Far East Journal of Mathematical Sciences (FJMS)
Volume 71, Number 1, 2012, Pages 105-120
Published Online: December 2012
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

FULLY PLANARIZED OF PEKERIS, ACCAD AND SHKOLLAR FLOW

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Abstract

Pekeris, Accad and Shkollar (PAS) dynamo (Pekeris et al. [7]) is one of the earliest successful kinematic dynamos. PAS dynamo has convergent solution at relatively low magnetic Reynolds number. Bachtiar, Ivers and James (BIJ) (Bachtiar et al. [2]) tried to planarized the PAS flow in order to produce a planar velocity dynamo. BIJ claim that it is impossible to planarize PAS flow using their formula. In this paper, we discuss why PAS flow cannot be planarized in detail. We also propose a modified PAS flow so that it can be planarized and still have a similar structure. We believe that our results are important for future works.

I. Introduction

It is believed that the Earth's Magnetic Field (EMF) can be explained by self excited dynamo process that take place in the interior of the Earth. Considering the Earth's interior and the properties of EMF, magnetohydrodynamic (MHD) is assumed as the best model to represent the Earth's dynamo.

2010 Mathematics Subject Classification: 65-XX.

Keywords and phrases: planar flow, PAS dynamo, kinematic dynamos, boundary condition.

Received March 5, 2012

In MHD, there are six equations need to be solved simultaneously. Many scholars think that it is not easy to solve MHD. The fundamental sub-problem of MHD is the kinematic dynamo problem. In kinematic dynamo, we only consider one equation, the magnetic induction equation, where the velocity field is specified. This equation is the essential part of MHD. It describes the interaction between magnetic field and the velocity field in order to produce self excited dynamo.

Pekeris, Accad and Shkollar (PAS) flow is one of the earliest successful flows that can maintain magnetic field. PAS flow is also an example of Beltrami flow. It has a strong helicity that is believed as an important factor to produce a dynamo process. Dudley and James [4] revisited this flow using their formula and got the same result as PAS. The advantage of PAS model is that the convergent of the solution is easy to get at relatively low truncation level. In other words, we can get the solution using a relatively small computer capacity.

On the other hand, some scholars found conditions that the dynamo process cannot be happened. This condition is called by *anti-dynamo theorem*. They usually provide the proof of the theorem for general situation. These results help many scholars in finding new kinematic dynamo models.

Planar velocity theorem (PVT) is an example of anti-dynamo theorem. It was proposed by Zel'dovich in 1957. PVT precludes the existence of dynamo process when the conductor is planar or parallel to plane. However, the proof of this theorem assumed that the conducting fluid occupies all space. Moffat [6] mentioned that the proof of PVT using finite volume is easy to construct. Unfortunately, he did not provide the proof.

In 2006, Bachtiar, Ivers and James (BIJ) claimed that it is impossible to prove PVT for finite volume analytically. Following their claim, BIJ evaluated 32 models for planar velocity dynamos with single harmonic flow numerically. They found one flow that might possible to maintain magnetic field. However, their result has convergent problem. Although it is an early indication, their result proposed that PVT theorem is now a conjecture.

To provide an additional proof, BIJ tried to planarize several well known flows that can maintain magnetic field. One of them is the PAS flow. Using their formula, they found that it is impossible to planarize PAS entirely. So, they only planarized part of PAS flow and found that the new flow can maintain magnetic field. Moreover, the solution of the new flow can be achieved at lower Reynolds number compare to the original PAS.

II. Mathematical Background

In kinematic dynamo problem, we consider the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = R\nabla \times (\mathbf{u} \times \mathbf{B}) + \nabla^2 \mathbf{B}, \tag{1}$$

where:

B is the magnetic field,

u is the velocity field,

$R(= UL/\eta)$ is the magnetic Reynolds number.

The equation describes the magnetic field changes due to the interaction between velocity field and magnetic field. This interaction must be dominant compare to the second term, the diffusivity of the magnetic field.

Bullard and Gellman [3] introduced a solving method for the induction equation. They decompose the fields into toroidal and poloidal components

$$\mathbf{B} = \mathbf{S} + \mathbf{T},$$

$$\mathbf{u} = \mathbf{s} + \mathbf{t}. \tag{2}$$

And expand the component using spherical harmonics expansion:

$$\mathbf{S} = \sum_{n,m} \mathbf{S}_n^m,$$

$$\mathbf{T} = \sum_{n,m} \mathbf{T}_n^m,$$

where

$$\mathbf{S}_n^m = \nabla \times \nabla \times (rS_n^m(r, t)Y_n^m(\theta, \varphi))$$

$$\mathbf{T}_n^m = \nabla \times (rT_n^m(r, t)Y_n^m(\theta, \varphi))$$

$$n = 1, 2, 3, \ldots; \quad m = -n, \ldots, n$$

$$Y_n^m = (-1)^m \left[\frac{2n+1}{2-\delta_m^0}\right]^{\frac{1}{2}} P_n^m(\cos\theta)e^{im\varphi} = (-)^m \overline{Y_n^{-m}}$$

$$Y_n^m = 0, \text{ for } n < |m|.$$

Y_n^m is the spherical harmonic function which involves the associated Legendre function P_n^m. We use the Schmidt's normalization for the Legendre function.

Using the above expansion, equation (1) will be transformed into the spectral form:

$$\left(\frac{\partial}{\partial t} - D_{n3}\right) = R\sum_{n1, n2}\{(t_{n1}S_{n2}S_{n3}) + (s_{n1}T_{n2}S_{n3}) + (s_{n1}S_{n2}S_{n3})\}$$

$$\left(\frac{\partial}{\partial t} - D_{n3}\right) = R\sum_{n1, n2}\{(t_{n1}T_{n2}T_{n3}) + (t_{n1}S_{n2}T_{n3}) + (s_{n1}S_{n2}T_{n3}) + (s_{n1}T_{n2}T_{n3})\},$$

where $D_n \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{n(n+1)}{r^2}$ and

$$(t_{n1}S_{n2}S_{n3}),\ (s_{n1}T_{n2}S_{n3}),\ (s_{n1}S_{n2}S_{n3}),\ (t_{n1}T_{n2}T_{n3}),$$

$$(t_{n1}S_{n2}T_{n3}),\ (s_{n1}S_{n2}T_{n3}),\ (s_{n1}T_{n2}T_{n3})$$

are the interaction terms. The details can be found in Bachtiar [1]. In solving the spectral form, we need to satisfy the rigid boundary condition:

$$s_n = 0 \quad \text{at} \quad r = 0.$$

III. Planar Velocity

The first planar velocity formula was probably introduced by Bachtiar et al. [2]. In their work, they defined planar flow as a flow without z-component. Using BIJ formula, it is known that a planar flow must consist of the following combination:

$$\mathbf{s}_n^m + \mathbf{t}_{n-1}^m + \mathbf{t}_{n+1}^m,$$

where

$$s_n^m = \frac{imf_n^m}{n(n+1)}, \tag{3}$$

$$t_{n-1}^m = \frac{\alpha_n}{n} d_{n+1} f_n^m, \tag{4}$$

$$t_{n+1}^m = -\frac{\alpha_{n+1}}{n+1} d_{-n} f_n^m, \tag{5}$$

and

$$\alpha_n := \sqrt{(n^2 - m^2)/(4n^2 - 1)},$$

$$d_n := \frac{d}{dr} + \frac{n}{r}.$$

If we start with a single harmonic function f_n^m, then we can get a planar flow using equations (3)-(5). Another way to construct planar flow is by transforming a given flow into planar flow. Bachtiar et al. [2] called this process as *planarization process*.

For example, if the given flow consists of $\mathbf{s}_2^2 + \mathbf{t}_2^2$, then the poloidal part will be planarized by adding $\mathbf{t}_3^2$. Meanwhile, the toroidal part will be planarized by adding $\mathbf{s}_3^2$ and $\mathbf{t}_4^2$. These addition poloidal-toroidal parts are defined according to equations (3)-(5).

Bachtiar et al. [2] also showed that the toroidal part of the planar flow have to satisfy the following integral:

$$\int_0^1 r^{n+2} t_n dr = 0. \tag{6}$$

The integral indicates that the flow needs to satisfy the differentiability at the origin and rigid boundary.

IV. Numerical Methods

Although kinematic dynamo is a subset of MHD, the solution is not easy to find. For the derivatives with respect to the radial direction, we divide the radial direction into J intervals and use the central difference scheme. We assume that the velocity field is time independent and the magnetic field can be expanded into $\mathbf{B} = \sum_\lambda \mathbf{B}_\lambda e^{\lambda t}$.

Using the above assumptions, the problem is reduced into ordinary eigenvalue problem: $A\mathbf{x} = \lambda\mathbf{x}$. To make the system finite, we need to determine the value of n in equation (2). If we arrange the form of magnetic field, then the matrix A can be formed into a sparse-banded matrix. Such arrangement will help us to optimize the use of computer's memory.

A successful dynamo will have a positive real part of eigenvalue which indicates that the magnetic field will grow and the self excited dynamo is established. The convergence of solution is indicated by the convergence of eigenvalues using different values of J and n. To maintain the sparse-banded form, we use the inverse iteration method in our routine.

V. PAS Planarization

As we mentioned above, PAS dynamo is one of the earliest successful kinematic dynamos. The solution is easy to converge at relatively low magnetic Reynolds number. PAS flow is defined as follows:

$$\mathbf{v} = 2\mathbf{Re}\{\mathbf{s}_2^2 + \mathbf{t}_2^2\},$$

where

$$s_2^2 = k\Lambda j_2(\Lambda r), \quad t_2^2 = \Lambda s_2^2,$$

$j_2(r)$ is the spherical Bessel function order two and Λ is one of the positive roots of $j_2(r)$.

Using different numerical approach, Dudley and James [4] reproduced similar result as PAS. Another characteristic of PAS flow is that this flow is a Beltrami flow, i.e.,

$$\mathbf{v} = k\nabla \times \mathbf{v}.$$

It means that PAS flow has a strong helicity. Strong helicity is believed to be an important aspect in producing dynamo process (Bachtiar [1]).

In 2006, Bachtiar et al. [2] tried to planarize PAS flow in order to get a planar velocity dynamo. They found that, using their formula, it is impossible to planarize the toroidal part of PAS. As a result, they introduced a new flow which is PAS flow with planarized poloidal part. They called this flow as *partly-planarized PAS flow*. They reported that this flow can produce a dynamo process. The following is the formula for partly-planarized PAS flow:

$$\mathbf{v} = 2\mathbf{Re}\{\mathbf{s}_2^2 + \mathbf{t}_3^2 + \mathbf{t}_2^2\},$$

where

$$s_2^2 = k\Lambda j_2(\Lambda r),$$

$$t_2^2 = \Lambda s_2^2,$$

$$t_3^2 = \frac{i}{\sqrt{7}}\left(\frac{d}{dr} - \frac{2}{r}\right)k\Lambda j_2(\Lambda r).$$

The problem with planarizing PAS is when we want to planarize the toroidal part. If we try to planarize the toroidal part, then we start with equation (4) to define f_3^2. Then, we use equations (3) and (4) to define s_3^2

and t_4^2. The form of f_3^2 is the following:

$$f_3^2(\Lambda r) = 3\Lambda k\sqrt{7}\, j_3(\Lambda r) + C.$$

If we impose the boundary condition to get the integral constant in equation, then the flow will fail the differentiability condition at the origin. It means that the flow will be undefined near the origin. This will give us a serious numerical problem.

On the other hand, if we impose the differentiability condition to get the integral constant, then the flow will fail the boundary condition. The f_3^2 becomes:

$$f_3^2(\Lambda r) = 3\Lambda k\sqrt{7}\, j_2(\Lambda r),$$

where $j_3(r)$ is the spherical Bessel function order three. Meanwhile, s_3^2 and t_4^2 are now defined as follows:

$$s_3^2 = \frac{i}{6} f_3^2,$$

$$t_4^2 = -\frac{1}{2\sqrt{21}}\left(\frac{d}{dr} - \frac{3}{r}\right) f_3^2.$$

From the above equations, we can conclude that the problem arises for the s_3^2 component. At $r = 1$, s_3^2 does not vanish.

VI. Discussion

Violating boundary condition, a model will be irrelevant to the Earth's dynamo because our Earth has solid mantle. Therefore, in this work, we propose an alternative way to avoid the above problem. Our new flows will preserve the Beltrami's structure but not entirely planar.

The main idea of our method is by replacing f_3^2 with polynomial near the boundary for s_3^2 component. We choose polynomials because they are easy to handle. Moreover, we also need to choose the appropriate polynomial so that the radial function of our flow will be continuous until its third derivative. Otherwise, we will offend the requirement of the central difference scheme.

Our polynomials are defined as follows:

$$P(r) = a_0(1-r) + b_0(1-r)^2 + c_0(1-r)^3 + d_0(1-r)^4. \tag{7}$$

The values of a, b, c and d depend on the point (r_0), in the radial direction, where we substitute f_3^2 with $P(r)$ so that the radial function of our flow is continuous until its third derivative. First, we have to choose our r_0. Then, we solve the following equations simultaneously to get the corresponding coefficient of $P(r)$:

$$f_3^2 = P(r),$$

$$\frac{d}{dr} f_3^2 = \frac{d}{dr} P(r),$$

$$\frac{d^2}{dr^2} f_3^2 = \frac{d^2}{dr^2} P(r),$$

$$\frac{d^3}{dr^3} f_3^2 = \frac{d^3}{dr^3} P(r).$$

Table 1. Coefficients of $P(r)$ polynomial

r_0	a_0	b_0	c_0	d_0	
0.9	–2.28	3.81	–2.84	0.79	$\times 10^5$
0.91	–2.28	3.81	–2.84	0.79	$\times 10^5$
0.92	–0.63	1.02	–0.74	0.20	$\times 10^6$
0.93	–1.12	1.82	–1.30	0.35	$\times 10^6$

0.94	–2.18	3.48	–2.47	0.65	$\times 10^6$
0.95	–4.7	7.42	–5.20	1.37	$\times 10^6$
0.96	–1.18	1.85	–1.28	0.33	$\times 10^7$
0.97	–3.87	5.99	–4.12	1.06	$\times 10^7$
0.98	–2.02	3.10	–2.10	0.53	$\times 10^8$
0.99	–3.34	5.06	–3.40	0.86	$\times 10^9$

Therefore, our new flows are defined as follows:

$$\mathbf{v} = 2\,\mathrm{Re}\{\mathbf{s}_2^2 + \mathbf{t}_3^2 + \mathbf{t}_2^2 + \mathbf{s}_3^2 + \mathbf{t}_4^2\},$$

where

$$s_2^2 = k\Lambda j_2(\Lambda r),$$

$$t_3^2 = \frac{i}{\sqrt{7}}\left(\frac{d}{dr} - \frac{2}{r}\right)k\Lambda j_2(\Lambda r),$$

$$t_2^2 = \frac{1}{3\sqrt{7}}\left(\frac{d}{dr} + \frac{4}{r}\right)f_3^2,$$

$$s_3^2 = \frac{i}{6}F_3^2,$$

$$t_4^2 = -\frac{1}{2\sqrt{21}}\left(\frac{d}{dr} - \frac{3}{r}\right)f_3^2,$$

$$f_3^2(\Lambda r) = 3\Lambda k\sqrt{7}\, j_3(\Lambda r),$$

$$\begin{aligned} F_3^2 &= f_3^2 \quad 0 \le r \le r_0 \\ &= P(r) \quad r_0 < r \le 1. \end{aligned}$$

These flows are continuous until the third derivative as we can see in the following figures. Because of space limitation, we only present $f_3^2(*)$ and $F_3^2(-)$ with $r_0 = 0.96, 0.97, 0.98, 0.99$.

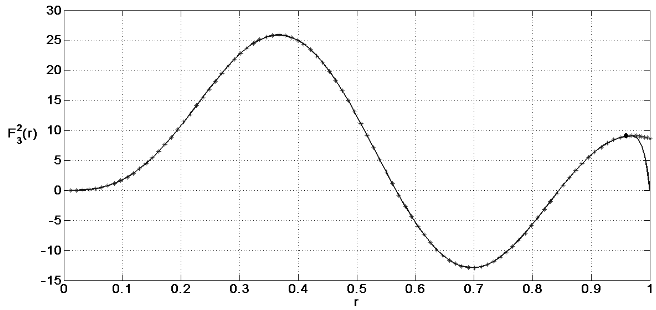

Figure 1. Plot of f_3^2 and F_3^2 with $r_0 = 0.96$.

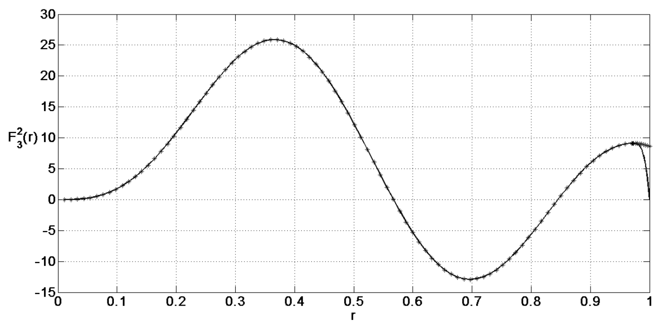

Figure 2. Plot of f_3^2 and F_3^2 with $r_0 = 0.97$.

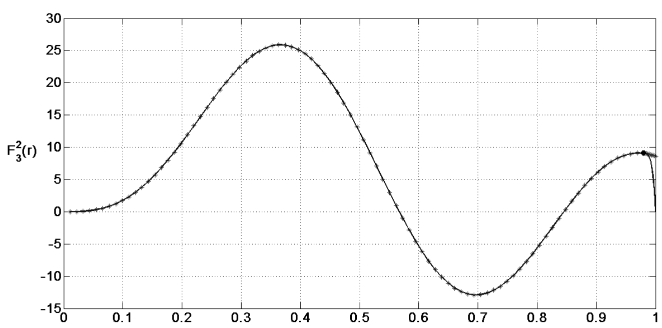

Figure 3. Plot of f_3^2 and F_3^2 with $r_0 = 0.98$.

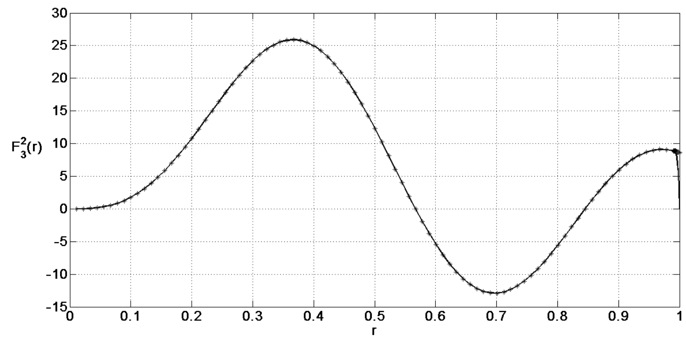

Figure 4. Plot of f_3^2 and F_3^2 with $r_0 = 0.99$.

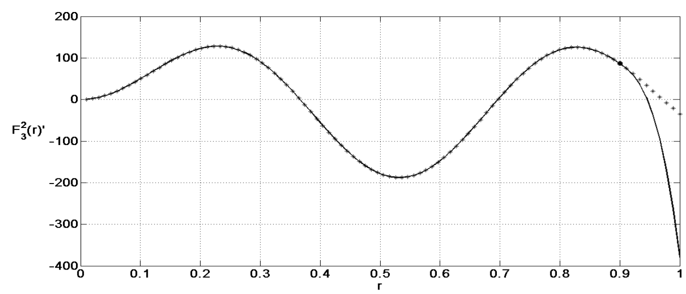

Figure 5. Plot of the first derivative of f_3^2 and F_3^2 with $r_0 = 0.9$.

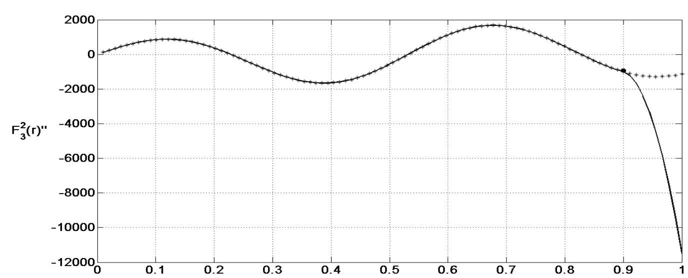

Figure 6. Plot of the second derivative of f_3^2 and F_3^2 with $r_0 = 0.9$.

As an alternative, we can also use the following polynomial:

$$A(r) = a_9(1 - r^{10}) + b_9(1 - r^{11}) + c_9(1 - r^{12})$$
$$+ d_9(1 - r^{13}) + e_9(1 - r^{14})$$

to substitute $P(r)$. So, our flow will be similar to the first one, except that:

$$s_3^2 = \frac{i}{6} G_3^2,$$

where

$$G_3^2 = f_3^2 \quad 0 \le r \le r_0$$
$$= A(r) \quad r_0 < r \le 1.$$

Table 2 consists of coefficients of $A(r)$, the graph of $A(r)$ is similar to the previous one. But, G_3^2 is continuous until its fourth derivative. This requirement is needed when we are going to use second derivative in central difference scheme.

Table 2. Coefficients of $A(r)$

r_0	a_9	b_9	c_9	d_9	e_9	
0.9	0.34	–1.39	2.13	–1.45	0.37	$\times 10^6$
0.91	0.58	–2.32	3.52	–2.38	0.60	$\times 10^6$
0.92	1.06	–4.21	6.30	–4.22	1.06	$\times 10^6$
0.93	0.21	–0.83	1.23	–0.81	0.20	$\times 10^7$
0.94	0.47	–1.82	2.67	–1.74	0.43	$\times 10^7$
0.95	1.20	–4.62	6.69	–4.33	1.05	$\times 10^7$
0.96	0.37	–1.43	2.05	–1.31	0.31	$\times 10^8$
0.97	1.63	–6.14	8.71	–5.52	1.32	$\times 10^8$
0.98	1.27	–4.74	6.65	–4.17	0.98	$\times 10^9$
0.99	0.41	–1.54	2.14	–1.33	0.31	$\times 10^{11}$

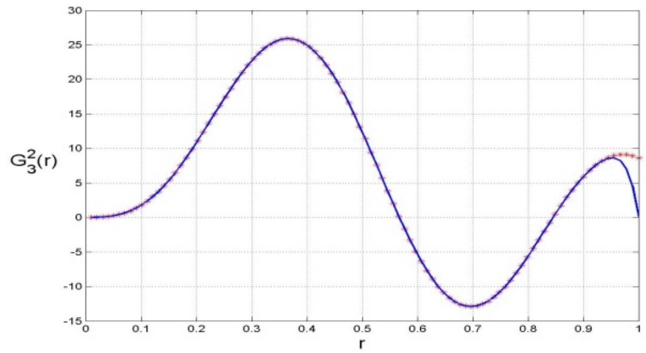

Figure 7. Plot of f_3^2 and G_3^2 with $r_0 = 0.9$.

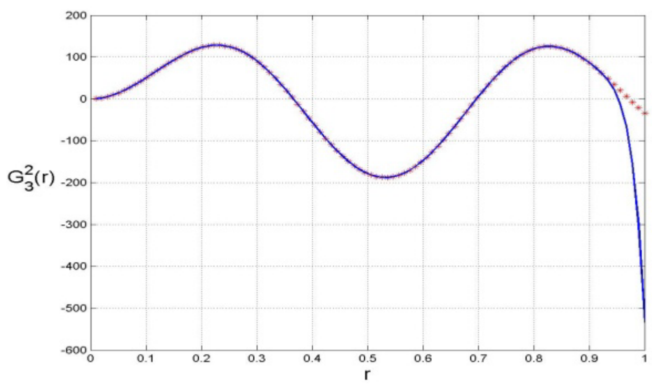

Figure 8. Plot of the first derivative of f_3^2 and G_3^2 with $r_0 = 0.96$.

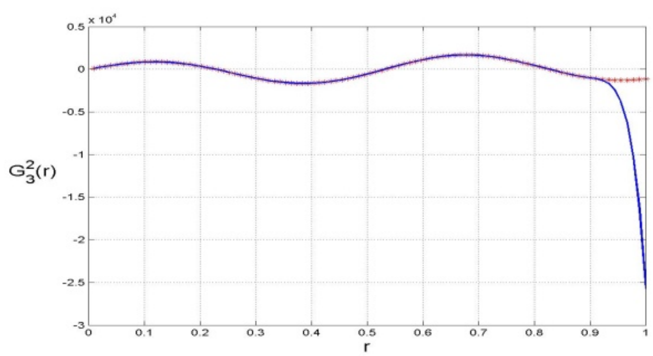

Figure 9. Plot of the second derivative of f_3^2 and G_3^2 with $r_0 = 0.9$.

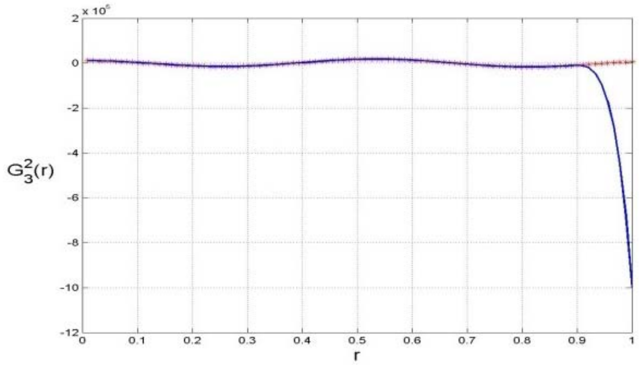

Figure 10. Plot of the third derivative of f_3^2 and G_3^2 with $r_0 = 0.9$.

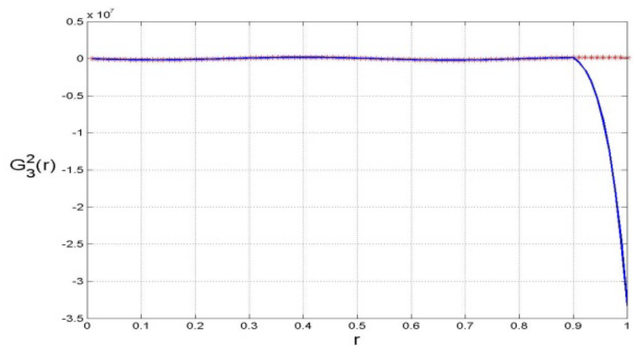

Figure 11. Plot of the fourth derivative of f_3^2 and G_3^2 with $r_0 = 0.9$.

These flows preserve the Beltrami's structure and more than 90% of these flows are planar flows.

V. Conclusions

By using our method, PAS flow can be transformed into a planar flow. Although it is not entirely planar, our new flows still preserve the Beltrami structure in some fashion, so these flows might have a strong helicity. As a result, we would expect that these flows can produce dynamo process.

Our future work will try to do computer simulation using our dynamo program in order to support the above conjecture. However, the simulation will probably be a tedious and complicated project.

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