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PSEUDO *k*-FLAT SEMIMODULES IN SEMIRINGS

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Abstract

In this paper, we principally introduce the concept of quasi-principally *k*-flat semimodules, on the basis of the theories of *k*-flat semimodules and quasi-principally modules, we get some good properties of quasi-principally *k*-flat semimodules, therefore generalize some properties of quasi-principally modules of rings and *k*-flat semimodules of semiring to quasi-principally *k*-flat semimodules of semirings.

1. Introduction

Throughout R will denote a semiring with identity 1; unless otherwise stated, all semimodules M will be left R-semimodules with $1 \cdot m = m$ for all $m \in M$, and all homomorphisms will be R-homomorphisms.

In this paper, we will use the following facts (cf. [1, 3, 9-12]:

(a) A semiring R is said to satisfy the *left cancellation law*, if and only

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if for all $a, b, c \in R$, $a + b = a + c \Rightarrow b = c$. A semimodule M is said to satisfy the *left cancellation law*, if and only if for all $m, m', m'' \in R$, $m + m' = m + m'' \Rightarrow m' = m''$.

(b) We say that a nonempty subset N of a left semimodule M is *subtractive* if and only if $m + m' \in N$ and $m \in N$ imply that $m' \in N$ for all $m, m' \in M$; a semiring R is called *completely subtractive* if R is a completely subtractive semimodule; and a left R-semimodule M is called *completely subtractive* if and only if for every subsemimodule N of M, N is subtractive.

(c) A semimodule M is said to be *free R-semimodule* if M has a basis over R.

(d) A semimodule M is said to be *semicogenerated* by C when there is a homomorphism $\varphi : M \to \prod_A C$ such that $\ker \varphi = 0$. A semimodule C is said to be *semicogenerator* when C semicogenerates every left R-semimodule M.

(e) Let $\alpha : A \to B$ be a homomorphism of semimodules. Then we shall define the subsemimodule Im α of B as follows:

$$\operatorname{Im} \alpha = \{b \in B : b + \alpha(a) = \alpha(a') \text{ for some } a, a' \in A\},$$

α is said to be *i-regular* if $\alpha(A) = \operatorname{Im} \alpha$; to be *$k$-regular* if for $a, a' \in A$, $\alpha(a) = \alpha(a')$ implies $a + k = a' + k'$ for some $k, k' \in Ker(\alpha)$; and to be *semimonomorphism* if $Ker(\alpha) = 0$, to be an *isomorphism* if it is injective and surjective, and to be *regular* if it is both i-regular and k-regular.

(f) An R-semimodule M is said to be *k-regular* if there exist a free R-semimodule F and a surjective R-homomorphism $\alpha : F \to P$ such that α is k-regular.

(g) The sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is called an *exact sequence* if $Ker(\beta) = \operatorname{Im}(\alpha)$ and is *proper exact* if $Ker(\beta) = \alpha(B)$.

(h) A proper exact sequence of the form $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ is said to be *left k-regular right regular* if α is k-regular, β is right regular.

(i) For any two R-semimodules N, M,

$$Hom_R(N, M) := \{\alpha : N \to M \mid \alpha \text{ is an } R\text{-homomorphism of semimodules}\}$$

is a semigroup under addition. If N, M and U are R-semimodules and $\alpha : M \to N$ is homomorphism, then $Hom(\alpha, I_U) : Hom_R(N, U) \to Hom_R(M, U)$ is given by $Hom(\alpha, I_U)\gamma = \gamma\alpha,$ where I_U is the identity on U.

(j) Let A be semimodule, given a symmetric subsemimodule R of $A \times A,$ we often denote (a, b) in R by $\langle a, b\rangle,$ we define a relation $\rho(R)$ on A as follows:

$x\rho(R)y$ if and only if there exists $\langle a, b\rangle \in R$ such that $x + a = y + b.$

(k) If A is a right R-semimodule, B is a left R-semimodule, let N be the semiring of non-negative integers, N be a commutative semiring with $1 \neq 0$, and T be an N-semimodule, then a function $\theta : A \times B \to T$ is R-balanced if and only if for all $a, a' \in A,$ for all $b, b' \in B,$ and for all $r \in R,$ we have

(1) $\theta(a + a', b) = \theta(a, b) + \theta(a', b),$

(2) $\theta(a, b + b') = \theta(a, b) + \theta(a, b'),$

(3) $\theta(ar, b) = \theta(a, rb).$

For a and a' in A, b and b' in B, and r in R, and where $\alpha[a, b]$ is the function from $A \times B$ to B which sends (a, b) to 1 and sends every other element of $A \times B$ to 0. Let $N[A \times B]$ be the free semimodule generated by the set $A \times B,$ whose element is written as $\alpha = \Sigma n_i(a_i, b_i).$ We denote by $N[A \times B] \times N[A \times B]$ the product of semimodules, whose element is written as $\langle \alpha, \beta\rangle,$ for some $\alpha, \beta \in N[A \times B]$ and whose addition is given by $(\alpha, \beta) + (\alpha', \beta') = (\alpha + \alpha', \beta + \beta').$ Define M as the subset of the semimodule

$N[A \times B] \times N[A \times B]$ which consists of all elements of the following forms:

(1) $\langle (a + a', b), (a, b) + (a', b) \rangle$, (2) $\langle (a, b) + (a', b), (a + a', b) \rangle$,

(3) $\langle (a, b) + ((a, b')), (a, b + b') \rangle$, (4) $\langle (a, b + b'), (a, b) + (a, b') \rangle$,

(5) $\langle (a\lambda, b), (a, \lambda b) \rangle$, (6) $\langle (a, \lambda b), (a\lambda, b) \rangle$,

where $a, a' \in A$, $b, b' \in B$ and $\lambda \in R$.

Let $N[M]$ be the subsemimodule of $N[A \times B] \times N[A \times B]$ generated by M, and $\rho(N(M))$ be a congruence on $N(A \times B)$ by [10]. The set $N(A \times B)/\rho(N(M))$ is a semimodule in 2.3 of [10] and define $A \otimes_R B = N(A \times B)/\rho(N(M))$. The semimodule $A \otimes_R B$ is called the *tensor product* of A and B over R.

(l) A left R-semimodule P is said to be *projective semimodule* if and only if for each surjective R-homomorphism $\varphi : M \to N$, the induced homomorphism $\overline{\varphi} : Hom_R(P, M) \to Hom_R(P, N)$ is surjective. Similarly, M is a pseudo projective semimodule if and only if for each surjective R-homomorphism $\alpha : M \to N$, the induced homomorphism $\overline{\alpha} : Hom(M, M) \to Hom(M, N)$ is surjective. Especially, if f is k-regular, then M is called *pseudo k-projective semimodules*.

(m) A semimodule V_R is flat relative to a semimodule $_RM$ (or that V is M-flat) if and only if for every subsemimodule $K \leq M$, the sequence $0 \to V \otimes_R K \xrightarrow{I_Y \otimes_R i_k} V \otimes_R M$ is proper exact (i.e., $Ker(I_V \otimes_R i_k) = 0$), where $I_V \otimes_R i_k(v \otimes k) = v \otimes i_k(k)$. A semimodule V_R, that is, flat relative to every left R-semimodule is called a *flat right R-semimodule*.

(n) A semimodule V_R is k-flat relative to a semimodule $_RM$ (or that V is Mk-flat) if and only if for every subsemimodule $K \leq M$, the sequence $0 \to V \otimes_R K \xrightarrow{I_Y \otimes_R i_k} V \otimes_R M$ is proper exact and $I_V \otimes_R i_k$ is k-regular (i.e., $I_V \otimes_R i_k$ is injective). A semimodule V_R, that is, k-flat relative to every left R-semimodule is called a *k-flat right R-semimodule*. Thus, if V_R is k-flat relative to ${}_RM$, then V_R is flat relative to ${}_RM$.

2. Pseudo *k*-flat Semimodules

In this section, we discuss the structure of pseudo k-flat semimodules.

Definition 2.1. A semimodule V_R is pseudo k-flat relative to a semimodule ${}_RM$ (or that V is pseudo Mk-flat) if and only if for every semimodule $M = M$, the sequence $0 \to V \otimes {}_RM \xrightarrow{I_Y \otimes {}_Ri_M} V \otimes {}_RM$ is proper exact (i.e., $I_V \otimes {}_Ri_k$ is injective). A semimodule V_R, that is, pseudo k-flat relative to every left R-semimodule is called a *pseudo k-flat right R-semimodule*.

Our next result shows that the class of pseudo k-flat semimodules is closed under direct sums.

Proposition 2.2. *Let $(V_\alpha)_{\alpha \in A}$ be an indexed set of right R-semimodule. Then $\oplus_A V_\alpha$ is pseudo k-flat if and only if each V_α is pseudo k-flat.*

Proof. Let M be a left R-semimodule. Consider the following commutative diagram:

$$\begin{tikzcd}
V_\alpha \otimes M \arrow[r, "I_{V_\alpha} \otimes i_M"] \arrow[d, shift left, "i_\alpha"] & V_\alpha \otimes M \arrow[d, shift right, "i_\alpha"'] \\
\oplus(V_\alpha \otimes M) \arrow[r, "I_{\oplus(V_\alpha \otimes M)}"] \arrow[u, shift left, "\pi_\alpha"] \arrow[d, leftrightarrow, "\varphi"'] & \oplus(V_\alpha \otimes M) \arrow[u, shift right, "\pi_\alpha"'] \arrow[d, leftrightarrow, "\varphi'"'] \\
(\oplus V_\alpha) \otimes M \arrow[r, "I_{\oplus V_\alpha} \otimes i_M"] & (\oplus(V_\alpha) \otimes M
\end{tikzcd} \tag{2.1}$$

where $\pi_\alpha : \oplus(V_\alpha \otimes M) \to V_\alpha \otimes M$ and $i_\alpha : V_\alpha \otimes M \to \oplus(V_\alpha \otimes M)$ are defined, respectively, by $\pi_\alpha : (v_\alpha \otimes m_\alpha) \to v_\alpha \otimes m_\alpha$ and $i_\alpha : v_\alpha \otimes m_\alpha \to (v_\alpha \otimes m_\alpha)$, where $v_i \otimes m_i = 0$ if $i \neq \alpha$ and $v_i \otimes m_i = v_\alpha \otimes m_\alpha$ if $i = \alpha$; φ and φ' are the isomorphisms of [9, Proposition 5.4] given by $\varphi[(v_\alpha) \otimes m]$

$= (v_\alpha \otimes m)$ and $\varphi'(v_\alpha \otimes m) = (v_\alpha \otimes i(m))$. Now suppose that $\oplus V_\alpha$ is pseudo k-flat. If

$$I_{V_\alpha} \otimes i_m(v_\alpha \otimes m) = 0[I_{V_\alpha} \otimes i_m((v_\alpha \otimes m)) = I_{V_\alpha} \otimes i_m((v'_\alpha \otimes m'))],$$

then by the above diagram, we have

$$(v_\alpha) \otimes i_M(m) = 0[(v_\alpha) \otimes i(m) = (v'_\alpha) \otimes i(m')].$$

Since $\oplus V_\alpha$ is pseudo k-flat, $(v_\alpha) \otimes m = 0[(v_\alpha) \otimes k = (v'_\alpha) \otimes k']$. Again, by (2.1), $(v_\alpha) \otimes i_M(m) = 0$ whence

$$v_\alpha \otimes m = 0[(v_\alpha \otimes k) = (v'_\alpha \otimes k'), \text{ whence } v_\alpha \otimes k = v'_\alpha \otimes k'].$$

Therefore, V_α is pseudo k-flat.

Conversely, suppose that V_α is pseudo k-flat for each $\alpha \in A$. If

$$I_{\oplus V_\alpha} \otimes i_M((v_\alpha) \otimes m)$$
$$= 0[I_{\oplus V_\alpha} \otimes i_M((v_\alpha) \otimes m) = I_{\oplus V_\alpha} \otimes i_M((v'_\alpha) \otimes m')],$$

then by the above diagram, we have $v_\alpha \otimes i(m) = 0[v_\alpha \otimes i(m) = v'_\alpha \otimes i(m')]$ for each $\alpha \in A$. Since V_α is pseudo k-flat, $v_\alpha \otimes m = 0[v_\alpha \otimes m = v'_\alpha \otimes m']$ for each α. Therefore, $(v_\alpha \otimes m) = 0[(v_\alpha \otimes m) = (v'_\alpha \otimes m')]$. Again, by (2.1), $(v_\alpha) \otimes m = 0[(v_\alpha) \otimes m = (v'_\alpha) \otimes m']$. Thus, $\oplus V_\alpha$ is pseudo k-flat.

Proposition 2.3. *Let M be a left R-semimodule. Then a right R-semimodule V is pseudo k-flat if and only if the functor $(V \otimes_R -)$ preserves the exactness of all left k-regular right regular short exact sequences with middle term M:*

$$0 \to {}_RM \xrightarrow{\alpha} {}_RM \xrightarrow{\beta} {}_RN \to 0. \tag{2.2}$$

Proof. "If" part. Let $0 \to {}_RM \xrightarrow{\alpha} {}_RM \xrightarrow{\beta} {}_RN \to 0$ be a left k-regular right regular exact sequence. Since V_R is ${}_RM$-pseudo k-flat, using [9, Theorem 5.5(2)], the sequence

$$0 \to V \otimes {}_R M \xrightarrow{I_Y \otimes \alpha} V \otimes {}_R M \xrightarrow{I_Y \otimes \beta} V \otimes {}_R N \to 0 \qquad (2.3)$$

is exact.

"Only if" part. Let ${}_R M = {}_R M$. Consider the following exact sequence:

$$0 \to M \xrightarrow{i_K} M \xrightarrow{\pi i_K(M)} M/i_K(M) \to 0. \qquad (2.4)$$

By hypothesis, $0 \to V \otimes {}_R M \xrightarrow{I_Y \otimes \alpha} V \otimes {}_R M$ is an exact sequence. Thus, V_R is ${}_R M$-pseudo k-flat.

Our next result gives a necessary and sufficient condition for a projective semimodule to be pseudo k-flat relative to a cancellable semimodule M.

Proposition 2.4. *Let V_R be pseudo projective and ${}_R M$ cancellable. Then V is pseudo k-flat if and only if the functor $(V \otimes {}_R -)$ preserves the exactness of all left k-regular right regular short exact sequences*

$$0 \to {}_R M \xrightarrow{\alpha} {}_R M \xrightarrow{\beta} {}_R N \to 0. \qquad (2.5)$$

Proof. "If" part. Let $0 \to {}_R M \xrightarrow{\alpha} {}_R M \xrightarrow{\beta} {}_R N \to 0$ be a left k-regular right regular exact sequence. Since V_R is ${}_R M$-pseudo k-flat, V_R is ${}_R M$-flat. By using Proposition 2.3, the sequence

$$0 \to V \otimes {}_R M \xrightarrow{I_Y \otimes \alpha} V \otimes {}_R M \xrightarrow{I_Y \otimes \beta} V \otimes {}_R N \to 0 \qquad (2.6)$$

is exact.

"Only if" part. Let ${}_R M = {}_R M$. Consider the following exact sequence:

$$0 \to M \xrightarrow{i_K} M \xrightarrow{\pi i_K(M)} M/i_K(M) \to 0. \qquad (2.7)$$

Since V is pseudo projective and ${}_R M$ cancellable, by using [10, Proposition 1.16], $I_V \otimes i_k$ is k-regular. By hypothesis, $0 \to V \otimes {}_R M \xrightarrow{I_Y \otimes \alpha} V \otimes {}_R M$ is an exact sequence. Thus, V is ${}_R M$ pseudo k-flat.

3. Flat via Injectivity

We will discuss the relation between the injectivity and flatness. By $(\cdot)^*$, we mean the functor $Hom_R(-, C)$, where C is a fixed injective semicogenerator cancellative N-semimodule.

Remark 3.1 [3]**.** If U is a right R-semimodule, then U^* is a left R-semimodule.

We state and prove the following lemma, analogous to the one on modules which is needed in the proof of Proposition 3.3.

Lemma 3.2 [3]**.** *Let R be a semiring, M and M' be left R-semimodules, and U be a right R-semimodule. Let T be a cancellative N-semimodule. If $\alpha : M' \to M$ is an R-homomorphism, then there exist N-isomorphisms φ and φ' such that the following diagram commutes*:

$$\begin{tikzcd}
Hom_R(M, Hom_N(U,T)) \arrow[r, "Hom_R(\alpha, I_{Hom_N(U,T)})"] \arrow[d, "\varphi"'] & Hom_R(M', Hom_N(U,T_N)) \arrow[d, "\varphi'"'] \\
Hom_N((U \otimes_R M),T) \arrow[r, "Hom_R(\alpha, I_{Hom_N(U,T)})"] & Hom_N(U \otimes_R M'),T)
\end{tikzcd} \tag{3.1}$$

Proposition 3.3. *Let M be a left R-semimodule.*

(1) *If the right R-semimodule V is pseudo k-flat, then V^* is M-injective.*

(2) *If V^* is M-injective, then V is pseudo k-flat.*

Proof. (1) Let K be a subsemimodule of M. Since V is pseudo k-flat, the sequence $0 \to V \otimes K \xrightarrow{I_Y \otimes i_K} V \otimes M$ is proper exact, and $I_V \otimes i_K$ is k-regular. By Lemma 3.2, we have the following commutative diagram:

$$\begin{tikzcd}
Hom_R(M, V^*) \arrow[r, "Hom_R(i_K, I_{V^*})"] \arrow[d, "\varphi'"'] & Hom_R(K, V^*) \arrow[r] \arrow[d, "\varphi"'] & 0 \\
(V \otimes M)^* \arrow[r, "Hom_R(I_V \otimes i_K, I_C)"] & (V \otimes K)^* \arrow[r] & 0
\end{tikzcd} \tag{3.2}$$

where φ and φ' are N-isomorphisms. It follows that the top row is proper exact if and only if the bottom row is proper exact, whence by [8, Proposition 3.1], V^* is injective.

(2) If V^* is injective, then

$$Hom_R(M, V^*) \xrightarrow{Hom_R(i_K, I_{V^*})} Hom_R(K, V^*) \to 0 \tag{3.3}$$

is proper exact. Again by the above diagram,

$$(V \otimes M)^* \xrightarrow{Hom_R(I_V \otimes i_K, I_C)} (V \otimes K)^* \to 0 \tag{3.4}$$

is proper exact. Hence, the sequence is exact. Since C is a semicogenerator, by [11, Proposition 4.1], the sequence $0 \to V \otimes K \to V \otimes M$ is exact. Hence, V is pseudo k-flat.

4. Cancellable Semimodules

In this section, we deal with cancellable semimodules, we characterize pseudo k-flat semimodules by means of left ideals.

Proposition 4.1. *The following statements about a cancellable right R-semimodule V are equivalent*:

(1) *V is pseudo k-flat*;

(2) *For each* (*finitely generated*) *left ideal $I \leq {}_RR$, the surjective N-homomorphism $\varphi : V \otimes_R I \to VI$ with $\varphi(v \otimes a) = va$ is a k-regular semimonomorphism*.

Proof. (1) $\Rightarrow$ (2) Since V is cancellable, by using [1, Proposition 14.16], $V \otimes_R R \cong V$. Consider the following commutative diagram:

$$\begin{array}{ccc} V \otimes_R I & \xrightarrow{I_V \otimes i_I} & V \otimes_R R \\ \varphi' \downarrow & & \varphi \downarrow \\ VI & \xrightarrow{I_{VI}} & V \end{array} \tag{4.1}$$

where θ is the isomorphism of [1, Proposition 14.16]. Since $\psi : VI \to VI$

given by $\psi(v, i) = vi$ is an R-balanced function, by using [1, Proposition 14.14], there is an exact unique N-homomorphism $\varphi : V \otimes I \to VI$ satisfying the condition $\varphi(v \otimes i) = \psi(v, i)$. Since V is pseudo k-flat relative to ${}_R R$, $I_V \otimes {}_R i_I$ is injective. If $\varphi(\Sigma v_i \otimes a_i) = \varphi(\Sigma v_i' \otimes a_i')$, then

$$\theta(I_V \otimes {}_R i_I)(\Sigma v_i \otimes a_i) = \theta(I_V \otimes {}_R i_I)(\Sigma v_i' \otimes a_i').$$

Since θ and $I_V \otimes {}_R i_I$ are injective, $\Sigma(v_i \otimes a_i) = \Sigma(v_i' \otimes a_i')$.

(2) $\Rightarrow$ (1) Again consider the above diagram. Let I be any left ideal of R and let $I_V \otimes {}_R i_I(\Sigma v_i \otimes a_i) = I_V \otimes {}_R i_I(\Sigma v_i' \otimes a_i')$, where $\Sigma(v_i \otimes a_i) = \Sigma(v_i' \otimes a_i') \in V \otimes {}_R I$. Let $K_1 = \Sigma R a_i$, $K_2 = \Sigma R a_i'$ and $K = K_1 + K_2$. Now

$$\theta(I_V \otimes {}_R i_I)(\Sigma v_i \otimes a_i) = \theta(I_V \otimes {}_R i_I)(\Sigma v_i' \otimes a_i'),$$

whence $\Sigma(v_i \otimes a_i) = \Sigma(v_i' \otimes a_i') \in V \otimes {}_R I$.

Now consider the following commutative diagram, where $i_K : K \to I$ is the inclusion map:

$$\begin{tikzcd} V \otimes K \arrow[r, "I_V \otimes i_K"] \arrow[d, "\varphi_K"'] & V \otimes I \arrow[d, "\theta"'] \\ VK \arrow[r, "I_{VK}"] & V \end{tikzcd} \tag{4.2}$$

By hypothesis, φ_k is monic. Thus, $\Sigma_i v_i \otimes a_i = \Sigma_i v_i' \otimes a_i'$ as an element of $V \otimes K$. Hence, $I_V \otimes {}_R i_I(\Sigma v_i \otimes a_i) = I_V \otimes {}_R i_I(\Sigma v_i' \otimes a_i') \in V \otimes I$ and $\Sigma_i v_i \otimes a_i = \Sigma_i v_i' \otimes a_i'$ as an element of $V \otimes I$. Therefore, $I_V \otimes {}_R i_I$ is monic. Hence, V is pseudo k-flat relative to ${}_R R$.

Proposition 4.2. *Let M be a cancellable left R-semimodule. Then R_R is pseudo k-flat.*

Proof. Let $i_K : K \to M$ be the inclusion homomorphism. By [1, Proposition 14.16], $R \otimes {}_R K \cong K$ and $R \otimes {}_R M \cong M$. Consider the following commutative diagram:

$$\begin{array}{ccc} R \otimes_R K & \xrightarrow{I_R \otimes i} & R \otimes_R M \\ \cong \big\downarrow & & \cong \big\downarrow \\ K & \xrightarrow{i_K} & M \end{array} \tag{4.3}$$

Since i_K is injective, $I \otimes_R i_K$ is injective.

Corollary 4.3. *Let M be a cancellable left R-semimodule. Then every free R-semimodule is pseudo k-flat.*

Proof. The proof is immediate from Propositions 2.3 and 4.2.

In module theory, every projective module is flat. Now we see that this is true for certain special semimodules.

Proposition 4.4. *Let M be a cancellable left R-semimodule, where R is a cancellative completely subtractive semiring. Then every k-regular projective R-semimodule P is pseudo k-flat.*

Proof. By using [12, Proposition 19], *P* is isomorphic to a direct summand of a free semimodule *F*. By Corollary 4.3, *F* is pseudo *k*-flat. Hence, by using Proposition 2.3, *P* is pseudo *k*-flat.

Corollary 4.5. *Let M be a k-regular left R-semimodule and R be a cancellative completely subtractive semiring. Then every k-regular projective R-semimodule P is pseudo k-flat.*

Proof. We only need to show that *M* is cancellable. Since *M* is *k*-regular, there exists a free *R*-semimodule *F* such that $\varphi : F \to M$ is surjective. Let $m_1 + m = m_2 + m$, where $m_1, m_2, m \in M$. Since φ is surjective, $\varphi(a_1) + \varphi(a) = \varphi(a_2) + \varphi(a)$, where $\varphi(a_1) = m_1$, $\varphi(a) = m$ and $\varphi(a_2) = m_2$. Since φ is *k*-regular, $a_1 + a + k_1 = a_2 + a + k_2$, where $k_1, k_2 \in Ker\varphi$. Since *F* is cancellable, $a_1 + k_1 = a_2 + k_2$. Hence $\varphi(a_1) = \varphi(a_2)$.

Proposition 4.6. *Let M be a cancellable left R-semimodule. If V is a free R-semimodule, then the following assertions hold*:

(a) *V is pseudo k-flat*;

(b) V^* *is M-injective*.

Proof. By using Corollary 4.3, V is pseudo k-flat.

(a) $\Rightarrow$ (b) The proof is immediate from Proposition 3.3.

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