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QUALITATIVE STUDY OF AN ADAPTIVE THREE SPECIES PREDATOR-PREY MODEL

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Abstract

In this paper, we propose an adaptive mathematical model for the three species: nutrient, algae, and small fish predator-prey system. We formulate the model for the system, investigate sufficient conditions for the existence of positive periodic solutions, and checking the validity of the model. Then we study the stability, and asymptotical behaviours of the solutions with respect to the parameters.

1. Introduction

No species of any animal can live in a complete isolation. Since all species must eat to stay alive, all must interact, if not with other animals, then with plants. The model, which we are going to construct and study, considers the situation where one population serves as food for another, and grows according to some reasonable set of biological laws.

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Recall from the single species population model that the exact description of the growth process is represented by $\frac{dN(t)}{dt} = R(N, t)N(t)$, where N is the population size at any time t, and R is the intrinsic rate of population growth (IRPG). The classical model for a predator-prey system of two interacting species is given by the Lotka-Volterra equations [2, 6]:

$$\dot{N}_1 = aN_1 - bN_1N_2,$$

$$\dot{N}_2 = -cN_2 + dN_1N_2.$$

In this paper, we will generalize this to the case of three interacting species. Many mathematical models have been established to describe the relationships between the species and the outer environment or among different species in biomathematics [2, 6, 7, 9-13]. Recently, close attention has been paid to the dynamics of a large number of mathematical models with diffusion, and many nice results have been obtained.

Chen and Wang [3] studied the qualitative properties of a diffusive predator-prey model subject to the homogeneous Neumann boundary condition. They proved under some hypotheses that the positive steady state is globally asymptotically stable.

Aranda et al. [1] studied a predator-prey model with Holling-Tanner functional response, the stability of the positive constant solution, and provided sufficient conditions for the global stability of the positive equilibrium by constructing a suitable Lyapunov function. Then they proved a number of existence and non-existence results about the non-constant steady states of the system.

Peng et al. [12] investigated a diffusion predator-prey model in a spatially degenerate heterogeneous environment. They obtained some results for the existence and non-existence of positive solutions.

Li et al. [8] established a predator-prey system with group defense and impulsive control strategy.

Ding and Jiang [4] established verifiable criteria for the existence of multiple periodic solutions in delayed Gause-type ratio-dependent predator-prey systems with numerical responses.

Ding and Jiang [5] used a new method to drive sufficient conditions for the existence of positive periodic solutions of general two-species semi-ratio-dependent predator-prey system in a two-patch environment.

Xiong and Zhang [16] investigated a discrete periodic stage-structured competitive model. They obtained sufficient and realistic conditions for the existence of a positive periodic solution of the system.

Xu et al. [17] considered a delayed three-species periodic Lotka-Volterra food chain model. They drove sufficient conditions for the existence, uniqueness, and global stability of positive periodic solutions, using Gains and Mawhin's continuation theorem and by constructing Lyapunov functional.

Shao [14] studied and analyzed the effects of prey-impulsive diffusion in two patches and delays on dynamics of a predator-prey system.

In this paper, we formulate the proposed model in Section 2, in Section 3, we state the assumptions for the existence of the solution, in Section 4, we study the stability and the asymptotical behaviours of the solutions with respect to the parameters. We conclude the paper in Section 5.

2. Formulating the Proposed Model

In this section, we formulate an adaptive mathematical model for the predator-prey interacting three species system which generalized the Lotka-Volterra of two interacting species system.

Consider the following three species:

N_1: Nutrient, which eaten by algae, measured by mg/litre, depend on sunlight and temperature.

N_2: Algae (phytoplankton), measured by cells/litre.

N_3: Small fish (zooplankton) measured by number/litre.

Therefore, a mathematical model for the growth of the three populations in a predator-prey situation characterized by the basic assumption [11]:

Total growth rate = Growth rate in isolation + Modification due to interaction.

To build the model, we assume that the IRPG for each species is independent of explicit time dependence, and varies only with population size. We will incorporate the following simple assumptions:

(1) In the absence of algae and small fish, only one species present (N_1) the nutrient population tends to grow without restrictions.

(2) The effect of the presence of algae on the nutrient population is to reduce the rate of growth in proportion to the algae population.

(3) In the absence of small fish and nutrient, the algae population tends to die off due to starvation.

(4) The effect of presence of algae on the small fish population is to increase the rate of growth in proportion to the algae population present.

The above assumptions suggest to consider the basic form for the covering equations:

$$\dot{N}_1 = R - QN_1 - a_1 \frac{N_1 N_2}{N_1 + b_1},$$

$$\dot{N}_2 = -QN_2 + a_2 \frac{N_1 N_2}{N_1 + b_1} - a_3 \frac{N_2 N_3}{N_2 + b_2},$$

$$\dot{N}_3 = -QN_3 + a_4 \frac{N_2 N_3}{N_2 + b_2}, \tag{1}$$

where R is the rate of nutrient growth, Q is the rate of the flow $(10^3$ litre/unit time), b_i, $i = 1, 2$ are positive constants, and a_i, $i = 1, 2, 3$ are positive parameters measure the competitive of one species over another. As a_1

increases, for example, this indicates that the other species become more effective in outwitting, subduing, or otherwise inhibiting its competitor.

3. The Solution of the Proposed Model

In this section, we investigate sufficient conditions for the existence of positive periodic solution for the proposed model (1).

Let

$$W = \alpha N_1 + N_2 + \beta N_3, \tag{2}$$

where $\alpha = \frac{a_2}{a_1}$ and $\beta = \frac{a_3}{a_4}$. Differentiating both the sides of (2) and substituting (1), we obtain:

$$\dot{W} = \alpha R - QW.$$

As $t \to \infty$, $W(t) \to \frac{\alpha R}{Q}$, therefore, the limit W lies in the set Ω defined by the intersection of the plane

$$\alpha N_1 + N_2 + \beta N_3 = \frac{\alpha R}{Q}, \tag{3}$$

with N_i's ≥ 0. This represents the steady state equation of the total mass. So, it will be sufficient to study the limiting surface $W = \frac{\alpha R}{Q}$. In this way, the three-dimensional problem can be written in terms of the two-dimensional $(N_1 N_2\text{-coordinates})$. That is,

$$\dot{N}_1 = R - QN_1 - a_1 \frac{N_1 N_2}{N_1 + b_1},$$

$$\dot{N}_2 = -QN_2 + a_2 \frac{N_1 N_2}{N_1 + b_1} - a_3 \frac{N_2}{N_2 + b_2}\left(\frac{\alpha R}{Q} - N_2 - \alpha N_1\right). \tag{4}$$

The set Ω is positively invariant. To see this, it suffices to check what happens at the boundaries of Ω. If $N_1 = 0$, then $\dot{N}_1 > 0$ so the flow is inward. If $N_2 = 0$, then $\dot{N}_2 = 0$, and if $N_3 = 0$ implies $\dot{N}_3 = 0$, so this flow along these boundaries is never outward.

To begin with, a glance at system (1) shows that $N_1 = 0, N_2 = 0$, and $N_3 = 0$ not a possible equilibrium. Neither $N_2 = 0$ and $N_3 \neq 0$. System (1) has several equilibriums. An easy computation shows that there are three possibilities for equilibriums in the positive quadrant. We are only interested in equations which lie in Ω and in long term behaviour. We know this ultimately, all projectors tend to Ω. The other three possibilities are:

Case 1. $\overline{N}_1 = \dfrac{R}{Q}$ and $\overline{N}_2 = \overline{N}_3 = 0$ one species present.

Case 2. $\overline{N}_1, \overline{N}_2 \neq 0, \overline{N}_3 = 0$ two species present.

Case 3. $\overline{N}_i \neq 0, i = 1, 2, 3$ all species present.

4. Stability Analysis

In this section, we study the stability and the asymptotical behaviours of the solutions with respect to the parameters case by case:

Case 1. Let $\overline{N}_1 = \dfrac{R}{Q} \geq 0$, $\overline{N}_2 = \overline{N}_3 = 0$, and $\alpha\overline{N}_1 = \dfrac{\alpha R}{Q}$ so this equilibrium lies in Ω for all values of Q. To study the stability of this point, let Q_1 be the value of Q for which $Q_1 = \dfrac{a_2\overline{N}_1}{b_1 + \overline{N}_1}$, where $\overline{N}_1 = \dfrac{R}{Q_1}$. Suppose first that we consider perturbations in Ω with $N_3 = 0, N_2 > 0$. Since (3) must be satisfied, it follows that N_1 is perturbed to a value $N_1 < R/Q$, as long as $\overline{N}_2$ is perturbed from zero.

Now, we assume that $Q > Q_1$. Then $\dfrac{a_2 N_1}{b_1 + N_1} < \dfrac{a_2 R/Q}{b_1 + R/Q} < \dfrac{a_2 R/Q_1}{b_1 + R/Q_1}$, which implies that

$$\dot{N}_2 = N_2\left(-Q + \frac{a_2 N_1}{b_1 + N_1}\right) < N_2(-Q + Q_1) < 0,$$

and so N_2 decreases to zero while N_1 increases to R/Q, by virtue of (3).

If $Q < Q_1$, then as long as $\frac{R}{Q_1} \leq N_1 < \frac{R}{Q}$ we obtain

$$\dot{N}_2 > N_2\left(-Q_1 + \frac{a_2 N_1}{b_1 + N_1}\right) > N_2\left(-Q_1 + \frac{a_2 R/Q_1}{b_1 + R/Q_1}\right) = 0,$$

and so we need the opposite conclusions, that N_2 increases away from zero and N_1 decreases to the value R/Q_1.

When perturbations; $N_2 = 0$ and $N_3 > 0$, then for all values of Q, one has $\dot{N}_3 = -QN_3 < 0$, so that N_3 decreases while N_1 increases. These various possibilities are displayed below:

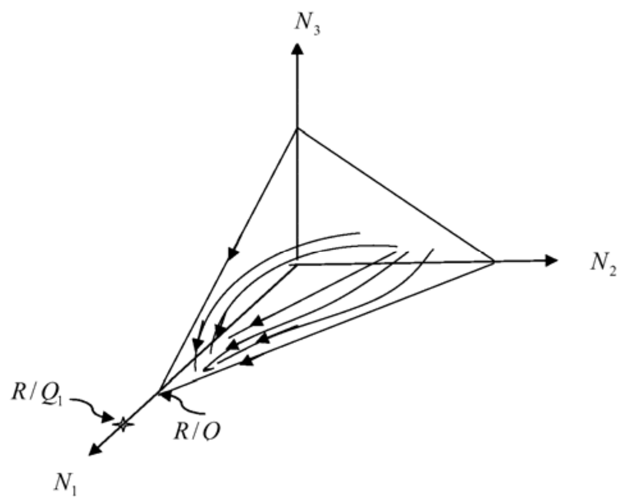

Figure 1. A stable node at $(R/Q, 0, 0)$ when $Q > Q_1$.

When all perturbations are positive and $Q > Q_1$, then the argument above together with a glance at system (1) shows that $\dot{N}_2 < 0$. When N_2 is small enough, since $N_2 \to 0$, system (1) shows that $\dot{N}_3 < 0$ also. Thus, N_3

decreases substantially, and N_1 increases to R/Q. This is shown below and proves that $(R/Q, 0, 0)$ is a stable node when $Q > Q_1$.

A more direct verification of stability for situation $Q > Q_1$, is obtained by considering the stability matrix of the 2 by 2 system (4):

$$M = \begin{pmatrix} -Q - a_1 \dfrac{b_1 N_2}{(N_1 + b_1)^2}, & -a_1 \dfrac{N_1}{N_1 + b_1} \\ a_2 \dfrac{b_1 N_2}{(N_1 + b_1)^2} + a_3 \dfrac{\alpha N_2}{N_2 + b_2}, & -Q + a_2 \dfrac{N_1}{N_1 + b_1} + \dfrac{a_3 N_2}{N_2 + b_2} - \dfrac{a_3 b_2}{(N_2 + b_2)^2}\left(\dfrac{\alpha R}{Q} - N_2 - \alpha N_1\right) \end{pmatrix}.$$

When evaluated at $(R/Q, 0, 0)$, we obtain

$$\overline{M} = \begin{pmatrix} -Q, & -a_1 \dfrac{R/Q}{R/Q + b_1} \\ 0, & -Q + a_2 \dfrac{R/Q}{R/Q + b_1} \end{pmatrix},$$

since $-Q + a_2 \dfrac{R/Q}{R/Q + b_1} < 0$, $\text{trace}(\overline{M}) < 0$.

Also, clearly, $\text{Det}(\overline{M}) > 0$. Then $(R/Q, 0, 0)$ is a stable node.

Case 2. Let $\overline{N}_1, \overline{N}_2 \neq 0, \overline{N}_3 = 0$. Since $\overline{N}_2 \neq 0$,

$$\frac{a_2 \overline{N}_1}{b_1 + \overline{N}_1} = Q \tag{5}$$

is equilibrium for N_1. If $Q \geq Q_1$, then $\dfrac{a_2 N_1}{b_1 + N_1}$ increases with N_1, which follows from (5) that $\overline{N}_1 \geq R/Q_1$. From (3), we obtain

$$\alpha \frac{R}{Q} = \alpha \overline{N}_1 + \overline{N}_2 \geq \alpha \frac{R}{Q_1} + \overline{N}_2 > \alpha \frac{R}{Q_1}.$$

This contradicts that $Q \geq Q_1$ unless $\overline{N}_2 \leq 0$. Thus, equilibrium cannot lie in Ω unless $Q < Q_1$. It is clear from what we just shown that $\overline{N}_1 < R/Q_1$.

In the discussion of Case 1 for $Q < Q_1$, we saw that N_3 moves away from the point $(R/Q, 0, 0)$, whereas on the plane $W = 0$. Suppose N_1 is a perturbation about $\overline{N}_1$ which satisfies $\overline{N}_1 < N_1$, then

$$\dot{N}_2 = N_2\left(-Q + \frac{a_2 N_1}{b_1 + N_1}\right) > N_2\left(-Q + \frac{a_2 \overline{N}_1}{b_1 + \overline{N}_1}\right) = 0,$$

by virtue of (5). Similarly, when $N_1 < \overline{N}_1$, then $\dot{N}_1 < 0$. This is shown in Figure 2:

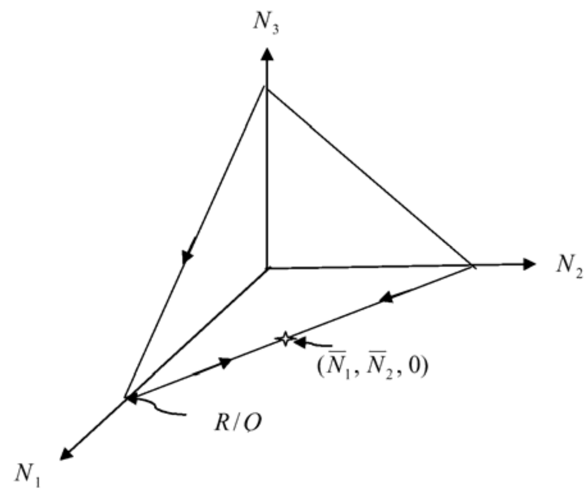

Figure 2. A stable node.

We now permit perturbation $N_3 > 0$. However, in order to proceed further, we first pause to introduce the equilibrium which applies to Case 3.

Case 3. $N_i \neq 0$, $i = 1, 2, 3$. All species present.

From the first and the third equations of (1), we get

$$Q = \frac{a_4 \overline{N}_2}{\overline{N}_2 + b_2} \quad \text{and} \quad \frac{\overline{N}_1 \overline{N}_2}{\overline{N}_1 + b_1} = \frac{R - Q\overline{N}_1}{a_1}, \tag{6a}$$

which can be written as

$$\alpha \overline{N}_1 + \frac{a_4 \overline{N}_1 \overline{N}_2}{Q(\overline{N}_1 + b_1)} = \frac{\alpha R}{Q}. \tag{6b}$$

Using (6b) in equation (3), we obtain

$$\overline{N}_2\left(\frac{a_4\overline{N}_1}{Q(\overline{N}_1 + b_1)} - 1\right) = \beta\overline{N}_3. \tag{6c}$$

It is apparent that for a non-trivial equilibrium in Ω, we must have

$$\frac{a_4\overline{N}_1}{\overline{N}_1 + b_1} > Q. \tag{6d}$$

Increase the value of Q, then $\overline{N}_2$ also increases while $\overline{N}_1$ remains strictly positive (see (6d)). However, $\alpha R/Q$ decreases and so (3) tells us that Q increases. There is an initial value Q_0 for which $\overline{N}_3$ is zero. Now, let $\widehat{N}_1$ and $\widehat{N}_2$ be the limiting non-zero values of the equilibrium as $Q \to Q_0$. Then $\widehat{N}_1$ and $\widehat{N}_2$ must satisfy, by virtue of (6a) and (6c),

$$Q_0 = \frac{a_4\widehat{N}_2}{\widehat{N}_2 + b_2} = \frac{a_2\widehat{N}_1}{\widehat{N}_1 + b_1} \quad \text{and} \quad \alpha\widehat{N}_1 + \widehat{N}_2 = \alpha R/Q_0. \tag{7}$$

So, there are two conclusions: first, in order to have a non-trivial equilibrium in Case 3, we must have $Q < Q_0$, and because of (7), $(\widehat{N}_1, \widehat{N}_2, 0)$ represent an equilibrium in Ω for Case 2. It follows that necessarily $Q_0 < Q_1$.

Now, we return to the situation in which there is perturbation $N_3 > 0$ from this equilibrium of Case 2. Suppose $Q > Q_0$. Then

$$\dot{N}_3 = N_3\left(-Q + a_4\frac{N_2}{N_2 + b_2}\right) < N_3\left(-Q_0 + a_4\frac{N_2}{N_2 + b_2}\right).$$

But $\left(a_2\frac{\overline{N}_1}{\overline{N}_1 + b_1}\right) = Q > Q_0 = a_2\frac{\hat{N}_1}{\hat{N}_1 + b_1}$, which implies $\hat{N}_1 < \overline{N}_1$. Also, since $\overline{N}_1 > \hat{N}_1$, equation (3) implies

$$\alpha\hat{N}_1 + \hat{N}_2 = \frac{\alpha R}{Q_0} > \frac{\alpha R}{Q} = \alpha\overline{N}_1 + \overline{N}_2 > \alpha\hat{N}_1 + \overline{N}_2,$$

which implies $\hat{N}_2 > \overline{N}_2$.

Now, if $\frac{a_4 N_2}{N_2 + b_2} > Q_0$, then this would imply that $\hat{N}_2 < N_2$ for all perturbations N_2 in Ω. In particular, then $\overline{N}_2 > \hat{N}_2$ which however, contradicts what we have just shown above. Therefore, $a_4 \frac{N_2}{N_2 + b_2} \leq Q_0$ for all perturbations, and so $\dot{N}_3 < N_3\left(-Q_0 + a_4 \frac{N_2}{N_2 + b_2}\right) < 0$.

Therefore, N_3 decreases to zero. Since the only other equilibrium in Ω is the saddle of Case 1 for $Q_0 < Q < Q_1$, it follows that the perturbed trajectory tends to the given equilibrium $(\overline{N}_1, \overline{N}_2, 0)$.

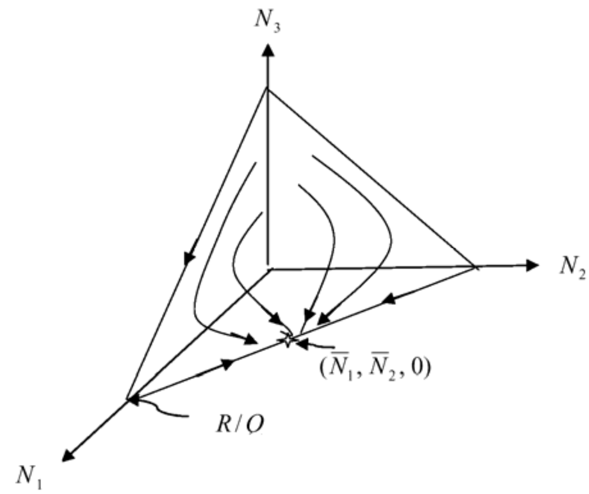

Figure 3. A stable node of Case 2, $(\overline{N}_1, \overline{N}_2, 0)$, when $Q_0 < Q$.

When $Q_0 > Q$, then the perturbation $N_3 > 0$ satisfies $\dot{N}_3 > N_3\left(-Q_0 + a_4 \frac{N_2}{N_2 + b_2}\right)$ and a similar argument to the one above shows that $\dot{N}_3 > 0$. Thus, N_3 increases away from zero and $(\overline{N}_1, \overline{N}_2, 0)$ is a saddle point in this case.

For more direct verification, we can use the stability matrix (M) of the 2 by 2 system (4). The stability matrix can be written as follows using the fact that $\beta = 1$ and (6a) at equilibrium:

$$\hat{M} = \begin{pmatrix} -Q - a_1 \dfrac{b_1\overline{N}_2}{(\overline{N}_1 + b_1)^2}, & -a_1 \dfrac{\overline{N}_1}{\overline{N}_1 + b_1} \\ a_2 \dfrac{b_1\overline{N}_2}{(\overline{N}_1 + b_1)^2} + a_3 \dfrac{\alpha\overline{N}_2}{\overline{N}_2 + b_2}, & -Q + a_2 \dfrac{\overline{N}_1}{\overline{N}_1 + b_1} + \dfrac{a_3\overline{N}_2}{\overline{N}_2 + b_2} - \dfrac{a_3 b_2}{(\overline{N}_2 + b_2)^2}\left(\dfrac{\alpha R}{Q} - \overline{N}_2 - \alpha\overline{N}_1\right) \end{pmatrix}.$$

For simplicity, let $\xi_1 = \dfrac{\overline{N}_1}{\overline{N}_1 + b_1}$, $\xi_2 = \dfrac{\overline{N}_2}{\overline{N}_2 + b_2}$ and $\xi_3 = \dfrac{\overline{N}_2}{\overline{N}_1 + b_1}$. Clearly, ξ_1, ξ_2 and ξ_3 are positive for non-trivial equilibrium in Ω. Then matrix $\hat{M}$ becomes

$$\hat{M} = \begin{pmatrix} -Q - a_1 b_1 \xi_3\left(-\dfrac{1}{\overline{N}_1 + b_1}\right), & -a_1\xi_1 \\ a_2 b_1 \xi_3\left(\dfrac{1}{\overline{N}_1 + b_1}\right) + a_3\alpha\xi_1, & -Q + a_2\xi_1 + a_3\xi_2 - \dfrac{a_3 b_2}{(\overline{N}_2 + b_2)^2}\left(\dfrac{\alpha R}{Q} - \overline{N}_2 - \alpha\overline{N}_1\right) \end{pmatrix},$$

$$\begin{aligned} \operatorname{Det}(\hat{M}) = {} & \left[-Q - a_1 b_1 \xi_3\left(\frac{1}{\overline{N}_1 + b_1}\right)\right] \\ & \cdot\left[-Q + a_2\xi_1 + a_3\xi_2 - \frac{a_3 b_2}{(\overline{N}_2 + b_2)^2}\left(\frac{\alpha R}{Q} - \overline{N}_2 - \alpha\overline{N}_1\right)\right] \\ & + a_1\xi_1\left[a_2 b_1 \xi_3\left(\frac{1}{\overline{N}_1 + b_1}\right) + a_3\alpha\xi_1\right]. \end{aligned}$$

Assume $a_3 = a_4$, then $a_3\xi_2 = Q$ by (6a). Also, $a_2\xi_3 = a_1\alpha\xi_3$, since $a_2 = \alpha a_1$. By a clear glance at the above equation, we can see that the $\operatorname{Det}(\hat{M}) > 0$. To determine the sign of the $\operatorname{trace}(\hat{M})$, we permit R to vary. From (6b), we have

$$\frac{R}{\overline{N}_1} = Q + \frac{a_1\overline{N}_2}{\overline{N}_1 + b_1}. \tag{8}$$

Since (6a) has a unique solution $\overline{N}_2$ for fixed Q, $\overline{N}_2$ does not vary, but (8) shows that R is monotonically increasing with $\overline{N}_1$, and hence $\overline{N}_1$ is strictly monotone in Ω. As $R \to \infty$, it follows that $\frac{R}{\overline{N}_1} \to Q$. Thus, for R large, $\overline{N}_1 \to R/Q$. Therefore, by virtue of (3), both $\overline{N}_2$ and $\overline{N}_3$ must be small by comparison with $\overline{N}_1$ for R large. Using (8), the upper left term in the $\widehat{M}$ matrix may be written as: $-\frac{R}{\overline{N}_1} + \frac{a_1\overline{N}_1\overline{N}_2}{(\overline{N}_1 + b_1)^2}$, which tends to $-Q$ as R increases. Therefore, the trace$(\widehat{M})$ is approximately $-Q + \frac{a_2\overline{N}_1}{\overline{N}_1 + b_1}$ for large R, and this is positive quantity, since $\overline{N}_1$ is large (see (6d)). Also, as R decreases, equation (3) shows that since $\overline{N}_2$ is fixed, $\overline{N}_1$ and $\overline{N}_2$ tend to zero as R approaches the value $\overline{N}_2 Q/\alpha$. In this case, the trace$(\widehat{M})$ evidently becomes negative, since $-R/N_1$ dominates the other terms. Therefore, there is a value R_0 at which Trace$(\widehat{M}) = 0$. Also, there is a range of values $\Psi : R_0 - \delta < R < R_0 + \delta$ for which trace$^2(\widehat{M}) - 4\text{Det}(\widehat{M}) < 0$ and so the eigenvalues of $\widehat{M}$ are complex at the equilibrium of Case 3 for Ψ.

When $R < R_0$, the phytoplankton and zooplankton co-exist at a stable equilibrium as shown in Figure 4:

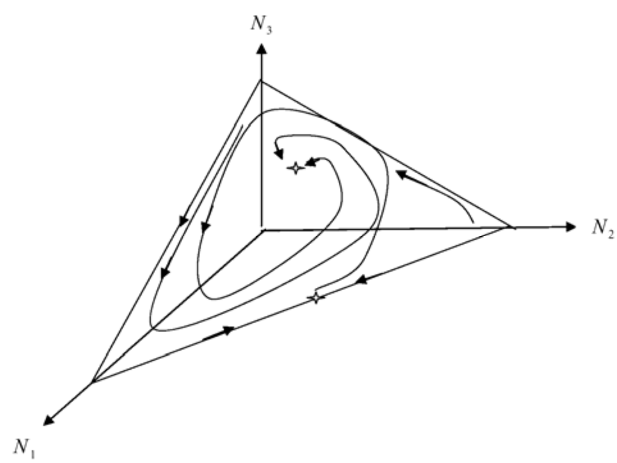

Figure 4. A stable equilibrium of Case 3 when $R < R_0$ and $Q < Q_0$.

When $R > R_0$, then this equilibrium is unstable. Since Ω is invariant under the flow, the Poincaré-Bendixon theorem gives a stable first cycle. We can also use the following fact: $\text{trace}(\widehat{M}) \neq 0$ (a calculation based on the fact that $\dfrac{\overline{N}_2\overline{N}_1}{\overline{N}_1 + b_1}$ and $\overline{N}_1$ are strictly increasing with R). The point R_0 gives rise to a bifurcation and for $R > R_0$ we obtain first cycles whose amplitude increases with R.

From the figures and the discussions above, we conclude that as nutrient level R increases, the stable equilibrium is disputed as a cycle. For R large, the amplitude of the cycle hugs the boundary of Ω which effectively means that one of the species will be driven to extinction. This is the "Paradox of Enrichment". When Q is large $(Q > Q_1)$ then, as we saw, both species also eliminated which is effectively a washing away of predator and prey by strong tidal flow.

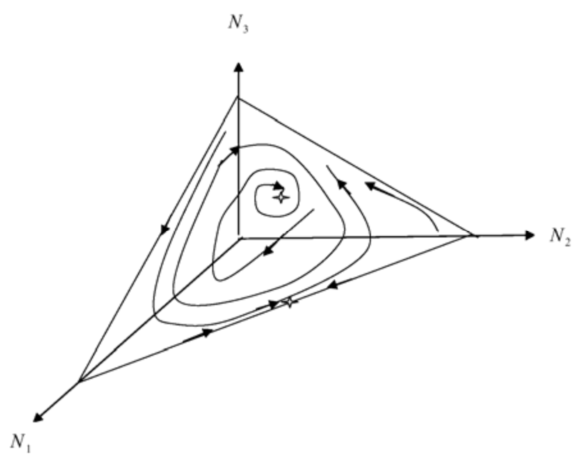

Figure 5. A stable equilibrium of Case 3 when $R > R_0$ and $Q < Q_0$.

We now wish to show that increasing R has the effect of increasing Q_0 so that $Q < Q_0$ becomes assured (and, conversely, decreasing R drives Q_0 to values below Q so this is an equilibrium of Case 3 is not possible). Therefore, a large inflow of nutrients can compensate for a high tidal outflow.

To prove this, note that $\alpha\hat{N}_1 + \hat{N}_2 = \alpha R/Q_0$ with the values of $\hat{N}_1$ and $\hat{N}_2$ is fixed by Q_0. If R increases to $\hat{R}$, then Q_0 changes to $\hat{Q}_0$. If $\hat{Q}_0 \leq Q_0$, then (7) shows that the corresponding $\hat{\hat{N}}_1$ and $\hat{\hat{N}}_2$ are also less than before. Therefore,

$$\frac{\alpha R}{Q_0} = \alpha\hat{N}_1 + \hat{N}_2 \geq \alpha\hat{\hat{N}}_1 + \hat{\hat{N}}_2 = (\alpha\hat{R}/\hat{Q}_0) \geq (\alpha\hat{R}/Q_0),$$

which implies that $R > \hat{R}$, a contradiction. Hence, $\hat{Q}_0 > Q_0$ which proves that Q_0 increases with R.

5. Conclusion

The model we built is not precise, at best, it is a part of qualitatively accurate model. We drive sufficient conditions for the existence of positive periodic solutions and check the validity of the model. We study the stability and the asymptotic behaviour of the solutions. The situations in which the predictions of our model fit well with observations and show the outbreak and collapse behaviour of the species and the competition play prominent roles.

Finally, the discussion shows that the rate of the flow Q determines the long term behaviour of the system depending on $0 < Q < Q_0$, $Q_0 < Q < Q_1$, and $Q_1 < Q < \infty$. We also saw that the rate of nutrient growth R affects the values of Q_0 and Q_1.

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