{M_1} = D_P$ and $T_{N_1} = D_Q$. Let J be a non-zero ideal of T. By (1), we have $J = JT_{M_1} \bigcap JT_{N_1}$, hence $J = JD_P \bigcap JD_Q$. Therefore, if I is a non-zero ideal of D, then $IT = (IT)D_P \bigcap (IT)D_Q$, hence $IT = ID_P \bigcap ID_Q$, i.e., $IT = I^\star$.

Lemma 3. *Let* $\Delta = \{P_\lambda \mid \lambda \in \Lambda\}$ *be a subset of* $\mathrm{Spec}(D)$ *with every* $P_\lambda \subseteq M$, *let* $\star = \star_\Delta$, *and set* $P = \bigcup_\lambda P_\lambda$.

(1) *We have* $D^\star = \bigcap_\lambda D_{P_\lambda} = D_P$.

(2) *Assume that* $M = P$. *Then, for every prime ideal* Q *of* D *with* $Q \subsetneqq M$, *we have* $Q \subseteq P_\lambda$ *for some* λ.

Proof. (1) Let $x \notin D_P$, and set $m = \dfrac{1}{x}$. Since D_P is a valuation domain, we have $m \in PD_P$, and $sm \in P$ for some $s \in D \backslash P$. Then $sm \in P_\lambda$ for some λ. Since $1 = mx = sm\dfrac{x}{s} \in P_\lambda\dfrac{x}{s}$, we have $\dfrac{x}{s} \notin D_{P_\lambda}$, hence $x \notin D_{P_\lambda}$. Therefore $\bigcap_\lambda D_{P_\lambda} = D_P$.

(2) Take an element $d \in M \backslash Q$. Then we have $\dfrac{d}{1} \in P_\lambda D_M$ for some λ and $\dfrac{d}{1} \notin QD_M$. Hence $P_\lambda D_M \supseteq QD_M$, hence $P_\lambda \supseteq Q$.

Lemma 4. *Let* $\Delta_1 = \{P_\lambda \mid \lambda \in \Lambda\}$ *be a subset of* $\mathrm{Spec}(D)$ *with every* $P_\lambda \subseteq M$, *let* $\Delta_2 = \{Q_\sigma \mid \sigma \in \Sigma\}$ *be a subset of* $\mathrm{Spec}(D)$ *with every* $Q_\sigma \subseteq N$, *set* $\Delta = \Delta_1 \bigcup \Delta_2$, *and let* $\star = \star_\Delta$. *Set* $P = \bigcup_\lambda P_\lambda$, *and set*

$Q = \bigcup_\sigma Q_\sigma$. *Assume that* P *and* Q *are incomparable*.

(1) *We have* $D^\star = \bigcap_\lambda D_{P_\lambda} \bigcap_\sigma D_{Q_\sigma} = D_P \bigcap D_Q$.

(2) $D^\star = D$ *if and only if* $P = M$ *and* $Q = N$.

Proof. (1) follows from Lemma 3(1).

(2) $T := D_P \bigcap D_Q$ is a Prüfer domain with maximal ideals $PD_P \bigcap T$ and $QD_Q \bigcap T$. Assume that $D^\star = D$. Then $T = D$ by (1). Hence $PD_P \bigcap D = M$ and $QD_Q \bigcap D = N$. Therefore $P = M$ and $Q = N$.

Lemma 5. *Let* $\Delta' = \{P_a \mid a \in A\}$ *be the set of prime ideals* P *of* D *with* $P \subsetneqq M$, *and let* $\Delta'' = \{Q_b \mid b \in B\}$ *be the set of prime ideals* Q *of* D *with* $Q \subsetneqq N$. *Let* $\star$ *be a spectral semistar operation on* D *with* $D^\star = D$. *Then one of the following four conditions holds*:

(1) $\star$ *is defined by* $\{M, N\}$.

(2) $\star$ *is defined by* $\Delta' \cup \{N\}$ *with* $M = \bigcup_a P_a$.

(3) $\star$ *is defined by* $\{M\} \cup \Delta''$ *with* $N = \bigcup_b Q_b$.

(4) $\star$ *is defined by* $\Delta' \cup \Delta''$ *with* $M = \bigcup_a P_a$ *and* $N = \bigcup_b Q_b$.

Proof. We confer Lemmas 2, 3 and 4. Let $\star = \star_\Delta$. By Lemma 3(1), we have $\Delta = \Delta_1 \bigcup \Delta_2$, $\Delta_1 = \{P_\lambda \mid \lambda \in \Lambda\}$ with every $P_\lambda \subseteq M$ and $\Delta_2 = \{Q_\sigma \mid \sigma \in \Sigma\}$ with every $Q_\sigma \subseteq N$. Set $P = \bigcup_\lambda P_\lambda$ and $Q = \bigcup_\sigma Q_\sigma$. Then P and Q are not comparable. By Lemma 4, we have $P = M$ and $Q = N$. The following four cases may occur: (1) $\Delta_1 \ni M$ and $\Delta_2 \ni N$, (2) $\Delta_1 \not\ni M$ and $\Delta_2 \ni N$, (3) $\Delta_1 \ni M$ and $\Delta_2 \not\ni N$, (4) $\Delta_1 \not\ni M$ and $\Delta_2 \not\ni N$.

The case (1): $I^\star = I$ for every $I \in \mathrm{F}(D)$.

The case (2): We may assume that $\Delta_1 = \Delta'$. We have $M^\star = D$ and $N^\star = N$.

The case (3): We may assume that $\Delta_2 = \Delta''$. We have $M^\star = M$ and $N^\star = D$.

The case (4): We may assume that $\Delta_1 = \Delta'$ and $\Delta_2 = \Delta''$. We have $M^\star = D$ and $N^\star = D$.

The semistar operation in (1) (resp., (2), (3), (4)) of Lemma 5 is denoted by $\star_1$ (resp., $\star_2, \star_3, \star_4$). Easily, we have $(D_M)^{\star_i} = D_M$ for every $1 \leq i \leq 4$.

3. Proof of Theorem

By [7] and [5], it is sufficient to prove the necessity. Thus, throughout the section, we assume that D is an infinite dimensional, non-local Prüfer domain with quotient field K. Let M be a maximal ideal of D with $\mathrm{ht}(M) = \infty$, and let N be another maximal ideal of D. Then $T := D_M \cap D_N$ is an infinite dimensional Prüfer domain with exactly two maximal ideals.

Therefore, we may assume that *D is an infinite dimensional Prüfer domain with exactly two maximal ideals M and N*. We may assume that $\mathrm{ht}(M) = \infty$. Set $V = D_M$, and set $W = D_N$.

Proposition 1. *Assume that M and N are independent. Then there is a non-spectral semistar operation $\star$ on D with $D^\star = D$.*

Proof. Set $I^\star = I$ for every $I \in \mathrm{F}(D)$, and set $J^\star = K$ for every $J \in \overline{\mathrm{F}}(D) \backslash \mathrm{F}(D)$. Then $\star$ is a semistar operation on D with $D^\star = D$. By Lemma 1, we have $V^\star = K$. Suppose that $\star$ is spectral. By Lemma 5, we have $\star = \star_i$ for some $1 \leq i \leq 4$. By the definition of $\star_i$, we have $V^{\star_i} = V$; a contradiction.

Proposition 2. *Assume that M and N are dependent, and assume that* $\left|\operatorname{Spec}\left(\frac{D}{P_0}\right)\right| < \infty$, *where* P_0 *is the largest prime ideal of D contained in* $M \cap N$. *Then there is a non-spectral semistar operation* $\star$ *on D with* $D^\star = D$.

Proof. Let $\star := \mathrm{v}$ be the v-semistar operation on D, i.e., $J^{\mathrm{v}} = D :_K (D :_K J)$ for every $J \in \overline{\mathrm{F}}(D)$. By Heinzer [1, Theorem 5.1], the restriction of $\star$ to $\mathrm{F}(D)$ differs from the identity mapping. Suppose that $\star$ is spectral, and let $\star = \star_\Delta$. We confer Lemma 5. Since $\left|\operatorname{Spec}\left(\frac{D}{P_0}\right)\right| < \infty$, almost all $P_a \subseteq P_0$, and almost all $Q_b \subseteq P_0$. Hence, neither (2) nor (3) nor (4) of Lemma 5 occurs. It follows that $\star$ is defined by $\{M, N\}$; hence the restriction of $\star$ to $\mathrm{F}(D)$ is the identity mapping; a contradiction.

Proposition 3. *Assume that M and N are dependent, and assume that* $\left|\operatorname{Spec}\left(\frac{D}{P_0}\right)\right| = \infty$, *where* P_0 *is the largest prime ideal of D contained in* $M \cap N$. *Then D has an infinite number of non-spectral semistar operations* $\star$ *with* $D^\star = D$.

Proof. By Lemma 5, the number of spectral semistar operations on D with $D^\star = D$ is less than or equal to 4. On the other hand, since $\left|\operatorname{Spec}\left(\frac{D}{P_0}\right)\right| = \infty$, D has an infinite number of star operations by Theorem 1. Every member $\star \in \operatorname{Star}(D)$ can be uniquely extended to a semistar operation $\star'$ on D ([3, Lemma 2]). Since $D^{\star'} = D$, the proof is complete.

Propositions 1, 2 and 3 complete the proof of our Theorem.

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