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SYMMETRIC SOLUTIONS OF A NONLINEAR ELLIPTIC PROBLEM

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Abstract

In this paper, we study a nonlinear elliptic problem with Neumann boundary conditions on [–1, 1].

1. Introduction

The maximum principle is one of the most used tools in the study of some differential equations of elliptic type. It is a generalization of the following well known theorem of the elemental calculus "If f is a function of

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class C^2 in $[a, b]$ such that the second derivative is positive on (a, b), then the maximum value of f attains at the ends of $[a, b]$". It is important to point out that the maximum principle gives information about the global behavior of a function over a domain from the information of qualitative character in the boundary and without explicit knowledge of the same function. The maximum principle allows us, for example, to obtain uniqueness of solution of certain problems with conditions of the Dirichlet and Neumann type. Also it allows to obtain a priori estimates for solutions. These reasons make interesting the study of the maximum principle on several forms and its generalizations and the Hopf lemma. For example, a geometric version of the maximum principle allows us to compare locally surfaces that coincide at a point. On the other hand, the maximum principle and the Aleksandrov reflection principle have been used to prove symmetries with respect to some point, some plane, symmetries of domain and to determine asymptotic-symmetric behavior of the solutions of some elliptic problems. See Serrin [9], Gidas et al. [4, 5], Caffarelli et al. [3], and Berestycki and Nirenberg [2]. The first person used this technique was Serrin [9]. Serrin proved that: "If u is a positive solution of the problem

$$\Delta u = -1 \text{ on } \Omega$$

which is zero on the boundary and its outer normal derivative on the boundary is constant, then Ω is a ball and u is radially symmetric with respect to the center of Ω". Using the ideas of Serrin and a version of the maximum principle for functions that do not change of sign; Gidas et al. in [4] proved that: "If Ω is a ball, $f \in C^1(\mathbb{R})$ and u is a positive solution of the problem,

$$\Delta u + f(u) = 0 \text{ on } \Omega$$

which is zero on the boundary, then u is radially symmetric with respect to the center of the ball". Using the method of reflection and a version of maximum principle for thin domains Berestycki and Nirenberg in [2] made a generalization of the paper [4].

Our proof shows that the technique used in [2], [4], [9] for the study of symmetries of solutions of the elliptic problem with Dirichlet condition, can be applied in elliptic problems with Neumann conditions.

2. Main Result

Theorem 2.1. *Let* $u \in C^2((-1, 1)) \cup C^0([-1, 1])$ *be a solution of*

$$\begin{cases} a(x)u''(x) + b(x)u'(x) + \alpha(x)u(x) = f(u(x)), & \text{on } (-1, 1), \\ u'(1) = -u'(-1), \end{cases}$$

where $a, \ \alpha : [-1, 1] \to \mathbb{R}$ *are bounded functions and symmetric with respect to the origin such that* $a(x) > 0$ *and* $\alpha(x) \leq 0$ *for all* $x \in [-1, 1]$, $f \in C'(\mathbb{R})$ *is strictly increasing and* $b : [-1, 1] \to \mathbb{R}$ *is a bounded function and odd. Then u is symmetric with respect to the origin.*

Proof. Define the reflected function of u in $[-1, 1]$ by

$$v(x) = u(-x), \quad x \in [-1, 1].$$

Hence, $v'(x) = -u'(-x), v''(x) = u''(-x)$. Then v satisfies

$$\begin{cases} a(x)v''(x) + b(x)v'(x) + \alpha(x)v(x) = f(v(x)), & \text{on } (-1, 1), \\ v'(1) = -v'(-1). \end{cases}$$

Define

$$w(x) = u(x) - v(x).$$

Then w satisfies

$$\begin{cases} a(x)w''(x) + b(x)w'(x) + \alpha(x)w(x) = f(u(x)) - f(v(x)), & \text{on } (-1, 1), \\ w'(-1) = w'(1) = 0. \end{cases}$$

Since w is continuous in $\overline{\Omega}$, there are $x_M, x_m \in \overline{\Omega}$ such that

$$w(x_m) = \min_{\overline{\Omega}} w \quad \text{and} \quad w(x_M) = \max_{\overline{\Omega}} w.$$

Suppose that x_M or $x_m \in (-1, 1)$, then if $x_M \in (-1, 1)$, $w(x_M) \geq 0$ since $w(0) = 0$. Further $w''(x_M) \leq 0$, $w'(x_M) = 0$. Therefore

$$f(u(x_M)) - f(v(x_M)) \leq 0.$$

Since *f* is strictly increasing,

$$w(x_M) \leq 0.$$

Then

$$w(x_M) = 0.$$

Therefore

$$w(x) \leq 0 \text{ for all } x \in [-1, 1].$$

By definition of *w*, we conclude that

$$w \equiv 0 \text{ on } [-1, 1].$$

So *u* is symmetric with respect to the origin. If $x_m \in (-1, 1)$, then using a similar argument we demonstrate that $w \equiv 0$ on $[-1, 1]$ and we obtain the same conclusion. We will prove that x_m, x_M do not belong to $\partial[-1, 1]$. Suppose now that $x_m, x_M \in \{-1, 1\}$ and $w(x_m) < w(x) < w(x_M)$ for all $x \in (-1, 1)$, then $w(x_M) > 0$ and $w(x_m) < 0$. If $x_m = -1$ and $x_M = 1$, then

$$w(x) \leq 0 \text{ in } (-1, \alpha) \text{ and } w(x) \geq 0 \text{ in } (\beta, 1),$$

where $\alpha, \beta \in (-1, 1)$ are such that α is the first zero of *w* and β is the last. Since *f* is strictly increasing,

$$\begin{cases} a(x)w'' + b(x)w' + \alpha(x)w \leq 0, & \text{in } (-1, \alpha), \\ w'(-1) = 0, \end{cases}$$

and

$$\begin{cases} a(x)w'' + b(x)w' + \alpha(x)w \geq 0, & \text{in } (\beta, 1), \\ w'(1) = 0. \end{cases}$$

Applying the maximum principle and the Hopf lemma,

$$w'(-1) > 0, \quad w'(1) > 0,$$

since w is not constant which contradicts the fact that

$$w'(-1) = w'(1) = 0.$$

Hence, this case is impossible. It happens equally to $x_m = 1$ and $x_M = -1$. In conclusion, we have that $w \equiv 0$ on $[-1, 1]$ and therefore u is symmetric with respect to $x = 0$.

References

[1] A. D. Aleksandrov, Uniqueness theorems for surfaces in the large, V. Vestnik Leningrad. Uni. 13 (1958), 5-8.

[2] H. Berestycki and L. Nirenberg, On the method of moving planes and the sliding method, Bol. Soc. Brasil. Mat. 22(1) (1991), 1-37.

[3] L. A. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. 42 (1989), 271-297.

[4] B. Gidas, Wei-Ming Ni and L. Nirenberg, Symmetry and related properties via maximum principle, Comm. Math. Phys. 68 (1979), 209-243.

[5] B. Gidas, Wei-Ming Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, Mathematical Analysis and Applications, Part A, in: Adv. Math. Suppl. Stud. Vol. 7A, Academic Press, New York, 1981, pp. 369-402.

[6] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[7] F. John, Partial Differential Equations, Springer-Verlag, New York, 1982.

[8] M. H. Protter and H. F. Weinberger, Maximum Principle in Differential Equations, Springer-Verlag, New York, 1984.

[9] J. Serrin, A symmetry problem in potential theory, Arch. Rational Mech. Anal 43 (1971), 304-318.