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SOME REMARKS ON CONVEX FUZZY MULTIVALUED FUNCTIONS

Dawid Kotrys

Department of Mathematics and Computer Science
University of Bielsko-Biała
Willowa 2, 43-309 Bielsko-Biała, Poland
e-mail: dkotrys@ath.bielsko.pl

Abstract

In this paper, we introduce a definition of convex fuzzy multivalued function and then, we explore some basic properties of them.

1. Introduction

In the first part of this note, we introduce a definition of convex fuzzy multivalued function and give some conditions under which Jensen convex multivalued functions are convex. In the second part, we present some necessary and sufficient conditions for the existence of a convex fuzzy multivalued function H, such that for any two given fuzzy multivalued functions F and G, the graphs of them satisfy $GrF \subset GrH \subset GrG$.

Let (X, τ_X) be a topological vector space over $\mathbb{R}$, where $\mathbb{R}$ denotes the real numbers. For the space X, we define a fuzzy set in X to be a function $A : X \to [0; 1]$. Here $A(x)$ means the degree of membership of element

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$x \in X$ in the fuzzy set A. The set of all $x \in X$ for which $A(x) > 0$ we call the *support* of the fuzzy set A and denote by $supp(A)$.

The intersection of two fuzzy sets A, B and the union of the fuzzy sets A_i $(i \in I)$ are defined on X by the rules: $(A \cap B)(x) = \min\{A(x), B(x)\}$ and $(\bigcup_{i \in I} A_i)(x) = \sup_{i \in I} A_i(x)$.

Definition 1 [3]**.** A family δ of fuzzy sets in X is called a *fuzzy topology* for X (and the pair (X, δ) is called a *fuzzy topological space*) if

(1) 0 and 1 belong to δ,

(2) if A, B belong to δ, then so does $A \cap B$,

(3) if A_i belongs to δ for each $i \in I$, then so does $\bigcup_{i \in I} A_i$.

The elements A of δ are called *open* and their complements $(1 - A)$ *closed*.

Recall that a function $f : X \to [0; 1]$ is lower semicontinuous (l.s.c.) if for all $\alpha \in [0; 1]$, the sets $\{x \in X : f(x) > \alpha\}$ are open.

For a topological vector space (X, τ_X), the l.s.c. function family associated with τ is defined by

$$\delta(\tau_X) = \{A : X \to [0; 1], A \text{ is l.s.c.}\}.$$

It is easy to verify that $\delta(\tau_X)$ satisfies the conditions (1)-(3) of Definition 1 (continuous real-valued functions on X are l.s.c., the supremum of any family $[0; 1]$-valued l.s.c. functions on X is also l.s.c., the infimum of any finite family of $[0; 1]$-valued l.s.c. functions on X is also l.s.c.). It means $\delta(\tau_X)$ is a fuzzy topology on X. It is called the *l.s.c. fuzzy topology associated with* τ_X. Throughout this and the next section by fuzzy topology will be understood the l.s.c. fuzzy topology $\delta(\tau_X)$.

Definition 2. An *open fuzzy neighborhood* of a fuzzy set A in X is any fuzzy open set U $(U \in \delta(\tau_X))$ for which $A \subset U$. We write $A \subset U$ if $A(x) \leq U(x)$ for each $x \in X$.

Let (X, τ_X) and (Y, τ_Y) be topological vector spaces. Additionally on the space Y, we define fuzzy topology $\delta(\tau_Y)$ associated with τ_Y.

Definition 3. A *fuzzy multivalued function* from a space X into a space Y assigns to each $x \in X$ a fuzzy set $F(x)$ in Y. As in [2] we denote it by $F : X \rightsquigarrow Y$.

Moreover, we may identify F with the fuzzy set $GrF : X \times Y \to [0; 1]$, where $GrF(x, y) := F(x)(y)$. The fuzzy set GrF is called the *graph of the fuzzy function F* and its support is defined by

$$suppGrF = \{(x, y) \in X \times Y : F(x)(y) > 0\}.$$

Definition 4. A fuzzy multivalued function $F : X \rightsquigarrow Y$ is called *fuzzy closed valued* if $F(x)$ is closed for all x, it means $1 - F(x)$ is a membership of $\delta(\tau_Y)$.

Definition 5. By the *upper inverse image* of a fuzzy set U in Y under a fuzzy multivalued function $F : X \rightsquigarrow Y$, we understand the set $F^+(U) = \{x \in X : F(x) \subset U\}$.

Here the inclusion $F(x) \subset U$ means the fuzzy set inclusion $F(x)(y) \leq U(y)$ for all $y \in Y$.

Definition 6. A fuzzy multivalued function $F : X \rightsquigarrow Y$ is *upper semicontinuous* at the point x_0 if for every fuzzy open neighborhood V of $F(x_0)$ there exists an open neighborhood U_{x_0}, such that for every $x \in U_{x_0}$, the inclusion $F(x) \subset V$ holds.

The fuzzy multivalued function F is *upper semicontinuous* on X if it is upper semicontinuous at every point of X.

2. Convexity of Fuzzy Multivalued Functions

In this section, we introduce a definition of convex fuzzy multivalued function and explore some basic properties. We say that a fuzzy set $A : X \to [0; 1]$ is *convex* if for all $\lambda \in [0; 1]$ and $x_1, x_2 \in X$ the inequality $A(\lambda x_1 + (1 - \lambda x_2)) \geq \min\{A(x_1, A(x_2))\}$ is satisfied. For more details, we refer to [9]. As before, let $F : X \rightsquigarrow Y$ be fuzzy multivalued function between two topological vector spaces. Additionally, we define the fuzzy topology $\delta(\tau_Y)$ on the space *Y*. Similarly, as in the case of fuzzy sets, we introduce a definition of convex fuzzy multivalued functions as follows:

Definition 7. A fuzzy multivalued function $F : X \rightsquigarrow Y$ is called *convex* if

$$F(\lambda x_1 + (1 - \lambda)x_2)(\lambda y_1 + (1 - \lambda)y_2) \geq \min\{F(x_1)(y_1), F(x_2)(y_2)\}, \quad (1)$$

for each $\lambda \in [0; 1]$ and for all x_1, x_2 from *X*, y_1, y_2 from *Y*.

We say that a fuzzy multivalued function is $\frac{1}{2}$-*convex* (or *Jensen convex*) if inequality (1) holds for $\lambda = \frac{1}{2}$.

The above definition is a natural translation of the definition of convex multivalued functions to the language of fuzzy multivalued functions. Convexity of multivalued function is defined as follows:

$$\forall_{x_1, x_2 \in X} \ \forall_{\lambda \in [0; 1]} \ \ F(\lambda x_1 + (1 - \lambda)x_2) \supset \lambda F(x_1) + (1 - \lambda)F(x_2). \quad (2)$$

For fuzzy multivalued function, inclusion (2) means that for each $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $\lambda \in [0; 1]$ holds

$$F(\lambda x_1 + (1 - \lambda)x_2)(\lambda y_1 + (1 - \lambda)y_2)$$
$$\geq (\lambda F(x_1) + (1 - \lambda)F(x_2))(\lambda y_1 + (1 - \lambda)y_2). \quad (3)$$

The addition on the right hand side of the inequality (3) means addition of two fuzzy sets. It is defined below

$$(A+B)(z) := \sup_{z=x+y} \min\{A(x),\ B(y)\},$$

$$supp(A+B) = \{x+y : x \in suppA,\ y \in suppB\}. \tag{4}$$

Multiplication of a fuzzy set A by a real number α is defined in the following way:

$$\alpha A(\alpha x) = A(x),$$

$$supp(\alpha A) = \{\alpha x : x \in suppA\}. \tag{5}$$

According to (4) and (5), the inequality (3) can be written in the form

$$F(\lambda x_1 + (1-\lambda)x_2)(\lambda y_1 + (1-\lambda)y_2)$$

$$\geq (\lambda F(x_1) + (1-\lambda)F(x_2))(\lambda y_1 + (1-\lambda)y_2)$$

$$\geq \min\{\lambda F(x_1)(\lambda y_1),\ (1-\lambda)F(x_2)((1-\lambda)y_2)\}$$

$$= \min\{F(x_1)(y_1),\ F(x_2)(y_2)\},$$

which gives (1).

As a consequence of the above definitions, we obtain the following characterization of convex fuzzy multifunctions.

Proposition 8. *Given a fuzzy multifunction* $F : X \rightsquigarrow Y$, *the following conditions are equivalent*:

(1) *F is convex fuzzy multifunction*;

(2) *GrF is a convex fuzzy set in* $X \times Y$;

(3) *The sets* $\Gamma_\alpha^F := \{(x, y) \in X \times Y : F(x)(y) \geq \alpha\}$ *are convex for each* $\alpha \in (0; 1]$.

Proof. (1) $\Rightarrow$ (2) This implication is obvious, because of the definition of the graph *GrF*.

(2) $\Rightarrow$ (3) We take an arbitrary $\alpha \in (0; 1]$ and $(x_1, y_1), (x_2, y_2) \in \Gamma_\alpha^F$. Then, for all $\lambda \in [0; 1]$ we have

$$GrF(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2)$$
$$\geq \min\{GrF(x_1, y_1), GrF(x_2, y_2)\} \geq \alpha.$$

It means that every point $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)$ for each $\lambda \in [0; 1]$ also belongs to Γ_α^F.

(3) $\Rightarrow$ (1) Let $\alpha = F(x_1)(y_1) \leq F(x_2)(y_2)$. Then $(x_2, y_2) \in \Gamma_\alpha^F$ and of course $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in \Gamma_\alpha^F$. Finally, we obtain

$$F(\lambda x_1 + (1-\lambda)x_2)(\lambda y_1 + (1-\lambda)y_2)$$
$$\geq \alpha = F(x_1)(y_1) = \min\{F(x_1)(y_1), F(x_2)(y_2)\}. \qquad \square$$

As we mentioned above, a fuzzy multivalued function from a space X into a space Y assigns to each $x \in X$ a fuzzy set $F(x)$ in Y. We say that $F(x)_\alpha$ is the α-*cut* of the fuzzy set $F(x)$ on the level $\alpha \in [0; 1]$ if

$$F(x)_\alpha(y) = \begin{cases} F(x)(y), & \text{for } F(x)(y) \geq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 9. *Let $(X, \tau_X), (Y, \tau_Y)$ be topological vector spaces with fuzzy topology $\delta(\tau_Y)$ defined on the space Y. Let additionally (Y, τ_Y) be regular space. About fuzzy multivalued function $F : X \rightsquigarrow Y$, we assume that it is upper semicontinuous and fuzzy closed valued. Then the graph GrF is a fuzzy closed set.*

Proof. Fuzzy set *GrF* will be closed, if for each $\alpha \in [0; 1]$ the set

$$\Gamma_\alpha^F = \{(x, y) \in X \times Y : F(x)(y) \geq \alpha\}$$

is closed. In the case $\alpha = 0$ the proof is trivial. So we take an arbitrary $\alpha \in (0; 1]$ and arbitrary $(x_0, y_0) \notin \Gamma_\alpha^F$. Then $y_0 \notin F(x_0)_\alpha$. This notation means that $y_0 \notin supp(F(x)_\alpha)$. Because the space (Y, τ_Y) is regular, then by the definition, the fuzzy topology $\delta(\tau_Y)$ is regular too. The fuzzy set $F(x_0)_\alpha$ is closed and $y_0 \notin F(x_0)_\alpha$ thus using regularity of the fuzzy topology $\delta(\tau_Y)$ there exist fuzzy open sets V, W such that

$$V \cap W = 0 \quad (suppV \cap suppW = \varnothing),$$

$$y_0 \in V \quad \text{and} \quad F(x_0)_\alpha \subset W.$$

Using the upper semicontinuous fuzzy multivalued function F we have $x_0 \in U = F^+(W)$ and $U \in \tau_X$. Finally, $(x_0, y_0) \in U \times suppV \cap \Gamma_\alpha = \varnothing$. This completes the proof. □

It is well known that a $\frac{1}{2}$-convex, continuous function defined on an interval $I \subset \mathbb{R}$ is convex ([4, p. 149] or [6, p. 218]). Nikodem moved this classical result to the theory of set-valued functions in [5, p. 30]. We will write analogous theorem for fuzzy multivalued functions.

Theorem 10. *Let the assumptions of Lemma* 9 *on X and Y be satisfied. If a fuzzy multivalued function* $F : X \rightsquigarrow Y$ *is* $\frac{1}{2}$*-convex, upper semicontinuous and fuzzy closed valued, then F is convex.*

Proof. From $\frac{1}{2}$-convexity for each $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ we have

$$F\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right)\left(\frac{1}{2}y_1 + \frac{1}{2}y_2\right) \geq \min\{F(x_1)(y_1), F(x_2)(y_2)\}.$$

By induction, we can obtain

$$F\left(\frac{1}{2^p}x_1 + \cdots + \frac{1}{2^p}x_{2^p}\right)\left(\frac{1}{2^p}y_1 + \cdots + \frac{1}{2^p}y_{2^p}\right)$$

$$\geq \min\{F(x_1)(y_1), \ldots, F(x_{2^p})(y_{2^p})\}.$$

Let now $x_1 := x_1 = \cdots = x_n$, $x_2 := x_{n+1} = \cdots = x_{2^p}$ and $y_1 := y_1 = \cdots = y_n$, $y_2 := y_{n+1} = \cdots = y_{2^p}$. Then for $n < 2^p$, we have

$$F\left(\frac{n}{2^p}x_1 + \left(1 - \frac{n}{2^p}\right)x_2\right)\left(\frac{n}{2^p}y_1 + \left(1 - \frac{n}{2^p}\right)y_2\right) \geq \min\{F(x_1)(y_1), F(x_2)(y_2)\}.$$

If we denote the dyadic number $\frac{n}{2^p}$ by q_n, we will have

$$F(q_n x_1 + (1 - q_n)x_2)(q_n y_1 + (1 - q_n)y_2) \geq \min\{F(x_1)(y_1), F(x_2)(y_2)\}. \quad (6)$$

Let us take an arbitrary $q \in [0; 1]$ and a sequence of dyadic numbers (q_n) convergent to q. Condition (6) means that for every $n \in \mathbb{N}$

$$(q_n x_1 + (1 - q_n)x_2, q_n y_1 + (1 - q_n)y_2) \in \Gamma_\alpha^F$$

with $\alpha = \min\{F(x_1)(y_1), F(x_2)(y_2)\}$. By Lemma 9, the set Γ_α^F is closed, so passing with (q_n) to the limit, we get

$$(qx_1 + (1 - q)x_2, qy_1 + (1 - q)y_2) \in \Gamma_\alpha^F.$$

This means that

$$F(qx_1 + (1 - q)x_2)(qy_1 + (1 - q)y_2) \geq \min\{F(x_1)(y_1), F(x_2)(y_2)\}$$

and proves that F is convex. □

3. A Sandwich with Convexity for Fuzzy Multivalued Functions

In [1], Baron et al. proved that two real functions f, g defined on an interval $I \subset \mathbb{R}$ can be separated by convex functions if and only if they satisfy

$$\forall_{x_1, x_2 \in I} \ \forall_{\lambda \in [0;1]} \quad f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda g(x_1) + (1 - \lambda)g(x_2).$$

Next, Sadowska proved the analogous theorem for set-valued functions in [7].

The aim of this section is to present conditions under which for the given fuzzy multivalued functions *F* and *G*, there exists a convex fuzzy multivalued function *H* such that $F \subset H \subset G.$

We mentioned above that a fuzzy multivalued function *F* may be identified with the graph *GrF* defined as $GrF(x, y) := F(x)(y).$ So if we talk about the graph of *F*, we can think about the function *F* and conversely.

We start with a simple lemma.

Lemma 11. *Let* $H, G : \mathbb{R} \rightsquigarrow \mathbb{R}$ *be fuzzy multivalued functions. Then* $GrH \subset GrG$ *(it means* $GrH(x, y) \leq GrG(x, y)$ *for each* $(x, y) \in \mathbb{R} \times \mathbb{R}$*) if and only if* $\Gamma_\alpha^H \subset \Gamma_\alpha^G$ *for all* $\alpha \in [0; 1],$ *where*

$$\Gamma_\alpha^H = \{(x, y) \in \mathbb{R}^2 : H(x)(y) \geq \alpha\},$$

$$\Gamma_\alpha^G = \{(x, y) \in \mathbb{R}^2 : G(x)(y) \geq \alpha\}.$$

Theorem 12. *Let* $F, G : \mathbb{R} \rightsquigarrow \mathbb{R}$ *be given fuzzy multivalued functions such that for each* $\alpha \in [0; 1],$ Γ_α^F *is the union of two connected subsets of* $\mathbb{R}^2$. *Then for each* $x_1, x_2 \in X, y_1, y_2 \in Y, \lambda \in [0, 1],$ *F and G satisfy*

$$G(\lambda x_1 + (1 - \lambda)x_2)(\lambda y_1 + (1 - \lambda)y_2) \geq \min\{F(x_1)(y_1), F(x_2)(y_2)\} \quad (7)$$

if and only if there exists a convex fuzzy multivalued function $H : \mathbb{R} \rightsquigarrow \mathbb{R}$ *such that* $GrF \subset GrH \subset GrG.$

Proof. We assume that *F* and *G* satisfy (7). We recall that the convex hull of a fuzzy set *A* is the smallest convex fuzzy set containing *A* and is denoted by $conv(A)$ [2]. Let $\mathcal{F}$ be the family of all convex fuzzy sets containing *GrF*. Then

$$conv(GrF) = \bigcap_{A \in \mathcal{F}} A = \inf_{A \in \mathcal{F}} A.$$

Consider the fuzzy set *GrH* defined as follows:

$$GrH(x,\ y) := conv(GrF(x,\ y)) = \bigcap_{A\in\mathcal{F}} A(x,\ y) = \inf_{A\in\mathcal{F}} A(x,\ y).$$

It is easy to check

$$GrF \subset GrH(GrF(x,\ y) \leq GrH(x,\ y) \text{ for all } (x,\ y) \in \mathbb{R}^2).$$

Indeed, $GrF \subset conv(GrF)$. Moreover, GrH is convex fuzzy set. It remains to show that $GrH \subset GrG$. Using Lemma 11, it is sufficient to verify that for each $\alpha \in (0; 1]$ holds $\Gamma_\alpha^H \subset \Gamma_\alpha^G$. We set $\alpha_0 \in (0; 1]$ and we take an arbitrary $(x,\ y) \in \Gamma_{\alpha_0}^H$. The question is

$$(x,\ y) \in \Gamma_{\alpha_0}^H \Rightarrow (x,\ y) \in \Gamma_{\alpha_0}^G\ ?$$

Since Γ_α^F is the union of two connected subsets of $\mathbb{R}^2$ for each $\alpha \in [0; 1]$, according to [8, p. 169], $(x,\ y) \in \Gamma_{\alpha_0}^H$ can be expressed in the following form:

$$\Gamma_{\alpha_0}^H \ni \lambda(x_1,\ y_1) + (1-\lambda)(x_2,\ y_2),$$

where $(x_1,\ y_1)$ and $(x_2,\ y_2)$ belong to $\Gamma_{\alpha_0}^F$. Finally, using inequality (7), we have

$$GrG(x,\ y) = G(x)(y) \geq \min\{F(x_1)(y_1),\ F(x_2)(y_2)\} \geq \alpha_0.$$

It shows that $(x,\ y) \in \Gamma_{\alpha_0}^G$.

The converse implication is obvious. □

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