Far East Journal of Mathematical Sciences (FJMS)
Volume 70, Number 2, 2012, Pages 251-260
Published Online: October 2012
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

MODELLING TEMPERATURES IN SHANGHAI USING FRACTIONAL BROWNIAN MOTION

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Abstract

In this study, daily average temperatures in Shanghai over the last twenty years are modelled with a view towards application to weather derivatives. For this purpose, a mean-reverting Ornstein-Uhlenbeck (OU) process driven by Fractional Brownian Motion (FBM) is used. The estimated Hurst parameter shows that temperature dynamics deviate from the assumptions of Brownian motion and that option prices using FBM are significantly higher compared to the model with an OU process driven by Brownian motion. The motivation for using FBM is the long-range temporal dependence and the normality of temperature fluctuations observed for Shanghai temperatures. Standard call and put options on a temperature index (Heating/Cooling Degree Days [HDDs/CDDs]) for Shanghai are priced using a Monte Carlo simulation of the proposed model with fitted parameters.

I. Introduction

Weather derivatives, which emerged in the US energy industry in 1997

2010 Mathematics Subject Classification: 60G22, 91B72, 91B30, 91B84, 78M31.

Keywords and phrases: fractional Brownian motion, Ornstein-Uhlenbeck process, heating/cooling degree days, weather derivatives, Monte Carlo simulation.

Received May 26, 2012

and have since been adopted worldwide, are a new class of financial instrument that can be used for hedging against weather-related risks. The income-smoothing effect of weather derivatives is a major reason for the development of this new class of financial instrument.

In a weather derivatives contract, the underlying variable is a weather-related index, which is not a tradable asset. This means that pricing models are not based on no-arbitrage principles, as can be done with standard stock options. Therefore, the valuation of a weather derivative is generally carried out following the expected discounted value approach.

Alaton et al. [1] gave the definitions of daily temperatures and Heating/Cooling Degree Days (HDDs/CDDs). Weather derivatives based on HDD and CDD indices, which are the most commonly traded weather indices on the Chicago Mercantile Exchange (CME), are priced in this study.

The dynamic modelling of daily average temperatures is a popular approach in the literature on weather derivatives. Its major advantage is the flexibility to price the most general class of payoff functions based on daily temperatures. Due to the mean-reverting behaviour of temperature dynamics, the Ornstein-Uhlenbeck (OU) process is a natural choice. The first study on modelling and pricing weather derivatives in China is given by Göncü [4]. Benth and Šaltytė-Benth [2] proposed a more general model using a Fourier approximation to model seasonal volatility. Another study for modelling temperatures in China and Turkey is given by Göncü [5, 6] with the Greeks derived for standard call and put options. FBM is used for the first time in modelling and pricing weather derivatives in the study by Brody et al. [3]. To the best of the author's knowledge, this article is the first study to apply FBM to modelling daily temperatures in a Chinese city.

II. Temperature Data

Our data consists of daily average temperatures observed in Shanghai during the period 1990 to 2009. Figure 1 shows a plot of these temperatures for the given twenty years. The same dataset is used in an earlier study by Göncü [4], in which the long-term mean temperature dynamics are given with the following fitted values:

$$T_t^{m,\,Shanghai} = 16.2926 + 3.0 \times 10^{-4} t + 11.9794 \sin\left(\frac{2\Pi}{365} t - 1.9280\right). \quad (1)$$

The seasonal model given in equation (1) captures the deterministic component of the temperature dynamics. After removing the deterministic component, one can focus on the noise or so-called random fluctuations around the long-term temperatures. In Figure 2, a histogram of daily temperature fluctuations around the long-term mean is plotted. This figure shows that the 'noise' component of random fluctuations can be assumed to be normally distributed.

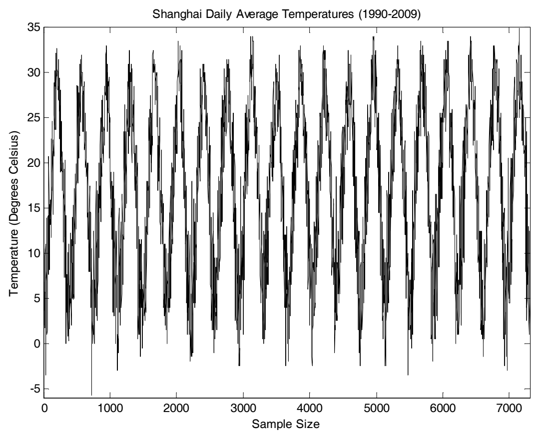

Figure 1. Daily average temperatures in Shanghai from 1990 to 2009.

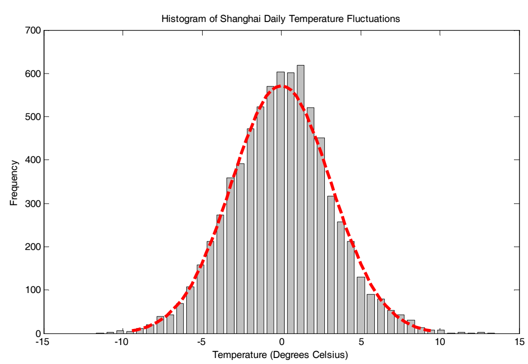

Figure 2. Histogram of daily de-trended and de-seasonalized temperature fluctuations around long-term mean with fitted normal distribution.

III. Fractional Brownian Motion

In probability theory, a normalized FBM is a continuous-time Gaussian process $B_H(t)$ on $[0, T]$ which starts at zero for all t in $[0, T]$ and has the following covariance function

$$E[B_H(t)B_H(s)] = (|t|^{2H} + |s|^{2H} - |t-s|^{2H})/2, \tag{2}$$

where H is a real number in $(0, 1)$, called the *Hurst parameter* and associated with the FBM. It was introduced by Mandelbrot and van Ness [7]. The value of H determines the type of FBM process: (i) If $H = 0.5$, then the process is in fact a Brownian motion or Wiener process; (ii) If $H > 0.5$, then the increments of the process are positively correlated; (iii) If $H < 0.5$, then the increments of the process are negatively correlated.

In Figure 3, three sample paths are simulated with Hurst parameters equal to 0.25, 0.50 and 0.75, respectively. As can be seen, a very low H value, say 0.25, gives an anti-persistent process, whereas a Hurst parameter of 0.75 yields a persistent process. In simple terms, we can consider that positive correlation with previous values of the process makes it more difficult to return back to where it started.

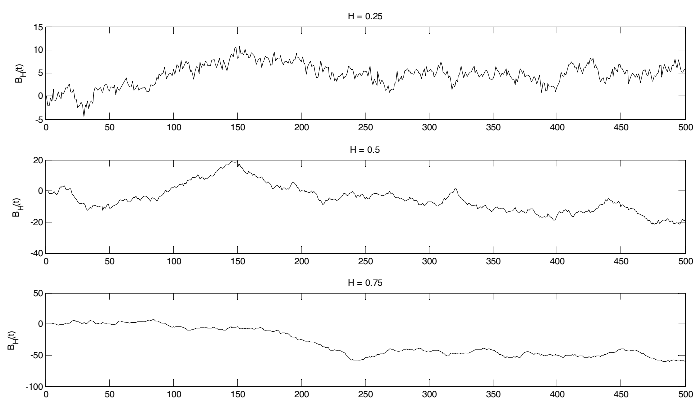

Figure 3. Simulations of Fractional Brownian Motion (FBM) with Hurst parameters 0.25 (highly anti-persistent), 0.5 (standard Brownian Motion [BM]), and 0.75 (highly persistent).

IV. A Mean-reverting Model Driven by Fractional Brownian Motion

As suggested by Brody et al. [3] and Syroka and Toumi [8], temperature exhibits long-range temporal correlations or so-called 'long-memory', which motivates the use of FBM as the driving process for temperature dynamics. Different methods have been developed for identifying the presence of long-memory in temperature dynamics. In the present article, we use a simple and efficient method of identification known as the Surface Temperature (ST) method, introduced by Syroka and Toumi [8]. The underlying principle is to analyse the temperature fluctuations remaining after the deterministic components has been removed, and then to quantify how the variability of these fluctuations depends on time. The is done using the statistic

$$\sigma(T) \equiv \sqrt{\frac{1}{N}\sum_{i=1}^{N} \overline{X}_i^2}, \tag{3}$$

which represents the root-mean-square fluctuation of $\overline{X}_i$ (see Brody et al. [3], for details).

For completely random and uncorrelated fluctuations, we have $\sigma(T) \approx T^{-1/2}$. On the other hand, for a smooth fluctuation with little randomness, variability is constant. Hence, an exponent γ, in other words, (T^{γ}), between 0 and $-1/2$ suggests the existence of temporal correlation between daily temperatures.

After removing the deterministic component given in equation (1), we apply the ST method to our Shanghai temperature data.

In Figures 4 and 5, the best-fit exponents are given as $\gamma = -0.32$ for timescales from 5 to 30 days and $\gamma = -0.45$ for timescales from 30 days to 730 days, respectively. By the definition of FBM, if we have $H = 0.5$, then FBM becomes equivalent to standard Brownian Motion (BM). Therefore, for comparison, we plotted the best-fit line corresponding to the standard

Brownian motion process with $H = 0.5.$ The FBM process with $H = 0.68$ corresponds to a strong temporal dependence compared to the BM process with $H = 0.5.$ As given in the properties of FBM, if $H > 1/2,$ then the increments of FBM are positively correlated, suggesting that there exists temporal dependence in temperatures. Considering the same ST analysis for timescales from 5 to 730 days, we estimated the Hurst parameter to be equal to 0.56, as given in Figure 6.

We observe two facts: (i) temperature fluctuations around the long-term mean temperatures follow approximately a normal distribution, (ii) temperature dynamics show long-range temporal dependence. These two facts motivate the use of FBM rather than BM as the driving process for the mean-reverting OU process for daily temperatures. Hence, we use the following model:

$$dT_t = \left(\frac{dT_t^m}{dt} + a(T_t^m - T_t) \right) dt + \sigma_t dW_t^H, \tag{4}$$

where $a \in \Re$ determines the speed of mean reversion and W_t^H is an FBM. The term $\frac{dT_t^m}{dt} = B + \omega C \cos(\varpi t + \phi)$ is used to ensure that the process reverts to the long-term mean temperatures. The long-term mean temperature dynamics for Shanghai temperatures are given in equation (1).

The model in equation (4) is often used with standard Brownian motion instead of FBM as a basis for Monte Carlo simulations. Some examples can be found in Alaton et al. [1] and Göncü [4, 5]. However, using Brownian motion, which corresponds to $H = 1/2,$ does not take into account the low-frequency variability of weather.

V. Numerical Results and Conclusion

We consider the most common type of weather derivatives for our numerical examples. HDD and CDD contracts are priced using a Monte Carlo simulation. The calibrated parameters and the definitions of HDD and CDD contracts can be found in the study by Göncü [4, 5].

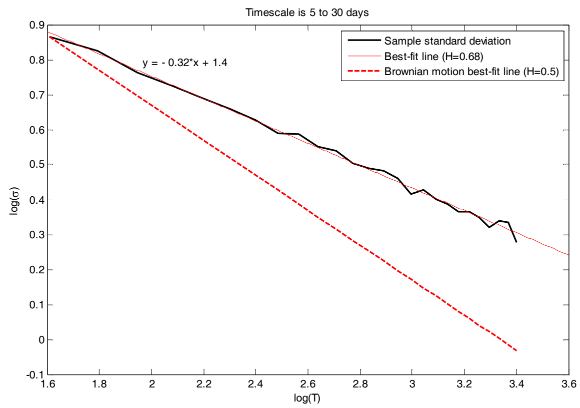

Figure 4. ST analysis of Shanghai temperatures from 1991 to 2010. The best-fit exponent for timescales from 5 to 30 days is −0.32, corresponding to $H = 0.68$. For comparison, a line with slope equal to −0.5, corresponding to increments of standard Brownian motion with $H = 0.5$, is plotted. At shorter timescales (from 5 to 30 days), we observe strong short-term persistence of synoptic weather conditions.

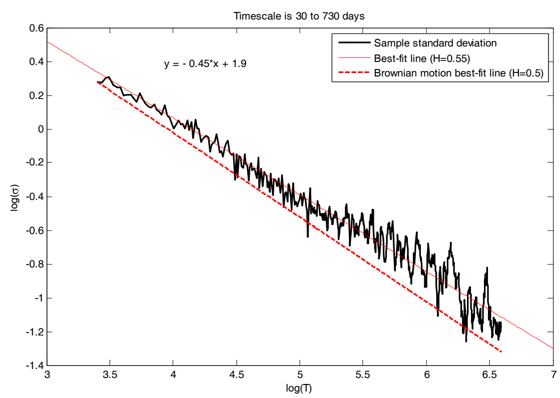

Figure 5. ST analysis of Shanghai temperatures from 1991 to 2010. The best-fit exponent for timescales from 30 to 730 days is −0.45, corresponding to $H = 0.55$. For comparison, a line with slope equal to −0.5, corresponding to increments of standard Brownian motion with $H = 0.5$, is plotted timescales from 5 to 730 days is −0.44, corresponding to $H = 0.56$. For comparison, a line with slope equal to −0.5, corresponding to increments of standard Brownian motion with $H = 0.5$, is plotted.

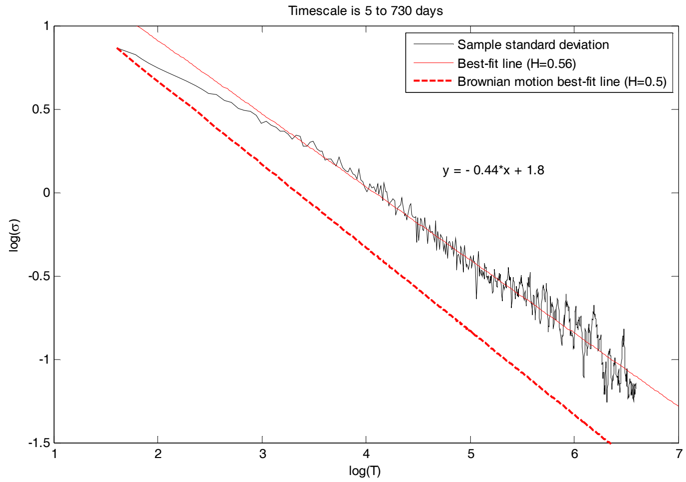

Figure 6. ST analysis of Shanghai temperatures from 1991 to 2010.

In Table 1, each of the price estimates are obtained by a Monte Carlo simulation of 100 000 sample paths. We see that prices are higher under the model given in equation (4), which is driven by an FBM. In particular, we see that at low price levels, the relative price difference is very large. These results show the importance of capturing the long-range temporal dependence in daily temperatures.

Table 1. Pricing of Heating Degree Day (HDD) and Cooling Degree Day (CDD) options for Shanghai using a Monte Carlo simulation. The contract period for the HDD option is January 1-31, whereas for CDD option, the contract period is July 1-31

	HDD Strike			CDD Strike		
	375	400	425	325	350	375
Call						
$H = 0.50$	7.446	2.296	0.528	44.477	22.388	8.076
$H = 0.56$	9.521	3.667	1.151	46.173	24.108	9.858
Put						
$H = 0.50$	26.811	46.084	69.028	0.635	4.069	14.542
$H = 0.56$	28.996	47.540	71.278	1.234	5.586	15.771

In Figures 7 and 8, we plot the call and put option prices with respect to different Hurst parameter values. The model with Brownian motion corresponds to $H = 1/2$, and any deviation from Brownian motion in the underlying dynamics causes significant pricing differences.

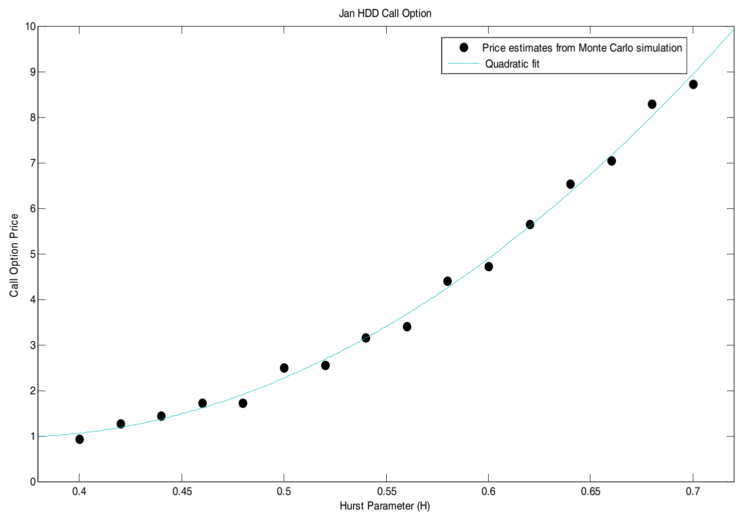

Figure 7. Price estimates of an HDD call option for Shanghai with strike level of 425 HDDs.

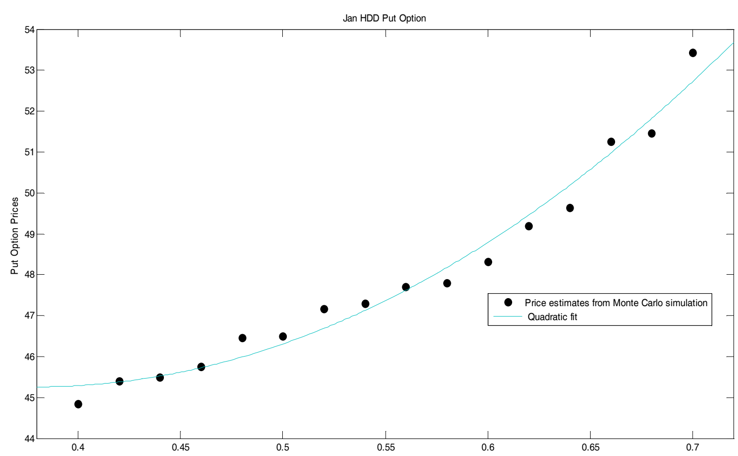

Figure 8. Price estimates of an HDD put option with strike level of 425 HDDs.

In this study, we modelled daily average temperatures in Shanghai for a period of twenty years using a mean-reverting stochastic process driven by Fractional Brownian Motion (FBM). The use of FBM is motivated by two observations: (i) the fluctuations of temperatures around the long-term mean obey a Gaussian law, (ii) there exists long-term temporal dependence in temperatures. Motivated by these two facts, we use an FBM-driven OU process for pricing temperature-based weather derivatives for Shanghai. The results show that assuming a standard Brownian motion significantly underestimates the price of standard call and put options on HDDs and CDDs. In an incomplete market setting with limited hedging opportunities, capturing temperature dynamics properly is crucial. To the best of the author's knowledge, this study is the first to propose the use of FBM for modelling weather derivatives in a Chinese city.

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