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ASYMPTOTICS OF THE KLEIN-GORDON EQUATION

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Abstract

We propose a simple method for constructing asymptotics of eigenvalues for the Klein-Gordon equation in the presence of a shallow potential well. Using Green functions, we reduce the initial problem to an integral equation and then applying the method of Neumann series to solve it.

1. Introduction

In [5], we find the Klein-Gordon equation

$$\Phi_{tt} - \Delta\Phi + m^2\Phi = 0, \quad m > 0,$$

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Δ is the Laplacian of dimension *n*, perturbed by a potential $U = U(x)$ to

$$\Phi_{tt} - \Delta\Phi + m^2\Phi + U\Phi = 0.$$

Looking for the solution of the equation, $\Phi = \exp(i\omega t)\Psi(x)$, where ω is the frequency, for Ψ we obtain

$$(-\Delta + m^2 + U)\Psi = E\Psi, \quad E = \omega^2, \tag{1.1}$$

in the case when U describes a shallow potential well $(\text{i.e., } U = \varepsilon V(x),$ $V(x) \in C_0^\infty(\mathbb{R}^n),$ and $\varepsilon \to 0).$ On the other hand, it is well known that the Schrödinger equation

$$(-\Delta + U)\Psi = E\Psi, \tag{1.2}$$

in the case when U describes a shallow potential well, has exactly one eigenvalue $E_0 = -\beta^2, \beta \in \mathbb{R}$ below the essential spectrum $[0, \infty)$ in the case when $\int_{\mathbb{R}^n} V(x)dx \leq 0$ and the dimension *n* of the configuration space is 1 or 2. This was established for $n = 1$ and in the radially symmetric case for $n = 2$ already in the famous book of Landau and Lifshitz [4] and then was proved in the general case in dimension 2 by Simon [6]. Close results on the limiting behavior of the resolvent can be encountered in [1, 3]. In [9], a different method was used for obtaining asymptotics of eigenfunctions. It was based on an explicit construction of approximate eigenfunctions. It happens that this construction is completely elementary when they pass to the momentum representation. Also, this method is equally efficient for the Schrödinger equation and Klein-Gordon equation. The last problem was studied by several authors (we mention, for example, [2, 5, 7]).

In this paper, using Green functions and Neumann series we obtain solutions for the Klein-Gordon equation.

2. Mathematical Formulation

The mathematical formulation of the problem under consideration is as follows: we look for non-trivial solutions $\varphi \in L^2(\mathbb{R})$ of the problem

$$-\varphi_{xx}(x) + m^2\varphi(x) + \varepsilon V(x)\varphi(x) = E\varphi(x), \tag{2.1}$$

where $\varepsilon \to 0$ and V is such that $\int_{-\infty}^{\infty} V(x)dx \leq 0$ and, for simplicity, V has compact support, then $V(x) = 0$ for $|x| > R$ with R sufficiently large. Given that the multiplication operator by a function of compact support is compact in L^2, the continuous spectrum of (2.1) coincides with the continuous spectrum of the non-perturbed equation $(\varepsilon = 0)$ and the last is the interval $[m^2, \infty)$. As in [9], formulas that appear here are based on the following: for $E = -\beta^2 + m^2$, we say that the solution of (2.1) for $|x| > R$ is given by $\varphi(x) \sim e^{-\beta|x|}$. We obtain (for small β) a function that is "almost constant", when $\beta \to 0$. The main result is as follows.

Theorem 2.1. *Let* $\int V(x)dx < 0$. *Then the unique negative eigenvalue of the problem* (2.1) *is given by* $E = m^2 - \beta^2(\varepsilon)$, *where*

$$\beta = -\frac{\varepsilon}{2}\int V(x)dx + O(\varepsilon^2). \tag{2.2}$$

3. Main Results and Solution of the Integral Equation

Considering the problem (2.1) and taking $E = m^2 - \beta^2(\varepsilon)$, we can find the Green function (see [8])

$$G(x - \xi) = \frac{1}{2\beta}e^{-\beta|x-\xi|}. \tag{3.1}$$

Taking the singular part of the Green function (3.1) as $\beta \to 0$, we have

$$G_s = \frac{1}{2\beta}.$$

Now we have the regular part of the Green function

$$G_r = G - G_s$$

which is analytic in β for small β. Therefore, looking for the solution of the problem (2.1) in the form

$$\varphi(x) = \int G(x - \xi) A(\xi) d\xi. \tag{3.2}$$

For A, we obtain

Lemma 3.1. *$A(x)$ satisfies*

$$A(x) = \varepsilon \int W(x, \xi) A(\xi) d\xi \tag{3.3}$$

$x \in \mathbb{R}$. *Here $W(x, \xi)$ is given by*

$$W(x, \xi) = -V(x) G(x - \xi). \tag{3.4}$$

Definition 3.2. Ω denotes the space of C_0^∞ functions on $\mathbb{R}$, with the norm in L_2.

Assume that $A(\xi)$ belongs to Ω (later we will show that this is indeed the case). By the Green function, equation (3.3) takes the form

$$A(x) - \varepsilon \int N(x, \xi) A(\xi) d\xi = -\varepsilon \frac{1}{2\beta} A_0 V(x). \tag{3.5}$$

Here $N(x, \xi)$ is given by

$$N(x, \xi) = -V(x) G_r(x - \xi), \tag{3.6}$$

and

$$A_0 = \int A(\xi) d\xi. \tag{3.7}$$

Definition 3.3. Define the integral operator $T_\beta : \Omega \to \Omega$ by the formula

$$[T_\beta \varphi(\xi)](x) = \int N(x, \xi) \varphi(\xi) d\xi, \quad x \in \mathbb{R}. \tag{3.8}$$

Remark 3.4. $[T_\beta \varphi(\xi)](x) \in \Omega$ (the integrand is analytic) and T_β is well-defined. $[T_\beta \varphi(\xi)](x)$ is analytic in β:

$$G_r = \frac{1}{2} \sum_{m=0}^{\infty} \frac{|x - \xi|^{m+1}}{(m+1)} \beta^m. \tag{3.9}$$

Furthermore, T_β is bounded. Indeed,

$$\| T_\beta \varphi \| = \left\| \int N(z, \zeta) \varphi(\zeta) d\zeta \right\| \leq C \| \varphi \|$$

for some adequate constant C. Therefore, $\varepsilon \| T_\beta \| < 1$ for ε sufficiently small. Now equation (3.5) can be rewritten as

$$[(1 - \varepsilon T_\beta) A(\xi)](x) = -\varepsilon \frac{1}{2\beta} A_0 V(x). \tag{3.10}$$

Given that εT_β is a contraction operator, we can invert the operator $(1 - \varepsilon T_{\beta, \xi \to x})$

$$A(x) = (1 - \varepsilon T_{\beta, \xi \to x})^{-1} \left[- \varepsilon \frac{1}{2\beta} A_0 V(\xi) \right](x). \tag{3.11}$$

Thus, because of Remark 3.4, we have a uniformly convergent series of analytic functions in x on $\mathbb{R}$. Therefore, $A(x)$ is analytic in $x \in \mathbb{R}$. From equation (3.11), integrating and multiplying by β we obtain the secular equation for β:

$$\beta = \int (1 - \varepsilon T_{\beta, \xi \to x})^{-1} \left[- \varepsilon \frac{1}{2} V(\xi) \right](x) dx. \tag{3.12}$$

Consider the function

$$F(\beta, \varepsilon) = \beta + \int (1 - \varepsilon T_{\beta, \xi \to x})^{-1} \left[\varepsilon \frac{1}{2} V(\xi) \right](x) dx. \tag{3.13}$$

Substituting the Neumann series instead of $(1 - \varepsilon T_\beta)^{-1}$ in equation (3.13), we obtain

$$F(\beta, \varepsilon) = \beta + \varepsilon \frac{1}{2} \int \sum_{l=0}^{\infty} \varepsilon^l [T^l_{\beta, \xi \to x} V(\xi)](x) dx. \tag{3.14}$$

Remark 3.5. Let us observe that $[T^l_{\beta, \xi \to x} V(\xi)](x)$ is analytic in β. Then the function $F(\beta, \varepsilon)$ is analytic in each argument, and by Hartogs' theorem, it is analytic in $\mathbb{C}^2$. Furthermore, $F(0, 0) = 0$, $[\partial_\beta F](0, 0) = 1$. By the implicit function theorem, the solution $\beta(\varepsilon)$ for β of the secular equation (3.12), which tends to zero as $\varepsilon \to 0$, exists, is unique and is given by equation (2.2). We have $[\partial_\varepsilon F](0, 0) = \frac{1}{2} \int V(x) dx$, $[\partial^2_{\beta, \varepsilon} F](0, 0) = 0$, $[\partial^2_\beta F](0, 0) = 0$. Thus, we have

$$F(\beta, \varepsilon) = \beta + \varepsilon \frac{1}{2} \int V(x) dx + O(\varepsilon^2). \tag{3.15}$$

Thus, we have by the implicit function theorem that

$$\beta(\varepsilon) = -\varepsilon \frac{1}{2} \int V(x) dx + O(\varepsilon^2),$$

we obtain (2.2). Furthermore, from equation (3.11), we have

$$A(x) = \sum_{l=0}^{\infty} \varepsilon^l T^l_{\beta(\varepsilon), \xi \to x} \left[-\varepsilon \frac{1}{2\beta(\varepsilon)} A_0 V(\xi) \right](x). \tag{3.16}$$

Thus, substituting (3.16) in (3.2), we obtain $\varphi(x)$. Theorem 2.1 is proven.

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