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THE ATTRACTORS OF MODIFIED NAVIER-STOKES EQUATION IN R^2

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Abstract

This paper concerns the modified Navier-Stokes equation with nonlinear coefficient in R^2 introducing the proof of existence and uniqueness of solution of the equations. We use Theorem 1.4 to get the existence of the attractors. In this paper, we will discuss the Navier-Stokes equations that add a nonlinear term $\eta = \beta|u|^{\alpha}(a_1, a_2)$, a_1, a_2 are fixed, where $0 < \alpha < 1$.

1. Introduction and Preliminaries

We consider the Navier-Stokes equations of viscous incompressible fluid and add a nonlinear term. We describe fluid motion under the following assumptions: constant density (ρ), constant viscosity (μ), continuity (incompressible flow), $\nabla \cdot u = 0$, $u = (u_1, u_2, u_3, ..., u_n)$. The motion of an incompressible viscous fluid in R^n, $n \geq 2$, is described by the Navier-

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Stokes equations:

$$\rho \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \mu \Delta u + f, \tag{1.1}$$

$$div\, u = 0, \tag{1.2}$$

where u is the velocity vector, and let $\rho = 1$, p is the pressure that maintains the incompressibility of a fluid at (t, x), f represents volume forces. Equation (1.2), i.e., $div\, u = 0$, is the incompressibility condition. As a nonlinear system of partial differential equations, u and p are regarded as unknown functions. The constant $\mu > 0$ is the kinematic viscosity constant, $u \cdot \nabla$ denotes the covariant derivative along the flow trajectories, that is, the directional derivative in the direction u, Δu is the usual Laplace on u, and $\mu \Delta u$ represents the stress applied to the fluid. As usual, we use $\nabla \cdot = div$ denote the divergence operator. We will discuss $n = 2$. There are many results for the attractors of Navier-Stokes equation. For example, see [1, 3, 11], but no papers study the attractor of this equation that adds a nonlinear term η. So we will study this problem. We will not change other conditions. Equation (1.1) can be written as:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \mu \Delta u + f + \beta |u|^{\alpha}(a_1, a_2) = \mu \Delta u + f + \eta. \tag{1.3}$$

We need the following preliminaries:

Definition 1.1. An attractor is a set $\mathscr{A} \subset H$ that enjoys the following properties:

(1) $\mathscr{A}$ is an invariant set $(S(t)\mathscr{A} = \mathscr{A}, \forall t \geq 0)$.

(2) $\mathscr{A}$ possesses an open neighborhood $\mathscr{U}$ such that for every u_0 in $\mathscr{U}$, $S(t)u_0$ converges to $\mathscr{A}$ as $t \to \infty$:

$$dist(S(t)\mathscr{A}, u_0) \to 0 \text{ as } t \to \infty.$$

The distance in (2) is understood to be the distance of a point to a set

$$d(x, \mathscr{A}) = \inf_{y \in \mathscr{A}} (x, y),$$

$d(x, y)$ denoting the distance of x to y in H. If $\mathscr{A}$ is an attractor, then the largest open set $\mathscr{U}$ that satisfies (2) is called the *basin of attraction* of $\mathscr{A}$.

We now show how to prove the existence of an attractor when the existence of an absorbing set is known. Further assumptions on the semigroup $S(t)$ are necessary at this point and we will make one of two following assumptions (1.4), (1.5):

The operators $S(t)$ are uniform for t large. By this, we mean that for every bounded set $\mathscr{B}$, there exists t_0 which may depend on $\mathscr{B}$ such that

$$\bigcup_{t \geq t_0} S(t)\mathscr{B} \tag{1.4}$$

is relatively compact H.

Alternatively, if H is a Banach space, then we may assume that $S(t)$ is the perturbation of an operator satisfying (1.4) by a (nonnecessarily linear) operator which converges to 0 as $t \to \infty$. We formulate this assumption more precisely:

H is a Banach space and for every t, $S(t) = S_1(t) + S_2(t)$, where the operators $S_1(\cdot)$ are uniformly compact for t large (i.e., satisfy (1.4)) and $S_2(t)$ is a continuous mapping from H into itself such that the following holds:

For every bounded set $C \subset H$,

$$r_c = \sup_{\varphi \in C} |S_2(t)\varphi|_H \to 0 \text{ as } t \to \infty. \tag{1.5}$$

Of course, if H is a Banach space, then any family of operators satisfying (1.4) also satisfies (1.5) with $S_2 = 0$.

Theorem 1.2. *We assume that H is a metric space and that the operators* $S(t)$ *are given and satisfy*

The evolution of the dynamical system is described by a family of operators $S(t)$, $t \geq 0$, that map H into itself and enjoy the usual semigroup properties:

$$\begin{cases} S(t+s) = S(t)S(s), \quad \forall s, t \geq 0, \\ S(0) = I \text{ (Indentity } H) \end{cases} \tag{1.6}$$

$(S(t)$ is continuous (nonlinear) operator from H into itself$)$ and (1.4) or (1.5). We also assume that there exist an open set $\mathscr{U}$ and a bounded set $\mathscr{B}$ of $\mathscr{U}$ such that $\mathscr{B}$ is absorbing in $\mathscr{U}$.

Then the ω-limit set of $\mathscr{B}$, $\mathscr{A} = \omega(\mathscr{B})$, is a compact attractor which attracts the bounded sets of $\mathscr{U}$. It is the maximal bounded attractor in $\mathscr{U}$ (for the inclusion relation).

Furthermore, if H is a Banach space, U is convex, and the mapping $t \to S(t)u_0$ is continuous from R_+ into H, for every in H; then $\mathscr{A}$ is connected too.

In equations (1.2), (1.3), we will add boundary condition. That is:

The nonslip boundary condition. The boundary Γ is solid and at rest; thus

$$u = 0 \text{ on } \Gamma.^1 \tag{1.7}$$

The space-periodic case. Here $\Omega = (0, L_1) \times (0, L_2)$ and

$$u, p \text{ and the first derivatives of } u \text{ are } \Omega\text{-periodic.}^2 \tag{1.8}$$

Remark 1. If Γ is solid but not at rest, then the nonslip boundary condition is $u = \varnothing$ on Γ, where $\varphi = \varphi(x, t)$ is the given velocity of Γ.

Remark 2. That is, u and p take the same values at corresponding points of Γ.

Furthermore, we assume in this case that the average flow vanishes

$$\int_\Omega udx = 0. \tag{1.9}$$

u satisfies the initial condition

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{1.10}$$

where u_0 is given. For the mathematical setting of this problem, we consider a Hilbert space H which is a close subspace of $L^2(\Omega)^n (n = 2 \text{ here})$. In the nonslip case,

$$H = \{u \in L^2(\Omega)^n,\ div\, u = 0,\ u \cdot \nu = 0 \text{ on } \Gamma\} \tag{1.11}$$

ν being the unit outward normal on Γ and in the periodic case

$$H = \{u \in \dot{L}^2(\Omega)^n,\ div\, u = 0,\ u_{i|\Gamma_i} = u_{i|\Gamma_{i+n}},\ i = 1, ..., n\}.^3 \tag{1.12}$$

We refer the reader to [7] for more details on these spaces and, in particular, a trace theorem showing that the trace of $u \cdot \nu$ on Γ exists and belongs to $H^{-\frac{1}{2}}(\Gamma)$ when $u \in L^2(\Omega)^n$ and $div\, u \in L^2(\Omega)$. The space H is endowed with the scalar product and the norm of $L^2(\Omega)^n$ denoted by $(\cdot, \cdot)$ and $|\cdot|_{L^2}$.

Remark 3. Γ_i and Γ_{i+n} are the faces $x_i = 0$ and $x_i = L_i$ of Γ. The condition $u_{i|\Gamma_i} = -u_{i|\Gamma_{i+n}}$ expresses the periodicity of $u \cdot \nu$; $\dot{L}^2(\Omega)^n$ is the space of $u \in L^2(\Omega)^n$ satisfying (1.9).

Another useful space is V, a closed subspace of $H^1(\Omega)^n$:

$$V = \{u \in H_0^1(\Omega)^n,\ div\, u = 0\} \tag{1.13}$$

in the nonslip case and, in the space-periodic case,

$$V = \{u \in \dot{H}^1_{per}(\Omega)^n,\ div\, u = 0\}, \tag{1.14}$$

where $\dot{H}^1_{per}$ is defined in [6]. In both cases, v is endowed with the scalar product $((u, v)) = \sum_{i,j=1}^{n} \left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j} \right)$ and the norm $\| u \| = ((u, u))^{\frac{1}{2}}$.

We denoted by A the linear unbounded operator in H which is associated with V, H and the scalar product $((u, v)) = (Au, v), \ \forall u, v \in V$. The domain of A in H is denoted by $D(A)$; A is self-adjoint positive operator in H. Also, A is an isomorphism from $D(A)$ onto H. The space $D(A)$ can be fully characterized by using the regularity theory of linear elliptic systems (see [7, 9]):

$$D(A) = H^2(\Omega)^n \cap V$$

and

$$D(A) = H^2_{per}(\Omega)^n \cap V$$

in the nonslip and periodic cases; furthermore, $| Au |_{L^2}$ is on $D(A)$ a norm equivalent to that induced by $H^2(\Omega)^n$.

Let V' be the dual of V. Then H can be identified to a subspace of V' (see [8]) and we have

$$D(A) \subset V \subset H \subset V', \tag{1.15}$$

where the inclusions are continuous and each space is dense in the following one.

In the space-periodic case, we have $Au = -\Delta u, \ \forall u \in D(A)$, while in the nonslip case, we have

$$Au = -P\Delta u, \ \ \forall u \in D(A),$$

where P is the orthogonal projector in $L^2(\Omega)^n$ on the space H. We can also say that $Au = f, \ u \in D(A), \ \ f \in H$, is equivalent to saying that there exists $p \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u + grad\ p = f & \text{in } \Omega, \\ div\, u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The operator A^{-1} is continuous from H into $D(A)$ and since the embedding of $H^1(\Omega)$ in $L^2(\Omega)$ is compact, the embedding of V in H is compact. Thus, A^{-1} is a self-adjoint continuous compact operator in H, and by the classical spectral theorems, there exist a sequence λ_j,

$$0 < \lambda_1 \leq \lambda_2, \ldots, \lambda_j \to \infty$$

and a family of elements w_j of $D(A)$ which is orthonormal in H, and such that

$$Aw_j = \lambda_j w_j,\ \forall j. \tag{1.16}$$

The weak form of the Navier-Stokes equations due to Leray [5] involves only u, as $\alpha = 0$. It is obtained by multiplying (1.1) by a test function v in V and integrating over Ω. Using the Green formula (1.2) and the boundary conditions, we find that the term involving p disappears and there remains

$$\frac{d}{dt}(u, v) + \mu(u, v) + b(u, u, v) = (f, v) + (\eta, v), \tag{1.17}$$

where

$$b(u, v, w) = \sum_{i, j=1}^{n} \int_\Omega u_i \frac{\partial v_i}{\partial x_j} w_j \tag{1.18}$$

whenever the integrals make sense. Actually, the form b is trilinear continuous on $H^1(\Omega)^n$ $(n = 2)$ and, in particular, on V. We have the following inequalities giving various continuity properties of b:

$$|b(u, v, w)|$$

$$\le c_1 \times \begin{cases} |u|_{L^2}^{\frac{1}{2}} \| u \|^{\frac{1}{2}} \| v \|^{\frac{1}{2}} |Av|_{L^2}^{\frac{1}{2}} |w|_{L^2}, \ \forall u \in V,\ v \in D(A),\ w \in H, \\ |u|_{L^2}^{\frac{1}{2}} |Au|_{L^2}^{\frac{1}{2}} \| v \| |w|_{L^2}, \ \forall u \in D(A),\ v \in V,\ w \in H, \\ |u|_{L^2} |Aw|_{L^2}^{\frac{1}{2}} \| v \| |w|_{L^2}^{\frac{1}{2}}, \ \forall u \in H,\ v \in V,\ w \in D(A), \\ |u|_{L^2}^{\frac{1}{2}} \| u \|^{\frac{1}{2}} |w|_{L^2}^{\frac{1}{2}} \| v \| \| w \|^{\frac{1}{2}}, \ \forall u, v, w \in V, \end{cases} \tag{1.19}$$

where $c_1 > 0$ is an appropriate constant.

Proof. First, we will prove the first inequality:

$$\begin{aligned} |b(u, v, w)| &= \left| \int_\Omega \sum_{i,j=1}^n u_i \frac{\partial v_i}{\partial x_j} w_j dx \right| \\ &\le \int_\Omega \left| \sum_{i=1}^n \left(u_i \sum_{j=1}^n v_{ix_j} w_j \right) \right| dx \\ &\le \int_\Omega \left| \sum_{i=1}^n |Dv_i| |w| |u_i| \right| dx \\ &\le \int_\Omega |w| \sum_{i=1}^n |Dv_i| |u_i| dx \\ &\le c_1 \left(\int_\Omega |Dv|^2 |w|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega |u|^2 dx \right)^{\frac{1}{2}} \\ &\le c_1 \left[\left(\int_\Omega |u|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega |u|^2 dx \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} |wDv|_{L^2} \\ &\le c_1 |u|_{L^2}^{\frac{1}{2}} \| u \|^{\frac{1}{2}} |w|_{L^2} \| v \|^{\frac{1}{2}} |Av|_{L^2}^{\frac{1}{2}}, \end{aligned}$$

where

$$|Dv|_{L^2} \le \|v\|^{\frac{1}{2}} |D^2 v|^{\frac{1}{2}}_{L^2}. \qquad \square$$

The proof of other inequality are same as this proof. So we will not give proof.

An alternative form of (1.17) can be given using the operator A and bilinear operator B from $V \times V$ into V' defined by

$$(B(u, v), w) = b(u, v, w), \quad \forall u, v, w \in V. \tag{1.20}$$

We also set

$$B(u, u) = B(u), \quad \forall u \in V',$$

and we easily see that (1.17) is equivalent to the equation

$$\frac{du}{dt} + \mu Au + B(u) = f + \eta \tag{1.21}$$

while (1.10) can be rewritten as

$$u(0) = u_0. \tag{1.22}$$

We assume that *f* is independent of *t* so that the dynamical system associated with (1.21) is autonomous

$$f(t) = f \in H, \forall t. \tag{1.23}$$

Existence and uniqueness results for (1.21), (1.22) are well known (see [2, 3]). The following theorem collects several classical results.

Lemma 1.3 (The uniform Gronwall lemma)**.** *Let g, h, y be three positive locally integrable functions on* $(t_0, +\infty)$ *such that* y' *is locally integrable on* $(t_0, +\infty)$ *and which satisfy*

$$\frac{dy}{dt} \le gy + h \;\; for \;\; t \ge t_0,$$

$$\int_t^{t+r} g(s)ds \le d_1, \int_t^{t+r} h(s)ds \le d_2, \int_t^{t+r} y(s)ds \le d_3 \;\; for \;\; t \ge t_0,$$

where $r,\ d_1,\ d_2,\ d_3$ *are positive constants*. *Then*

$$y(t+r) \leq \left(\frac{d_3}{r} + d_2\right)\exp(d_1), \quad \forall t \geq t_0.$$

Theorem 1.4. *Under the above assumptions, for* f *and* u_0 *given in* H, *there exists a unique solution* u *of* (1.20), (1.21) *satisfying*

$$u \in \mathscr{C}([0, T]; H) \cap L^2(0, t; V), \quad \forall T > 0.$$

Furthermore, u *is analytic in* t *with values in* $D(A)$ *for* $t > 0$, *and the mapping*

$$u_0 \mapsto u(t)$$

is continuous from H *into* $D(A)$, $\forall t > 0$.

Finally, if $u_0 \in V$, *then*

$$u \in \mathscr{C}([0, T]; V) \cap L^2(0, t; D(A)), \quad \forall T > 0.$$

Some indications for the proof of Theorem 1.4 will be given in Section 3. This theorem allows us to define the operators

$$S(t): u_0 \mapsto u(t).$$

These operators enjoy the semigroup properties (1.4) and they are continuous from H into itself and even from H into $D(A)$.

2. Existence of Absorbing Sets

2.1. Existence of an absorbing set in H

A first energy-type equality is obtained by taking the scalar product of (1.21) with u. Using the orthogonality property (see [6]),

$$b(u, v, v) = 0, \quad \forall u \in V, \quad \forall v \in H^1(\Omega)^n.^4 \tag{2.1}$$

Remark 4. More generally, $b(u, v, v) = 0$, $\forall u, v \in H^1(\Omega)^n$, and $div\, u = 0$ in Ω, $u \cdot \nu$ or $v = 0$ on Ω.

Proof.

$$
\begin{aligned}
b(u, v, v) &= \int_\Omega [(u \cdot \nabla)v] \cdot v dx \\
&= \int_\Omega u^T (\nabla v) v dx \\
&= \int_\Omega \sum_{i=1}^n (u_i v_{x_i}) v dx \\
&= \int_\Omega \sum_{i=1}^n u_i (v_{x_i} v) dx \\
&= \int_\Omega \sum_{i=1}^n v_{x_i} (u_i v) dx \\
&= \int_{\partial\Omega} \sum_{i=1}^n v(u_i \cdot v) - \int_\Omega \sum_{i=1}^n v \cdot (u_i \cdot v)_{x_i} dx \\
&= 0 - \int_\Omega \sum_{i=1}^n v \cdot (u_{ix_i} v + u_i v_{x_i}) dx \\
&= -\int_\Omega \sum_{i=1}^n u_{ix_i} (v \cdot v) + \sum_{i=1}^n u_i v \cdot v_{x_i} dx \\
&= 0 - b(u, v, v)
\end{aligned}
$$

so we get $b(u, v, v) = 0.$ □

We see that $B((u), u) = 0$ and there remains

$$\frac{1}{2}\frac{d}{dt}|u|_{L^2}^2 + \mu \| u \|^2 - (\eta, u) = (f, u) \le | f |_{L^2} | u |_{L^2}. \tag{2.2}$$

We know that

$$| u |_{L^2} \le \lambda_1^{-\frac{1}{2}} \| u \|, \quad \forall u \in V,$$

where λ_1 is the first eigenvalue of A. Hence, we can majorize the right-hand side of (2.2) by

$$\lambda_1^{-\frac{1}{2}} | f |_{L^2} \| u \| \leq \frac{\mu}{4} \| u \|^2 + \frac{1}{\mu\lambda_1} | f |_{L^2}^2$$

at the same time, in this paper, let $\max(a_1, a_2) = a$,

$$\begin{aligned}
(\eta, u) &= (\beta | u |^{\alpha} (a_1, a_2), u) \\
&\leq \int_{\Omega} 2a\beta | u |^{\alpha} u dx \\
&\leq \int_{\Omega} \frac{4a^2\beta^2}{\mu\lambda_1} | u |^{2\alpha} dx + \frac{\mu\lambda_1}{4} | u |^2 \\
&\leq \frac{4a^2\beta^2}{\mu\lambda_1} \left(\int_{\Omega} \alpha \frac{\mu}{16} | u |^2 dx + c' \right) + \frac{\mu\lambda_1}{4} | u |_{L^2}^2 \\
&\leq \frac{a^2\beta^2\alpha}{4} \| u \|^2 + c + \frac{\mu}{4} \| u \|^2
\end{aligned}$$

and we obtain

$$\frac{d}{dt} | u |_{L^2}^2 + \left(\mu - \frac{a^2\beta^2\alpha}{2} \right) \| u \|^2 \leq \frac{2}{\mu\lambda_1} | f |_{L^2}^2 + 2c, \tag{2.3}$$

$$\frac{d}{dt} | u |_{L^2}^2 + \lambda_1 \left(\mu - \frac{a^2\beta^2\alpha}{2} \right) | u |_{L^2}^2 \leq \frac{2}{\mu\lambda_1} | f |_{L^2}^2 + 2c. \tag{2.4}$$

When $\mu \geq \frac{a^2\beta^2\alpha}{2}$ using the classical Gronwall lemma, we obtain

$$\begin{aligned}
| u(t) |_{L^2}^2 \leq{}& | u_0 |_{L^2}^2 \exp\left(-\lambda_1 \left(\mu - \frac{a^2\beta^2\alpha}{2} \right) t \right) \\
&+ \left(\frac{2}{\mu^2\lambda_1^2} | f |_{L^2}^2 + \frac{2c}{\mu\lambda_1} \right) \left(1 - \exp\left(-\lambda_1 \left(\mu - \frac{a^2\beta^2\alpha}{2} \right) t \right) \right).
\end{aligned} \tag{2.5}$$

Thus,

$$\limsup_{t\to\infty} |u(t)|_{L^2} \le \rho_0, \quad \rho_0 = \sqrt{\frac{2}{\mu^2\lambda_1^2}|f|_{L^2}^2 + \frac{2c}{\mu\lambda_1}}. \tag{2.6}$$

We infer from (2.5) that the balls $B_H(0, \rho)$ of H with $\rho \ge \rho_0$ are positively invariants for semigroup $S(t)$, and these balls are absorbing for any $\rho \ge \rho_0$. We choose ρ_0' and denote by $\mathscr{B}_0$ the ball $B_H(0, \rho_0')$. Any set bounded in H is include in a ball $B(0, R)$ of H. It is easy to deduce from (2.5) that $S(t)\mathscr{B} \in \mathscr{B}_0$ for $t \ge t_0(\mathscr{B}, \rho_0')$, where

$$t_0 = \frac{1}{\lambda_1\left(\mu - \frac{a^2\beta^2\alpha}{2}\right)} \log \frac{R^2}{\rho_0'^2 - \rho_0^2}. \tag{2.7}$$

We then infer from (2.3) after integration in t that

$$\left(\mu - \frac{a^2\beta^2\alpha}{2}\right)\int_t^{t+r} \|u\|^2 ds \le \frac{2r}{\mu\lambda_1}|f|_{L^2}^2 + |u(t)|_{L^2}^2 + 2cr, \quad \forall r > 0. \tag{2.8}$$

With the use of (2.6), we conclude that

$$\begin{aligned}\limsup_{t\to\infty}\int_t^{t+r} \|u\|^2 ds \le{}& \frac{2r}{\mu\left(\mu - \frac{a^2\beta^2\alpha}{2}\right)\lambda_1}|f|_{L^2}^2 + \frac{2}{\mu^2\left(\mu - \frac{a^2\beta^2\alpha}{2}\right)\lambda_1^2}|f|_{L^2}^2 \\ &+ \frac{2c}{\mu\left(\mu - \frac{a^2\beta^2\alpha}{2}\right)\lambda_1} + \frac{2cr}{\left(\mu - \frac{a^2\beta^2\alpha}{2}\right)}\end{aligned} \tag{2.9}$$

and if $u_0 \in \mathscr{B} \subset B_H(0, R)$ and $t \ge t_0(\mathscr{B}, \rho_0')$, then

$$\begin{aligned}\int_t^{t+r} \|u\|^2 ds \le{}& \frac{2r}{\mu\left(\mu - \frac{a^2\beta^2\alpha}{2}\right)\lambda_1}|f|_{L^2}^2 \\ &+ \frac{2}{\mu - \frac{a^2\beta^2\alpha}{2}}\rho_0'^2 + \frac{2cr}{\mu - \frac{a^2\beta^2\alpha}{2}}.\end{aligned} \tag{2.10}$$

2.2. Existence of an absorbing set in V

In order to make our equation satisfy Theorem 1.4, we continue and show the existence of an absorbing set in V. For that purpose, we obtain another energy-type equation by taking the scalar product of (1.21) with Au, since

$$(Au, u') = ((u, u')) = \frac{1}{2}\frac{d}{dt}\| u \|^2,$$

we find

$$\frac{1}{2}\frac{d}{dt}\| u \|^2 + \mu | Au |_{L^2}^2 + (B(u), Au) - (\omega, Au) = (f, Au). \qquad (2.11)$$

We write

$$(f, Au) \le | f |_{L^2} | Au |_{L^2} \le \frac{\mu}{8} | Au |_{L^2}^2 + \frac{2}{\mu} | f |_{L^2}^2$$

and using the second inequality (1.19),

$$\begin{aligned} | B((u), Au) | &\le c_1 | u |^{\frac{1}{2}} \| u \| | Au |_{L^2}^{\frac{3}{2}} \\ &\le (\text{with the Young inequality } ^5) \\ &\le \frac{\mu}{8} | Au |_{L^2}^2 + \frac{2c_1'}{\mu^3} | u |_{L^2}^2 \| u \|^4. \end{aligned}$$

Remark 5. By the Young inequality,

$$ab \le \frac{\varepsilon}{p} a^p + \frac{1}{p' \varepsilon^{p'/p}} b^{p'}, \ \forall a, b, \varepsilon > 0, \ \forall p, 1 < p < \infty, \ p' = p/(p-1).$$

At the same time, we have

$$\begin{aligned} (\eta, Au) &= (\beta | u |^{\alpha} (a_1, a_2), Au) \\ &\le \int_\Omega 2a\beta | u |^{\alpha} Au dx \end{aligned}$$

$$\leq \frac{4a^2\beta^2}{\mu}\int_\Omega |u|^{2\alpha}dx + \frac{\mu}{4}|Au|_{L^2}^2$$

$$\leq \frac{4a^2\beta^2}{\mu}\left(\int_\Omega \alpha\frac{\mu}{16}|u|^2 dx + c'\right) + \frac{\mu}{4}|Au|_{L^2}^2$$

$$\leq \frac{\mu}{4}|Au|_{L^2}^2 + c + \frac{\alpha a^2\beta^2}{4}|u|_{L^2}^2.$$

Hence

$$\frac{d}{dt}\|u\|^2 + \mu|Au|_{L^2}^2 \leq \frac{4}{\mu}|f|_{L^2}^2 + \frac{\alpha a^2\beta^2}{2}|u|_{L^2}^2 + 2c + \frac{4c_1'}{\mu^3}|u|_{L^2}^2\|u\|^4 \quad (2.12)$$

and since

$$\|\varphi\| \leq \lambda_1^{-\frac{1}{2}}|A\varphi|_{L^2}, \quad \forall\varphi \in D(A), \quad (2.13)$$

we also have

$$\frac{d}{dt}\|u\|^2 + \lambda_1\mu\|u\|^2 \leq \frac{4}{\mu}|f|_{L^2}^2 + \frac{\alpha a^2\beta^2}{2}|u|_{L^2}^2 + 2c + \frac{4c_1'}{\mu^3}|u|_{L^2}^2\|u\|^4. \quad (2.14)$$

A priori estimate of u in $L^\infty(0, T; V)$, $\forall T > 0$, follows easily from (2.14) by the classical Gronwall lemma (Lemma 1.3), using the previous estimates on u. We are more interested in an estimate valid for large t. Assuming that u_0 belongs to a bounded set $\mathscr{B}$ of H and that $t \geq t_0(\mathscr{B}, \rho_0')$, t_0 as in (2.7), we apply the uniform Gronwall lemma to (2.14) with g, h, y replaced by

$$\frac{4c_1'}{\mu^3}|u|_{L^2}^2\|u\|^2, \quad \frac{4}{\mu}|f|_{L^2}^2 + \frac{\alpha a^2\beta^2}{2}|u|_{L^2}^2 + 2c, \quad \|u\|^2.$$

Thanks to (2.5) and (2.10), we estimate the quantities d_1, d_2, d_3 in Lemma 1.3 by

$$\begin{cases} d_1 = \dfrac{4c_1'}{\mu^3}\rho_0'^2 d_3, \\ d_2 = \dfrac{4r}{\mu}| f |_{L^2}^2 + \dfrac{r\alpha a^2\beta^2}{2}\rho_0'^2 + 2cr, \\ d_3 = \dfrac{2r}{\mu\left(\mu - \dfrac{a^2\beta^2\alpha}{2}\right)\lambda_1}| f |_{L^2}^2 + \dfrac{2}{\mu - \dfrac{a^2\beta^2\alpha}{2}}\rho_0'^2 + \dfrac{2cr}{\mu - \dfrac{a^2\beta^2\alpha}{2}} \end{cases} \tag{2.15}$$

and we obtain

$$\| u(t) \|^2 \leq \left(\frac{d_3}{r} + d_2\right)\exp(d_1) \text{ for } t \geq t_0 + r, \tag{2.16}$$

t_0 as in (2.7).

Let us fix $r > 0$ and denote by ρ_1^2 the right-hand side of (2.16). We then conclude that the ball $B_V(0, \rho_1)$ of V, denoted by $\mathscr{B}_1$, is an absorbing set in V for the semigroup $S(t)$. Furthermore, if $\mathscr{B}$ is any bounded set of H, then $S(t)\mathscr{B} \subset \mathscr{B}_1$ for $t \geq t_0(\mathscr{B}, \rho_0') + r$. This shows the existence of an absorbing set in V, namely $\mathscr{B}_1$, and also that the operators $S(t)$ are uniformly compact, i.e., (1.5) is satisfied.

3. Proof of Theorem 1.4

We prove the existence and uniqueness of solution for (1.20), (1.21) that belongs to $L^\infty(0, T; H) \cap L^2(0, T; V)$, $\forall T > 0$, is first obtained by the Faedo-Gakerkin method (see [8]). We implement this approximation procedure with the function w_j representing the eigenvalues of A (see (1.16)). For each m, we look for an approximate solution u_m of the form

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i$$

satisfying

$$\left(\frac{du_m}{dt}, w_j\right) + \mu a(u_m, w_j) + b(u_m, u_m, w_j)$$

$$= (f, w_j) + (\omega_m, w_j), \quad j = 1, ..., m, \tag{3.1}$$

$$u_m = P_m u_0, \tag{3.2}$$

where P_m is projector in H (or V) on the space spanned by $w_1, ..., w_m$. Since A and P_m commute, the relation (3.1) is also equivalent to

$$\frac{du_m}{dt} + \mu A u_m + P_m B(u_m) = P_m f + P_m \omega_m. \tag{3.3}$$

The existence and uniqueness of u_m on some interval $[0, T_m)$ is elementary and then $T_m = +\infty,$ because of the a priori estimate that we obtain for u_m. An energy equality is obtained by multiplying (3.1) by g_{jm} and summing these relations for $j = 1, ..., m.$ We obtain (2.2) exactly with u replaced by u_m and we deduce from this relation that

$$u_m \text{ remains bounded in } L^\infty(0, T; H) \cap L^2(0, T; V), \quad \forall T > 0. \tag{3.4}$$

Due to (2.1) and the last inequality (1.19),

$$\| B(\varphi) \|_V \le c_1 | \varphi |_{L^2} \| \varphi \|, \quad \forall \varphi \subset V. \tag{3.5}$$

Therefore, $B(u_m)$ and $P_m(u_m)$ remain bounded in $L^2(0, T; V')$ and by (3.3),

$$\frac{du_m}{dt} \text{ remains bounded in } L^2(0, T; V'). \tag{3.6}$$

By weak compactness, it follows from (3.4) that there exists $u \subset L^\infty(0, T; H) \cap L^2(0, T; V), \; \forall T > 0,$ and subsequence still denoted m such that

$$u_m \to u \text{ in } L^2(0, T; V) \text{ weakly and in } L^\infty(0, T; H) \text{ weak-star,}$$

$$\frac{du_m}{dt} \to \frac{du}{dt} \text{ in } L^2(0, T; V') \text{ weakly.} \tag{3.7}$$

Due to (3.6) and a classical compactness theorem (see, for instance, Temam [7]), we also have

$$u_m \to u \text{ in } L^2(0, T; H) \text{ strongly.} \tag{3.8}$$

This is sufficient to pass to the limit in (3.1)-(3.3) and we find (1.20), (1.21) at the limit. For (1.21), we simply observe that (3.7) implies that

$$u_m(t) \to u(t)$$

weakly in V' or even in H, $\forall t \in [0, T]$ (see [11]).

By (1.20) (or (3.7)), $\frac{du}{dt}$ belongs to $L^2(0, T; V')$ and by Lemma 3.2 of Chapter 2 (see [8]), u is in $\mathscr{C}([0, T]; H)$. The uniqueness and continuous dependence of $u(t)$ on u_0 (in H) follows by standard method using Lemma 3.2 of Chapter 2 (see [8]).

The fact that $u \subset L^\infty(0, T; V) \cap L^2(0, T; D(A)), \ \forall T > 0,$ is proved by deriving further a priori estimate on u_m. They are obtained by multiplying (3.1) by $\lambda_j g_{jm}$ and summing these relations for $j = 1, \dots, m.$ Using (1.16), we find a relation that is exactly (2.11) with u replaced by u_m. We deduce from this relation that

$$u_m \text{ remains bounded in } L^\infty(0, T; V) \cap L^2(0, T; D(A)), \quad \forall T > 0. \tag{3.9}$$

At the limit, we then find that u is in $L^\infty(0, T; V) \cap L^2(0, T; D(A))$. The fact that u is in $\mathscr{C}([0, T]; H)$, then follows from an appropriate application of Lemma 3.2 (see [8]).

Finally, the fact that u is analytic in t with values in $D(A)$ results from totally different methods, for which the reader is referred to [9] or [4]. However, this property was given for the sake of completeness and is never used here in an essential manner.

4. Maximal Attractor

Equation (1.3) is Navier-Stokes equation when $\beta = 0$, some people had proved the existence of absorbing sets and the existence of maximal attractor, the asymptotic attractor, the universal attractor, attractor in unbounded domain (see [4-6, 9]). In [10], some people prove the global attractor of N-S equation with linear dampness on the whole two-dimensional, that is, $\alpha = 0$. In this paper, from Section 2 and Section 3, we prove the existence of absorbing sets in bounded domain, at the same time, we give the proof of existence and uniqueness of solution of the equations. All the assumptions of Theorem 1.4 are satisfied and we deduce from this theorem the existence of a maximal attractor for modified Navier-Stokes equations. □

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