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SPECTRAL PROPERTIES OF CLASS A_k AND ALGEBRAICALLY CLASS A_k OPERATORS

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Abstract

If T is class A_k operator for a positive integer k and $0 \neq \lambda \in iso\,\sigma(T)$, then the Riesz-idempotent operator E_λ with respect to λ is self-adjoint and satisfies $E_\lambda H = \ker(T - \lambda) = \ker(T - \lambda)^*$. If T is algebraically class A_k operator, then Weyl's theorem holds for T and other Weyl type theorems are discussed.

1. Introduction and Preliminaries

Let $B(H)$ be the Banach algebra of all bounded linear operators on a non-zero complex Hilbert space H. By an operator T, we mean an element in $B(H)$. If T lies in $B(H)$, then T^* denotes the adjoint of T in $B(H)$. An operator T is said to be of *class A*, if $|T^2| \geq |T|^2$. An operator T is called

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paranormal if $\| T^2 x \| \geq \| Tx \|^2$, for every unit vector x in H. An operator T is called k-*paranormal* for positive integer k, if $\| T^{k+1} x \| \geq \| Tx \|^{k+1}$ for every unit vector x in H. An operator T is called *quasinormal* if $T(T^*T) = (T^*T)T$.

An operator T is called a *Fredholm operator* if the range of T denoted by $ran(T)$ is closed and both $\ker T$ and $\ker T^*$ are finite dimensional and is denoted by $T \in \Phi(H)$. An operator T is called *upper semi-Fredholm operator*, $T \in \Phi_+(H)$, if $ran(T)$ is closed and $\ker T$ is finite dimensional. An operator T is called *lower semi-Fredholm operator*, $T \in \Phi_-(H)$, if $\ker T^*$ is finite dimensional. The index of a semi-Fredholm operator T is an integer defined as $ind(T) = \dim \ker T - \dim \ker T^*$. An upper semi-Fredholm operator with index less than or equal to 0 is called *upper semi-Weyl operator* and is denoted by $T \in \Phi_+^-(H)$. A lower semi-Fredholm operator with index greater than or equal to 0 is called *lower semi-Weyl operator* and is denoted by $T \in \Phi_-^+(H)$. A Fredholm operator of index 0 is called *Weyl operator*. The set of all isolated eigenvalues of finite multiplicity of T is denoted by $\pi_{00}(T)$ and the set of all isolated eigenvalues of finite multiplicity of T in $\sigma_a(T)$ is denoted by $\pi_{00}^a(T)$.

The spectrum of T is denoted by $\sigma(T)$, where

$$\sigma(T) = \{\lambda \in C : T - \lambda I \text{ is not invertible}\}.$$

The approximate point spectrum of T is denoted by $\sigma_a(T)$, where

$$\sigma_a(T) = \{\lambda \in C : T - \lambda I \text{ is not bounded below}\}.$$

The essential spectrum of T is defined as

$$\sigma_e(T) = \{\lambda \in C : T - \lambda I \text{ is not Fredholm}\}.$$

The essential approximate point spectrum of T is defined as

$$\sigma_{ea}(T) = \{\lambda \in C : T - \lambda I \notin \Phi_+^-(H)\}.$$

The Weyl spectrum of T is defined as

$$w(T) = \{\lambda \in C : T - \lambda I \text{ is not Weyl}\}.$$

When the space is infinite dimensional, $w(0) = \{0\}$ and $w(T) = \{0\}$ if T is compact. Weyl has shown that $\lambda \in \sigma(T + K)$ for every compact operator K if and only if λ is not an isolated eigenvalue of finite multiplicity in $\sigma(T)$ for a Hermitian operator. We say that Weyl's theorem holds for T [6] if T satisfies the equality $\sigma(T) - w(T) = \pi_{00}(T)$ and a-Weyl's theorem holds for T [18] if T satisfies the equality $\sigma_a(T) - \sigma_{ea}(T) = \pi_{00}^a(T)$.

The ascent of T denoted by $p(T)$, is the least nonnegative integer n such that $\ker T^n = \ker T^{n+1}$. The descent of T denoted by $q(T)$, is the least nonnegative integer n such that $ran(T^n) = ran(T^{n+1})$. T is said to be of *finite ascent* if $p(T - \lambda) < \infty$, for all $\lambda \in C$. If $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$ (by [10, Proposition 38.3]). Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T. An upper semi-Fredholm operator with finite ascent is called *upper semi-Browder operator* and is denoted by $T \in B_+(H)$ while a lower semi-Fredholm operator with finite descent is called *lower semi-Browder operator* and is denoted by $T \in B_-(H)$. A Fredholm operator with finite ascent and descent is called *Browder operator*. Clearly, the class of all Browder operators is contained in the class of all Weyl operators. Similarly, the class of all upper semi-Browder operators is contained in the class of all upper semi-Weyl operators and the class of all lower semi-Browder operators is contained in the class of all lower semi-Weyl operators.

The Browder spectrum of T is defined as

$$\sigma_b(T) = \{\lambda \in C : T - \lambda I \text{ is not Browder}\}.$$

For an operator T, $p_{00}(T)$ is defined as

$$p_{00}(T) = \sigma(T) - \sigma_b(T).$$

We say that T satisfies property (w) if

$$\sigma_a(T) - \sigma_{ea}(T) = \pi_{00}(T)$$

and T satisfies property (b) if

$$\sigma_a(T) - \sigma_{ea}(T) = p_{00}(T).$$

An operator T is said to *have the single valued extension property* (*SVEP*) at $\lambda_0 \in C$, if for every open neighborhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T) f(\lambda) = 0$ for all $\lambda \in U$, is the function $f \equiv 0$. An operator T is said to have *SVEP*, if T has SVEP at every point $\lambda \in C$.

An operator T is called *polaroid* if $iso\, \sigma(T) \subseteq \pi(T)$, where $\pi(T)$ is the set of poles of the resolvent of T and $iso\, \sigma(T)$ is the set of all isolated points of $\sigma(T)$. An operator T is said to be *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T. An operator T is said to be *reguloid* if for every isolated point λ of $\sigma(T)$, $\lambda I - T$ is relatively regular. An operator T is known as relatively regular if and only if ker T and $T(X)$ are complemented. Also, polaroid $\Rightarrow$ reguloid $\Rightarrow$ isoloid.

In [16], we showed that class A_k operators form a proper subclass of k-paranormal operators, class A_k operators have finite ascent and satisfy Weyl's theorem.

In this paper, we prove that if T is class A_k operator for a positive integer k and $0 \neq \lambda \in iso\, \sigma(T)$, then the Riesz-idempotent operator E_λ with respect to λ is self-adjoint and satisfies $E_\lambda H = \ker(T - \lambda) = \ker(T - \lambda)^*$. If T is algebraically class A_k operator, then Weyl's theorem holds for T and $f(T)$, for every $f \in Hol(\sigma(T))$, T is polaroid and other Weyl type theorems are discussed.

2. Spectral Properties of Class A_k Operators

Definition 2.1. An operator $T \in B(H)$ is defined to be of class A_k, if $|T^{k+1}|^{\frac{2}{k+1}} \geq |T|^2$ for some positive integer k. If $k = 1$, then class A_k coincides with class A operator.

Example 2.2. Let H be the direct sum of a denumerable number of copies of two dimensional Hilbert space $R \times R$. Let A and B be two positive operators on $R \times R$. For any fixed positive integer n, define an operator $T = T_{A, B, n}$ on H as follows:

$$T((x_1, x_2, x_3, \ldots)) = (0, A(x_1), A(x_2), \ldots, A(x_n), B(x_{n+1}), \ldots).$$

Its adjoint T^* is given by

$$T^*((x_1, x_2, x_3, \ldots)) = (A(x_2), A(x_3), \ldots, A(x_n), B(x_{n+1}), \ldots).$$

For $n \geq k$, $T_{A, B, n}$ is of class A_k if and only if A and B satisfy

$$(A^{k-i+1}B^{2i}A^{k-i+1})^{\frac{1}{k+1}} \geq A^2, \quad i = 1, 2, \ldots, k.$$

If $A = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then $T = T_{A, B, n}$ is of class A_2.

Kubrusly and Duggal [13] have shown that k-paranormal operators are hereditarily normaloid. Since class A_k operators are k-paranormal, it follows that class A_k operators are hereditarily normaloid.

Theorem 2.3. *If T is class* A_k *operator for a positive integer k and for* $\lambda \in C$, $\sigma(T) = \lambda$, *then* $T = \lambda$.

Proof. If $\lambda = 0$, then since class A_k operator is normaloid, $T = 0$. Assume that $\lambda \neq 0$. Then T is an invertible normaloid operator with $\sigma(T) = \lambda$. $T_1 = \frac{1}{\lambda}T$ is an invertible normaloid operator with $\sigma(T_1) = \{1\}$. Hence T_1 is similar to an invertible isometry B (on an equivalent normed linear space) with $\sigma(B) = 1$ (by Theorem 2, [12]) T_1 and B being similar, 1 is an eigenvalue of $T_1 = \frac{1}{\lambda}T$ (by Theorem 5, [12]). Therefore, by Theorem 1.5.14 of [14], $T_1 = I$. Hence $T = \lambda$. □

Theorem 2.4. *If T is class A_k operator for a positive integer k and M is an invariant subspace of T, then the restriction $T_{|M}$ is also class A_k.*

Proof. Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection of H onto M and $T_1 = T_{|M}$. Then $TP = PTP$ and $T_1 = (PTP)_{|M}$.

Since T is of class A_k operator, $P\left(| T^{1+k} |^{\frac{2}{1+k}} - | T |^2 \right) P \geq 0$. By Hansen's inequality [9],

$$P\left(| T^{1+k} |^{\frac{2}{1+k}} \right) P = P(T^{*1+k}T^{1+k})^{\frac{1}{1+k}} P \leq (PT^{*1+k}T^{1+k}P)^{\frac{1}{1+k}}$$

$$= \begin{pmatrix} | T_1^{1+k} |^2 & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{1+k}} = \begin{pmatrix} | T_1^{1+k} |^{\frac{2}{1+k}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $\begin{pmatrix} | T_1^{1+k} |^{\frac{2}{1+k}} & 0 \\ 0 & 0 \end{pmatrix} \geq P\left(|T^{1+k}|^{\frac{2}{1+k}} \right) P \geq P| T |^2 P = \begin{pmatrix} | T_1 |^2 & 0 \\ 0 & 0 \end{pmatrix}$. Hence T_1 is also class A_k operator on M. □

Theorem 2.5. *If T is class A_k operator for a positive integer k, $0 \neq \lambda \in \sigma_p(T)$ and T is of the form $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\ker(T-\lambda) \oplus \overline{ran(T-\lambda)^*}$, then*

1. $T_2 = 0$ *and*

2. T_3 *is class* A_k.

Proof. Let P be the orthogonal projection of H onto $\ker(T-\lambda)$.

Since T is class A_k, T satisfies

$$|T^{k+1}|^{\frac{2}{k+1}} - |T|^2 \geq 0,$$

where k is a positive integer. Hence

$$P(|T^{k+1}|^{\frac{2}{k+1}} - |T|^2)P \geq 0,$$

where $P|T|^2 P = \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix}$ and $(P|T^{k+1}|^2 P) = \begin{pmatrix} |\lambda|^{2(k+1)} & 0 \\ 0 & 0 \end{pmatrix}$.

Therefore,

$$\begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix} = (P|T^{k+1}|^2 P)^{\frac{1}{k+1}} \geq P|T^{k+1}|^{\frac{2}{k+1}} P \geq P|T|^2 P = \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$P|T^{k+1}|^{\frac{2}{k+1}} P = \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix} = P|T|^2 P.$$

Hence $|T^{k+1}|^{\frac{2}{k+1}}$ is of the form $|T^{k+1}|^{\frac{2}{k+1}} = \begin{pmatrix} |\lambda|^2 & A \\ A^* & B \end{pmatrix}$ for some linear operators $A : \overline{ran(T-\lambda)^*} \to \ker(T-\lambda)$ and $B : \overline{ran(T-\lambda)^*} \to \overline{ran(T-\lambda)^*}$.

Since $\begin{pmatrix} |\lambda|^{2(k+1)} & 0 \\ 0 & 0 \end{pmatrix} = P(|T^{k+1}|^2)P = P\left(|T^{k+1}|^{\frac{2}{k+1}}\right)^{k+1}P$, we can easily show that $A = 0$. Therefore, $|T^{k+1}|^{\frac{2}{k+1}} = \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & B \end{pmatrix}$ and hence $|T^{k+1}|^2 = \begin{pmatrix} |\lambda|^{2(k+1)} & 0 \\ 0 & B^{(k+1)} \end{pmatrix}$.

This implies that $\lambda^k T_2 + \lambda^{k-1}T_2T_3 + \cdots + T_2T_3^k = 0$ and $B = |T_3^{k+1}|^{\frac{2}{k+1}}$. Therefore,

$$0 \leq |T^{k+1}|^{\frac{2}{k+1}} - |T|^2 = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix},$$

where $X = 0$, $Y = -\bar{\lambda}T_2$ and $Z = |T_3^{k+1}|^{\frac{2}{k+1}} - |T_2|^2 - |T_3|^2$.

A matrix of the form $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0$ if and only if $X \geq 0, Z \geq 0$ and $Y = X^{1/2}WZ^{1/2}$, for some contraction W. Hence $T_2 = 0$ and T_3 is class A_k. □

Corollary 2.6. *If T is class A_k operator for a positive integer k and $(T-\lambda)x = 0$ for $\lambda \neq 0$ and $x \in H$, then $(T-\lambda)^* x = 0$.*

Corollary 2.7. *If T is class A_k operator for a positive integer k, $0 \neq \lambda \in \sigma_p(T)$, then T is of the form $T = \begin{pmatrix} \lambda & 0 \\ 0 & T_3 \end{pmatrix}$ on $\ker(T-\lambda) \oplus \overline{ran(T-\lambda)^*}$, where T_3 is class A_k and $\ker(T_3 - \lambda) = \{0\}$.*

If $\lambda \in iso\,\sigma(T)$, then the spectral projection (or Riesz idempotent) E_λ of T with respect to λ is defined by $E_\lambda = \frac{1}{2\pi i}\int_{\partial D}(z-T)^{-1}dz$, where D is a closed disk with centre at λ and radius small enough such that

$D \cap \sigma(T) = \{\lambda\}$. Then $E_\lambda^2 = E_\lambda$, $E_\lambda T = TE_\lambda$, $\sigma(T_{|E_\lambda H}) = \{\lambda\}$ and $\ker(T - \lambda) \subset E_\lambda H$.

Theorem 2.8. *If T is a class* A_k *operator for a positive integer k and* $\lambda \in \sigma(T)$ *is an isolated point, then the Riesz idempotent operator* E_λ *with respect to* λ *satisfies* $E_\lambda H = \ker(T - \lambda)$. *Hence* λ *is an eigenvalue of T.*

Proof. Since $\ker(T - \lambda) \subseteq E_\lambda H$, it is enough to prove that $E_\lambda H \subseteq \ker(T - \lambda)$. Now $\sigma(T_{|E_\lambda H}) = \{\lambda\}$ and $T_{|E_\lambda H}$ is class A_k. Therefore, by Theorem 2.3, $T_{|E_\lambda H} = \lambda$. Hence $E_\lambda H = \ker(T - \lambda)$. □

Theorem 2.9 [11]**.** *If T is a class* A_k *operator for a positive integer k, then T has SVEP and* $p(\lambda I_T) \leq 1$ *for all* $\lambda \in C$. *Furthermore, both T and* T^* *are reguloid.*

Corollary 2.10. *If T is a class* A_k *operator for a positive integer k, then T is isoloid.*

Theorem 2.11. *Let T be a class* A_k *operator for a positive integer k and* $\lambda \neq 0$ *be an isolated point in* $\sigma(T)$. *Then the Riesz idempotent operator* E_λ *with respect to* λ *is self-adjoint and satisfies* $E_\lambda H = \ker(T - \lambda) = \ker(T - \lambda)^*$.

Proof. Without loss of generality, we assume that $\lambda = 1$. Let $T = \begin{pmatrix} 1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\ker(T - \lambda) \oplus \overline{ran(T - \lambda)^*}$. By Theorem 2.5, $T_2 = 0$ and T_3 is class A_k. Since $1 \in iso\,\sigma(T)$, either $1 \in iso\,\sigma(T_3)$ or $1 \notin \sigma(T_3)$. If $1 \in iso\,\sigma(T_3)$, since T_3 is isoloid, $1 \in \sigma_p(T_3)$ which contradicts $\ker(T_3 - \lambda) = \{0\}$ (by Corollary 2.7). Therefore, $1 \notin \sigma(T_3)$ and hence $T_3 - 1$ is invertible. Therefore, $T - 1 = 0 \oplus (T_3 - 1)$ is invertible on H and $\ker(T - 1) = \ker(T - 1)^*$. Also,

$$E_\lambda = \frac{1}{2\pi i}\int_{\partial D}(zI - T)^{-1}dz = \frac{1}{2\pi i}\int_{\partial D}\begin{pmatrix}(z-1)^{-1} & 0\\ 0 & (z-T_3)^{-1}\end{pmatrix}dz = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

Therefore, E_λ is the orthogonal projection onto $\ker(T - \lambda)$ and hence E_λ is self-adjoint. □

Theorem 2.12. *If T is a partial isometry and class* A_k *operator, then T is quasinormal.*

Proof. Since T is a partial isometry, $T = TT^*T$ [8]. This together with the definition of class A_k operator gives $T^{*k+1}T^{k+1} \geq (T^*T)^k \geq (T^*T)^{k-1} \geq \cdots \geq T^*T$.

Therefore,

$$\|Tx\|^2 = \langle T^*Tx, x\rangle \leq \langle |T^{k+1}|^2 x, x\rangle \leq \|T^{k+1}x\|^2$$
$$\leq \|T^k x\|^2 \leq \cdots \|T^2 x\|^2 \leq \|Tx\|^2.$$

Hence $\|T^2x\| = \|Tx\|$.

$$\|T^*T^2x - Tx\|^2 = \langle T^*T^2x, T^*T^2x\rangle - \langle T^*T^2x, Tx\rangle - \langle Tx, T^*T^2x\rangle + \langle Tx, Tx\rangle$$
$$= \langle T^2x, T^2x\rangle - \langle T^2x, T^2x\rangle - \langle T^2x, T^2x\rangle + \langle Tx, Tx\rangle$$
$$= \|Tx\|^2 - \|T^2x\|^2 = 0.$$

Hence $T^*TT = T = TT^*T$, i.e., T is quasinormal. □

3. Weyl Type Theorems for Algebraically Class A_k Operators

Definition 3.1. An operator T is defined to be of algebraically class A_k for a positive integer k, if there exists a non-constant complex polynomial $p(t)$ such that $p(T)$ is of class A_k.

Theorem 3.2. *If T is algebraically class* A_k *operator for some positive integer k and* $\sigma(T) = \mu_0$, *then* $T - \mu_0$ *is nilpotent.*

Proof. Since T is algebraically class A_k, there is a non-constant polynomial $p(t)$ such that $p(T)$ is class A_k for some positive integer k, then applying Theorem 2.3,

$$\sigma(p(T)) = p(\sigma(T)) = \{p(\mu_0)\} \text{ implies } p(T) = p(\mu_0).$$

Let $p(z) - p(\mu_0) = a(z - \mu_0)^{k_0}(z - \mu_1)^{k_1} \cdots (z - \mu_t)^{k_t}$, where $\mu_j \neq \mu_s$ for $j \neq s$. Then $0 = p(T) - p(\mu_0) = a(T - \mu_0)^{k_0}(T - \mu_1)^{k_1} \cdots (T - \mu_t)^{k_t}$. Since $T - \mu_1, T - \mu_2, ..., T - \mu_t$ are invertible, $(T - \mu_0)^{k_0} = 0$. Hence $T - \mu_0$ is nilpotent. □

If T is algebraically class A_k operator for some positive integer k, then there exists a non-constant polynomial $p(t)$ such that $p(T)$ is class A_k. By Theorem 4.3 [16], $p(T)$ is of finite ascent. Therefore, $(p(T))$ and hence T has SVEP ([14, Theorem 3.3.6]).

Theorem 3.3. *If T is algebraically class A_k operator for some positive integer k, then Weyl's theorem holds for T.*

Proof. Assume that $\lambda \in \sigma(T) - w(T)$. Then $T - \lambda$ is Weyl and not invertible.

Claim. $\lambda \in \partial\sigma(T)$. Assume on the contrary that λ is an interior point of $\sigma(T)$. Then there exists a neighborhood U of λ such that $\dim N(T - \mu) > 0$ for all μ in U. Hence by ([7, Theorem 10]), T does not have SVEP which is a contradiction. Hence $\lambda \in \partial\sigma(T) - w(T)$. Therefore, by punctured neighborhood theorem, $\lambda \in \pi_{00}(T)$.

Conversely, suppose that $\lambda \in \pi_{00}(T)$. Using the Riesz idempotent E_λ with respect to λ, we can represent T as the direct sum $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) - \{\lambda\}$. Then by Theorem 3.2, $T_1 - \lambda$ is nilpotent. Since $\lambda \in \pi_{00}(T)$, $T_1 - \lambda$ is a finite dimensional operator, so $T_1 - \lambda$ is Weyl. But since $T_2 - \lambda$ is invertible, $T - \lambda$ is Weyl. Hence $\lambda \in \sigma(T) - w(T)$. Therefore, $\sigma(T) - w(T) = \pi_{00}(T)$. □

By ([3, Theorem 2.16]), we get the following result.

Corollary 3.4. *If T is algebraically class* A_k *for some positive integer k, and* T^* *has SVEP, then a-Weyl's theorem and property* (w) *hold for T.*

Theorem 3.5. *If T is algebraically class* A_k *operator for some positive integer k, then* $w(f(T)) = f(w(T))$ *for every* $f \in Hol(\sigma(T))$.

Proof. Suppose that T is algebraically class A_k for some positive integer k. Then T has SVEP. Hence by [10, Proposition 38.5], $ind(T - \lambda) \leq 0$ for all complex numbers λ. Now to prove the result, it is sufficient to show that $f(w(T)) \subseteq w(f(T))$. Let $\lambda \in f(w(T))$. Suppose if $\lambda \notin w(f(T))$, then $f(T) - \lambda I$ is Weyl and hence $ind(f(T) - \lambda) = 0$. Let $f(z) - \lambda = (z - \lambda_1)(z - \lambda_2)\ldots(z - \lambda_n)g(z)$. Then $f(T) - \lambda = (T - \lambda_1)(T - \lambda_2)\cdots(T - \lambda_n)g(T)$ and $ind(f(T) - \lambda) = 0 = ind(T - \lambda_1) + ind(T - \lambda_2) + \cdots + ind(T - \lambda_n) + ind g(T)$. Since each of $ind(T - \lambda_i) \leq 0$, we get that $ind(T - \lambda_i) = 0$, for all $i = 1, 2, \ldots, n$. Therefore, $T - \lambda_i$ is Weyl for each $i = 1, 2, \ldots, n$. Hence $\lambda_i \notin w(T)$ and hence $\lambda \notin f(w(T))$, which is a contradiction. Hence the theorem. □

Theorem 3.6. *If T is algebraically class* A_k *operator for some positive integer k, then Weyl's theorem holds for* $f(T)$, *for every* $f \in Hol(\sigma(T))$.

Proof. For every $f \in Hol(\sigma(T))$,

$$\sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T) - \pi_{00}(T)) \text{ by ([15, Lemma])}$$

$$= f(w(T)) \text{ by Theorem 3.3}$$

$$= w(f(T)) \text{ by Theorem 3.5.}$$

Hence Weyl's theorem holds for $f(T)$, for every $f \in Hol(\sigma(T))$. □

Theorem 3.7. *If T or* T^* *is algebraically class* A_k *operator for some positive integer k, then* $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$.

Proof. For $T \in B(H)$, by [17], the inclusion $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ holds for every $f \in Hol(\sigma(T))$ with no restrictions on *T*. Therefore, it is enough to prove that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$.

Suppose if $\lambda \notin \sigma_{ea}(f(T))$, then $f(T) - \lambda \in \Phi_+^-(H)$, that is, $f(T) - \lambda$ is upper semi-Fredholm operator with index less than or equal to zero. Also, $f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2)\ldots(T - \alpha_n)g(T)$, where $g(T)$ is invertible and $\alpha_1\alpha_2, \ldots \alpha_n \in C$.

If *T* is algebraically class A_k for some positive integer *k*, then there exists a non-constant polynomial $p(t)$ such that $p(T)$ is class A_k. Then $p(T)$ has SVEP and hence *T* has SVEP. Therefore, $ind(T - \alpha_i) \le 0$ and hence $T - \alpha_i \in \Phi_+^-(H)$ for each $i = 1, 2, ..., n$. Therefore, $\lambda = f(\alpha_i) \notin f(\sigma_{ea}(T))$. Hence $\sigma_{ea}(f(T)) = g(\sigma_{ea}(T))$.

If T^* is algebraically class A_k for some positive integer *k*, then there exists a non-constant polynomial $p(t)$ such that $p(T^*)$ is class A_k. Then $p(T^*)$ has SVEP and hence T^* has SVEP. Therefore, $ind(T - \alpha_i) \ge 0$ for each $i = 1, 2, ..., n$. Therefore, $0 \le \sum_{i=1}^{n} ind(T - \alpha_i) = ind(f(T) - \lambda) \le 0$. Therefore, $ind(T - \alpha_i) = 0$ for each $i = 1, 2, ..., n$. Therefore, $T - \alpha_i$ is Weyl for each $i = 1, 2, ..., n$. $(T - \alpha_i) \in \Phi_+^-(H)$ and hence $\alpha_i \notin \sigma_{ea}(T)$. Therefore, $\lambda = f(\alpha_i) \notin f(\sigma_{ea}(T))$. Hence $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. □

Theorem 3.8. *If T is algebraically class* A_k *operator for some positive integer k, then T is polaroid.*

Proof. If $\lambda \in iso\,\sigma(T)$ using the spectral projection of *T* with respect to λ, we can write $T = T_1 \oplus T_2$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) - \{\lambda\}$. Since T_1 is algebraically class A_k operator and $\sigma(T_1) = \{\lambda\}$, by Theorem 3.2, $T_1 - \lambda I$ is nilpotent. Since $\lambda \notin \sigma(T_2)$, $T_2 - \lambda I$ is invertible. Hence both $T_1 - \lambda I$ and $T_2 - \lambda I$ and hence $T - \lambda I$ have finite ascent and descent. Hence λ is a pole of the resolvent of T. Hence T is polaroid. □

Corollary 3.9. *If T is algebraically class A_k operator for some positive integer k, then T is reguloid.*

Corollary 3.10. *If T is algebraically class A_k operator for some positive integer k, then T is isoloid.*

If T^* has SVEP, then by ([1, Lemma 2.15]), $\sigma_{ea}(T) = \sigma(T)$ and by ([2, Corollary 2.45]) $\sigma(T) = \sigma_a(T)$. Hence we get the following result.

Corollary 3.11. *If T is algebraically class A_k for some positive integer k and if in addition T^* has SVEP, then a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Corollary 3.12. *If T^* is algebraically class A_k for some positive integer k, then $w(f(T)) = f(w(T))$.*

By ([1, Theorem 2.17]), we get the following results.

Corollary 3.13. *If T is algebraically class A_k for some positive integer k, and T^* has SVEP, then property* (*b*) *holds for T.*

Corollary 3.14. *If T is algebraically class A_k for some positive integer k, Weyl's theorem, a-Weyl's theorem, then property* (*w*) *and property* (*b*) *hold for T^*.*

4. Generalized Weyl's Theorem

For an operator T and a nonnegative integer n, define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$. In particular, $T_{[0]} = T$. If for some integer n, $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then T is called an *upper* (resp. *lower*) *semi-B-Fredhom operator*. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a *B-Fredholm operator*. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. The index of a semi-B-Fredholm operator T is the index of semi-Fredholm operator $T_{[d]}$, where d is the degree of the stable iteration of T and defined as $d = \inf\{n \in N;$ for all $m \in N,\ m \geq n \Rightarrow (R(T^n) \cap N(T)) \subset (R(T^m) \cap N(T))\}$. T is called a *B-Weyl operator* if it is B-Fredholm of index 0. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by $\sigma_{BW}(T) = \{\lambda \in C : T - \lambda I$ is not a B-Weyl operator$\}$. We say that T satisfies generalized Weyl's theorem [4] if $\sigma(T) - \sigma_{BW}(T) = E(T)$, where $E(T)$ denotes the isolated eigenvalues of T with no restriction on multiplicity. An operator T is Drazin invertible, if it has finite ascent and descent.

Theorem 4.1. *If T is algebraically class A_k operator for some positive integer k, then generalized Weyl's theorem holds for T.*

Proof. Assume that $\lambda \in \sigma(T) - \sigma_{BW}(T)$. Then $T - \lambda$ is B-Weyl and not invertible. Then as in the necessary part of the proof of Theorem 3.3, we get $\lambda \in E(T)$.

Conversely, suppose that $\lambda \in E(T)$. Then λ is isolated in $\sigma(T)$. Using the Riesz idempotent E_λ with respect to λ, we can represent T as the direct sum $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) - \{\lambda\}$. Then by Theorem 3.2, $T_1 - \lambda$ is nilpotent. Since $\lambda \notin \sigma(T_2)$, $T_2 - \lambda$ is invertible. Hence both $T_1 - \lambda$ and $T_2 - \lambda$ have both finite ascent and descent. Hence $T - \lambda$ has both finite ascent and descent. Hence $T - \lambda$ is Drazin invertible. Therefore, by [5, Lemma 4.1], $T - \lambda$ is B-Fredholm of index 0. Hence $\lambda \in \sigma(T) - \sigma_{BW}(T)$. Therefore, $\sigma(T) - \sigma_{BW}(T) = E(T)$. □

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