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COEFFICIENT BOUNDS OF STARLIKE AND CONVEX FUNCTIONS OF COMPLEX ORDER INVOLVING GENERALIZED OPERATOR

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Abstract

In this paper, we derive some coefficient bounds for functions in the subclasses $SC_{\alpha, \beta, l}^{q, s, m}(b, \lambda_1, \lambda_2, \gamma)$ and $T_{\alpha, \beta, l}^{q, s, m}(b, \lambda_1, \lambda_2, \gamma, \mu)$ of A which consists of functions $f \in A.$ We also obtain a necessary and sufficient condition for functions to be in these subclasses.

1. Introduction

As usual, let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U}), \tag{1.1}$$

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which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$.

It is well known that for two functions *f*, given by (1.1) and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad (z \in \mathbb{U}),$$

the Hadamard product (convolution) of *f* and *g*, is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Let S stand for the well known subclass of A consisting of univalent functions.

The authors in [4] have, recently, introduced a new generalized operator $D_l^{m, \lambda_1, \lambda_2}(\alpha_i, \beta_j) f(z)$ as the following:

For complex parameters $\alpha_1, \alpha_2, ..., \alpha_q$ and $\beta_1, \beta_2, ..., \beta_s$, $(\beta_j \in \mathbb{C} - \mathbb{Z}_0, \mathbb{Z}_0 = \{0, -1, -2, ...\}; j = 1, 2, ..., s)$.

Let the generalized hypergeometric function ${}_qF_s(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z)$ be defined by

$${}_qF_s(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k, (\alpha_2)_k, ..., (\alpha_q)_k}{(\beta_1)_k, (\beta_2)_k, ..., (\beta_s)_k} \frac{z^k}{k!},$$

where

$$(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{U}),$$

for $\mathbb{N}$ denotes the set of all positive integers and $(x)_k$ is the Pochhammer symbol defined, in terms of the Γ function, by

$$(x)_k = \begin{cases} 1 & \text{for } k = 0, \\ x(x+1)(x+2)\cdots(x+k-1) & \text{for } k \in \mathbb{N} = \{1, 2, 3, ...\}. \end{cases}$$

Corresponding to a function $h_{q,s}(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z)$ defined by

$$h_{q,s}(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z)$$

$$=: z_q F_s(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z).$$

Dziok-Srivastava [8] introduced a convolution operator on A such that

$$H_{q,s}(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z) : A \to A$$

is defined by

$$H(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z)$$

$$= h(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z) * f(z)$$

$$= z + \sum_{k=2}^{\infty} \Gamma_k a_k z^k,$$

where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1}, (\alpha_2)_{k-1}, ..., (\alpha_q)_{k-1}}{(\beta_1)_{k-1}, (\beta_2)_{k-1}, ..., (\beta_s)_{k-1}(1)_{k-1}}.$$

Definition 1.1. Let

$$\varphi_l^{m,\lambda_1,\lambda_2}(z)(\alpha_i, \beta_j; z) = \sum_{k=2}^{\infty} \Gamma_k(\alpha_1)\delta_k z^k,$$

where

$$\delta_k = \frac{(1 + \lambda_1(k-1) + l)^{m-1}}{(1 + l)^{m-1}(1 + \lambda_2(k-1))^m}$$

$$(i = \{1, 2, ..., q\}, \ j = \{1, 2, ..., s\}, \ z \in \mathbb{U}),$$

and $(z \in \mathbb{U}, \lambda_2 \geq \lambda_1 \geq 0, l \geq 0, m \in \mathbb{Z})$, also $(x)_k$ is the Pochhammer symbol.

We define a linear operator $D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_j): A \to A$ by the following Hadamard product:

$$D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_j)f(z) := \varphi_l^{m,\lambda_1,\lambda_2}(z)(\alpha_i, \beta_j; z) * f(z)$$

$$D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_j)f(z) = z + \sum_{k=2}^{\infty} \delta_k \Gamma_k(\alpha_1) a_k z^k. \tag{1.2}$$

This operator $D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_j)f(z)$ includes various other linear operators which were considered in earlier works in the literature.

For $m = 1$ and $\lambda_2 = 0,$ we obtain

$$D_0^{1,0,0}(\alpha_i, \beta_j)f(z) = H_{q,s}(\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s; z),$$

which was given by Dziok-Srivastava [8].

For $\alpha_i = 1$ and $\beta_j = 1,$ we obtain

$$D_l^{m,\lambda_1,\lambda_2}(1, 1)f(z) = I^m(\lambda_1, \lambda_2, l, n)f(z)$$

as given in [3].

For $s = 1$ and $q = 2,$ we obtain the linear operator:

$$D_0^{1,0,0}(\alpha_1, \alpha_2, \beta_1)f(z) = F(\alpha_1, \alpha_2, \beta_1)f(z)$$

which was introduced by Hohlov [9]. Moreover, putting $\alpha_2 = 1,$ we obtain the Carlson-Shaffer operator [5]:

$$D_0^{1,0,0}(\alpha_1, 1, \beta_1)f(z) = L(\alpha_1, \beta_1)f(z).$$

Ruscheweyh [19] introduced an operator

$$D_0^{1,0,0}(\lambda + 1, 1, 1)f(z) = D^\lambda f(z).$$

Definition 1.2. Let the class S_b^* consisting of all analytic functions $f \in A$ satisfy the following inequality:

$$\Re\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0, \quad (b \in \mathbb{C}^* = \mathbb{C} - \{0\},\ z \in \mathbb{U}).$$

The class C_b consists of all analytic functions $f \in A$ satisfying

$$\Re\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > 0, \quad (b \in \mathbb{C}^* = \mathbb{C} - \{0\},\ z \in \mathbb{U}).$$

The function classes S_b^* and C_b were considered earlier by Nasr and Aouf [16-18] and Wiatrowski [21], respectively.

Furthermore, a function $f(z) \in A$ is said to be in the class $SC(b, \lambda, \gamma)$ if it satisfies the following inequality:

$$\Re\left\{1 + \frac{1}{b}\left(\frac{z[\lambda zf'(z) + (1-\lambda)f(z)]'}{\lambda zf'(z) + (1-\lambda)f(z)} - 1\right)\right\} > \gamma,$$

$$(b \in \mathbb{C}^* = \mathbb{C} - \{0\},\ z \in \mathbb{U}), \tag{1.3}$$

where $(f(z) \in A,\ 0 \le \lambda \le 1,\ 0 \le \gamma < 1)$.

The function class satisfying the inequality (1.3) was considered by [2].

Let $SC_{\alpha,\beta,l}^{q,s,m}(b, \lambda_1, \lambda_2, \gamma)$ denote the subclass of A consisting of functions $f(z)$ which satisfy the following condition:

$$\Re\left\{1 + \frac{1}{b}\left(\frac{z[D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_j)f(z)]'}{D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_j)f(z)} - 1\right)\right\} > \gamma, \quad (b \in \mathbb{C}^* = \mathbb{C} - \{0\}), \tag{1.4}$$

where $(z \in \mathbb{U},\ \lambda_2 \ge \lambda_1 \ge 0,\ l \ge 0,\ m \in \mathbb{Z})$ and $(f(z) \in A,\ 0 \le \gamma < 1,\ q \le s + 1,\ q, s \in \mathbb{N}_0,\ b \in \mathbb{C}^* = \mathbb{C} - \{0\})$.

Note that for $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, $\alpha_2 = 1$, $m = 1$, $\lambda_2 = 0$, $\gamma = 0$, the class $SC_{\alpha,\beta}^{q,s}(b, \lambda, \gamma)$ coincides the class S_b^*. Furthermore, for $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, $\alpha_2 = 1$, $l = 0$, $m = 2$, $\lambda_2 = 0$, $\gamma = 0$ and $\lambda_1 = 1$, we obtain the class C_b, and when $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, $\alpha_2 = 1$, $l = 0$, $m = 2$, $\lambda_2 = 0$, we obtain the class of $SC(b, \lambda, \gamma)$.

The main aim of the present investigation is to derive some coefficient bounds for functions in the subclass $T_{\alpha,\beta,l}^{q,s,m}(b, \lambda_1, \lambda_2, \gamma, \mu)$ of A which consists of functions $f(z) \in A$ satisfying the following nonhomogeneous Cauchy-Euler differential equation:

$$z^2 \frac{d^2\omega}{dz^2} + 2(1+\mu) z \frac{d\omega}{dz} + \mu(1+\mu)\omega = (1+\mu)(2+\mu) g(z), \tag{1.5}$$

where $(\omega = f(z) \in A,\ g(z) \in SC_{\alpha,\beta}^{q,s}(b, \lambda, \gamma),\ \mu \in R \backslash (-\infty, -1])$.

2. Main Results

Theorem 2.1. *Let* $f(z) \in A$ *be defined by* (1.1). *If the function* $f \in SC_{\alpha,\beta,l}^{q,s,m}(b, \lambda_1, \lambda_2, \gamma)$, *then*

$$|a_k| \le \frac{\prod_{j=0}^{k-2} [j + 2|b|(1-\gamma)]}{\delta_k \Gamma_k(\alpha_1)}, \quad (k \in \mathbb{N} \backslash \{1\} = \{2, 3, 4, \ldots\}). \tag{2.1}$$

Proof. Define the function $F(z)$ by

$$F(z) = D_l^{m,\lambda_1,\lambda_2}(\alpha_i, \beta_j) f(z) = z + \sum_{k=2}^{\infty} A_k z^k,$$

where $A_k = \delta_k \Gamma_k(\alpha_1) a_k$.

Thus, by setting

$$\frac{1+\frac{1}{b}\left(\frac{zF'(z)}{F(z)}-1\right)-\gamma}{1-\gamma} = p(z),$$

or, equivalently,

$$zF'(z) = [1 + b(1-\gamma)(p(z)-1)]F(z), \tag{2.2}$$

we get

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \quad (z \in \mathbb{U}). \tag{2.3}$$

Since

$$\Re\{p(z)\} > 0, \quad (0 \le \gamma < 1;\ b \in \mathbb{C}^*),$$

we conclude that

$$|\, p_k \,| \le 2, \quad (k \in \mathbb{N}).$$

We also find from (2.2) and (2.3) that

$$(k-1)A_k = b(1-\gamma)(p_1 A_{k-1} + p_2 A_{k-2} + \cdots + p_{k-1}).$$

In particular, for $k = 2, 3, 4,$ we have

$A_2 = b(1-\gamma)p_1$ implies $|\, A_2 \,| \le 2|\, b \,|(1-\gamma),$

$2A_3 = b(1-\gamma)(p_1 A_2 + p_2)$ implies $|\, A_3 \,| \le \dfrac{2|\, b \,|(1-\gamma)[1 + 2|\, b \,|(1-\gamma)]}{2!}$

and

$3A_4 = b(1-\gamma)(p_1 A_3 + p_2 A_2 + p_3)$ implies

$$|\, A_4 \,| \le \frac{2|\, b \,|(1-\gamma)[1 + 2|\, b \,|(1-\gamma)][2 + 2|\, b \,|(1-\gamma)]}{3!},$$

respectively. Using the principle of mathematical induction, we obtain

$$|A_k| \le \frac{\prod_{j=0}^{k-2}[j+2|b|(1-\gamma)]}{(k-1)!}, \quad (k \in \mathbb{N}\backslash\{1\} = \{2, 3, 4, \ldots\}).$$

Moreover, by the relationship between the functions $f(z)$ and $F(z)$, it is clear that

$$|a_k| \le \frac{\prod_{j=0}^{k-2}[j+2|b|(1-\gamma)]}{\delta_k \Gamma_k(\alpha_1)}, \quad (k \in \mathbb{N}\backslash\{1\} = \{2, 3, 4, \ldots\}).$$

By choosing suitable values of all the admissible parameters that we used in Theorem 2.1, we deduce the following corollaries.

Corollary 2.1 ([2]). *Let* $f(z) \in A$ *be defined by* (1.1). *If the function* $f \in SC(b, \lambda, \gamma)$, *then*

$$|a_k| \le \frac{\prod_{j=0}^{k-2}[j+2|b|(1-\gamma)]}{(k-1)![1+\lambda(k-1)]}, \quad (k \in \mathbb{N}\backslash\{1\} = \{2, 3, 4, \ldots\}).$$

Corollary 2.2 ([16]). *Let* $f(z) \in A$ *be defined by* (1.1). *If the function* $f \in S_b^*$, *then*

$$|a_k| \le \frac{\prod_{j=0}^{k-2}[j+2|b|]}{(k-1)!}, \quad (k \in \mathbb{N}\backslash\{1\} = \{2, 3, 4, \ldots\}).$$

Corollary 2.3 ([16]). *Let* $f(z) \in A$ *be defined by* (1.1). *If the function* $f \in C_b$, *then*

$$|a_k| \le \frac{\prod_{j=0}^{k-2}[j+2|b|]}{k!}, \quad (k \in \mathbb{N}\backslash\{1\} = \{2, 3, 4, \ldots\}).$$

Theorem 2.2. *Let* $f(z) \in A$ *be defined by* (1.1). *If the function* $f \in T_{\alpha,\beta,l}^{q,s,m}(b, \lambda_1, \lambda_2, \gamma, \mu)$, *then*

$$|\, a_k \,| \leq \frac{(1+\mu)(2+\mu)\prod_{j=0}^{k-2}[j + 2|\, b \,|(1-\gamma)]}{\delta_k \Gamma_k(\alpha_1)(k+\mu)(k+1+\mu)},$$

$$(k \in \mathbb{N}\backslash\{1\} = \{2, 3, 4, \ldots\}). \tag{2.4}$$

Proof. Let $f(z) \in A$ be defined by (1.1). Also, let

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in SC_{\alpha, \beta, l}^{q, s, m}(b, \lambda_1, \lambda_2, \gamma),$$

so

$$a_k \leq \frac{(1+\mu)(2+\mu)}{(k+\mu)(k+1+\mu)} b_k \quad (k \in \mathbb{N}^* = \mathbb{N}/\{0\},\ \mu \in R\backslash(-\infty, -1]).$$

Thus, by using Theorem 2.1, we readily obtain

$$|\, a_k \,| \leq \frac{(1+\mu)(2+\mu)\prod_{j=0}^{k-2}[j + 2|\, b \,|(1-\gamma)]}{\delta_k \Gamma_k(\alpha_1)(k+\mu)(k+1+\mu)},$$

$$(k \in \mathbb{N}\backslash\{1\} = \{2, 3, 4, \ldots\}),$$

which is precisely the assertion (2.4) of Theorem 2.2.

Some other works related to other generalized derivative or integral operators can be found in ([1, 6, 7, 10-15, 20]).

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