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KLEIN-GORDON EQUATION AS A BI-DIMENSIONAL MOMENT PROBLEM

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Abstract

We consider the solution of one-dimensional linear and nonlinear Klein-Gordon equations by first transforming them into bi-dimensional integral equations which are then handled as bi-dimensional moment problems. The integral equations are obtained by either Laplace transforming the linear PDE or by using Green identity for the linear

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as well as the nonlinear cases. The discretization of the so obtained integral equations results, for the linear and nonlinear problems, respectively, into a bi-dimensional Hausdorff problem and into a generalized moment problem (in which the kernel set $\{x^n y^m\}_{nm}$ has been replaced by sets $\{g_m(x, y)\}_m$ of more general linearly independent functions). In both cases, the corresponding inverse problem is numerically solved by approximating the associated finite moment problem by a truncated expansion.

I. Introduction

The Klein-Gordon (K-G) equation has been proved to be very useful in many scientific fields such as solid state physics, quantum field theory, chemical kinetics, nonlinear optics, fluid dynamics, mathematical biology and so on. In its more general (nonlinear and nonhomogeneous) version, it can be written (for just a one spatial dimension) as

$$u_{tt}(x, t) - cu_{xx}(x, t) - ag[u(x, t)] = bf(x, t), \tag{1.1}$$

where $u \equiv u(x, t)$ represents a wave displacement at position x and time t with $x > 0$, $t > 0$; $g(u)$ is a dispersive contribution, $f(x, t)$ is an external force and $c > 0$ and b are constants. The particular case $g[u(x, t)] \equiv u(x, t)$, where a is a constant, gives the nonhomogeneous linear Klein-Gordon equation

$$u_{tt}(x, t) - cu_{xx}(x, t) - au(x, t) = bf(x, t). \tag{1.2}$$

If we take $b = 0$ in both equations, then the corresponding homogeneous versions are recovered.

We assume that the function $u(x, t)$ is subjected to initial conditions:

$$u(x, t = 0) = \varphi_1(x), \quad u_t(x, t = 0) = \varphi_2(x) \tag{1.3}$$

and mixed (Cauchy) boundary conditions at the origin:

$$u(x = 0, t) = \varphi_3(t), \quad u_x(x = 0, t) = \varphi_4(t). \tag{1.4}$$

Also, without lost of generality, we will consider $c = 1$ throughout.

These PDE's have been numerically solved for diverse expressions of $g[u(x, t)]$, $f(x, t)$, $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(t)$ and $\varphi_4(t)$ by using a variety of techniques which include decomposition method [1, 2], iterative variational method [3], discrete difference approximation [4], Legendre spectral method [5, 6], the use of radial basis functions [7] and many others schemes [8, 9].

In this paper, we consider a different way to numerically solve the problem given by equation (1.1) or (1.2) with conditions (1.3) and (1.4): we first transform it into an integral equation which we then handle as a bi-dimensional moment problem. This approach was already suggested by Ang et al. [10] in relation with the heat conduction equation.

The work is organized as follows: In principle, we consider separately the linear and the nonlinear equations. Next section is devoted to the first one. There we transform equations (1.2), (1.3) and (1.4) into an integral equation by using Laplace transformation. The resulting integral equation is considered as a bi-dimensional Hausdorff moment problem which is regularized by solving a related finite problem as we did in [11] and also discuss in Appendix A. In Section III, the nonlinear Klein-Gordon equation is considered. There we use the Green identity to transform the PDE into the integral equation. Now we view the resulting equation as a bi-dimensional generalized moment problem of the type, we have discussed in [12] for just one-dimension and that we extend to involve two-dimension integrals in Appendix B. We also consider again the linear K-G equation as a particular case $(g(u) \equiv au)$ and show how the generalized moment problem transforms into the Hausdorff problem already seen in Section II. In all the cases, we illustrate the method with several examples.

II. Linear K-G Equation

We start considering the nonhomogeneous linear Klein-Gordon equation as given by equation (1.2) together with the conditions established by equations (1.3) and (1.4). This equation has a lot of applications in mathematical physics. For example, the homogeneous case $(b = 0)$ describes correctly a spinless pion.

The PDE is transformed into an integral equation by means of Laplace transform in the next subsection.

A. Laplace transform

Let us first assume $a > 0$ and define the Laplace transform:

$$\tilde{u}(\xi, t) \equiv L_x[u(x, t)] := \int_0^\infty u(x, t)e^{-\xi x}dx \tag{2.1}$$

so [13],

$$L_x[u_x(x, t)] = \xi\tilde{u}(\xi, t) - u(0, t),$$

$$L_x[u_{xx}(x, t)] = \xi^2\tilde{u}(\xi, t) - \xi u(0, t) - u_x(0, t).$$

Thus, applying Laplace transform to equation (1.2), we have

$$\tilde{u}_{tt}(\xi, t) - (a + \xi^2)\tilde{u}(\xi, t) = G(\xi, t), \tag{2.2}$$

where

$$G(\xi, t) = -\xi\varphi_3(t) - \varphi_4(t) + b\tilde{f}(\xi, t) \tag{2.3}$$

with $\tilde{f}(\xi, t) = L_x[f(x, t)]$.

Equations (2.2) and (2.3) define a second order ordinary differential equation in the variable t. To solve it, let us consider the associate homogeneous equation

$$\tilde{u}_{tt}(\xi, t) - (a + \xi^2)\tilde{u}(\xi, t) = 0$$

whose characteristic function is $r^2 - (a + \xi^2) = 0$, so it has the general solution

$$\tilde{u}_c(\xi, t) = c_1 e^{\sqrt{a+\xi^2}\,t} + c_2 e^{-\sqrt{a+\xi^2}\,t},$$

thus the general solution of equation (2.2) is of the form

$$\tilde{u}(\xi, t) = c_1 e^{t\sqrt{a+\xi^2}} + c_2 e^{-t\sqrt{a+\xi^2}\,t} + \tilde{u}_p(\xi, t), \tag{2.4}$$

where $\tilde{u}_p(\xi, t)$ is a particular solution of equation (2.2), which will depend in any particular case of the expression for $G(\xi, t)$, and c_1 and c_2 are constants which are determined from the system of algebraic equations derived by Laplace transforming the initial conditions (equation (1.3)):

$$\begin{cases} \tilde{u}(\xi, 0) = \tilde{\varphi}_1(\xi), \\ \tilde{u}_t(\xi, 0) = \tilde{\varphi}_2(\xi). \end{cases} \tag{2.5}$$

Here $\tilde{u}(\xi, 0) = L_x[u(x, 0)]$; $\tilde{u}_t(\xi, 0) = L_x[u_t(x, 0)]$ and $\tilde{\varphi}_i(\xi) = L_x[\varphi_i(x)]$ $(i = 1, 2)$. The direct way to obtain the desired solution is to apply, when it is possible, the inverse Laplace transform to $\tilde{u}(\xi, t)$: $u(x, t) = L_x^{-1}[\tilde{u}(\xi, t)]$.

Note that if $a < 0$, we can consider the Laplace transform with respect to the variable t:

$$\hat{u}(x, \tau) \equiv L_t[u(x, t)] := \int_0^\infty u(x, t)e^{-\tau t}dt$$

and, instead of equations (2.2), (2.3), we obtain the ODE

$$\hat{u}_{xx}(x, \tau) + (a - \tau^2)\hat{u}(x, \tau) = G(x, \tau),$$

where

$$G(x, \tau) = -\tau\varphi_1(x) - \varphi_2(x) + b\hat{f}(x, \tau)$$

and the general solution of the ordinary differential equation is

$$\tilde{u}(x, \tau) = c_1 e^{x\sqrt{\tau^2 - a}} + c_2 e^{-x\sqrt{\tau^2 - a}} + \hat{u}_p(x, \tau).$$

B. Hausdorff moment problem

In both cases, $a > 0$ or $a < 0$, if the inversion $u(x, t) = L_x^{-1}[\tilde{u}(\xi, t)]$ or $u(x, t) = L_t^{-1}[\hat{u}(x, \tau)]$, respectively, can be done analytically, then a closed expression for $u(x, t)$ is reached. Here we show a variant that allows, in general, for numerical solutions. To do this, we first consider a bi-dimensional integral equation by Laplace transforming $\tilde{u}(\xi, t)$ or $\hat{u}(x, \tau)$

with respect to the other variable. So, for $a > 0$, we obtain a bi-dimensional first kind Fredholm like integral equation for $u(x, t)$:

$$\int_0^{\infty}\int_0^{\infty} u(x, t)e^{-\xi x}e^{-\tau t}dxdt = \hat{\tilde{u}}(\xi, \tau) \tag{2.6}$$

with

$$\hat{\tilde{u}}(\xi, \tau) = \int_0^{\infty} \tilde{u}(\xi, t)e^{-\tau t}dt. \tag{2.7}$$

Here we show how the integral equation (2.6) can be solved by viewing it as a Hausdorff moment problem. Define $z_1 = e^{-x}$; $z_2 = e^{-t}$ and change variables $(x, t) \to (z_1, z_2)$. We obtain

$$\int_0^1\int_0^1 z_1^{\xi-1}z_2^{\tau-1}u(-\ln z_1, -\ln z_2)dz_1dz_2 = \hat{\tilde{u}}(\xi, \tau).$$

We can write for ξ and τ natural numbers:

$$\int_0^1\int_0^1 z_1^m z_2^n w(z_1, z_2)dz_1dz_2 = \mu_{mn}, \quad m, n = 0, 1, 2, ..., \tag{2.8}$$

where $\mu_{mn} \equiv \hat{\tilde{u}}(m + 1 + \alpha_1, n + 1 + \alpha_2)$ and

$$w(z_1, z_2) = z_1^{\alpha_1}z_2^{\alpha_2}u(-\ln z_1, -\ln z_2). \tag{2.9}$$

Here, α_1 and α_2 are conveniently chosen numbers so that the moments μ_{mn} are well defined. Equations (2.8) and (2.9) represent a bi-dimensional Hausdorff moment problem for $w(z_1, z_2)$. We have studied this problem in [11]. There we first consider the relative finite moment problem, say equation (2.8) but with $m, n = 0, 1, 2, ..., N$; $(N \in \mathbb{N})$ whose solution is expanded

$$w(z_1, z_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\lambda_{ij}P_{ij}(z_1, z_2),$$

where $P_{ij}(z_1, z_2) = P_i(z_1)P_j(z_2)$ with $P_i(z)(i = 0, 1, 2, ...)$ the Legendre polynomials defined in $[0, 1]$ and the coefficients λ_{ij} are

$$\lambda_{ij} = \int_0^1\int_0^1 w(z_1, z_2)P_{ij}(z_1, z_2)dz_1dz_2 \quad (i, j = 0, 1, 2, ...).$$

Then we estimate $w(z_1, z_2)$ by truncating the expansion:

$$w(z_1, z_2) \approx w_N(z_1, z_2) = \sum_{i=0}^{N}\sum_{j=0}^{N}\lambda_{ij}P_{ij}(z_1, z_2), \tag{2.10}$$

where the coefficients λ_{ij} are explicitly given by

$$\lambda_{ij} = \sum_{k_1=0}^{i}\sum_{k_2=0}^{j} c_{ik_1}c_{jk_2}\mu_{k_1k_2} \quad (i, j = 0, 1, 2, ..., N) \tag{2.11}$$

with

$$c_{ik} = \sqrt{2i+1}(-1)^k\binom{i}{k}\binom{i+k}{k}. \tag{2.12}$$

In order that this method of truncated expansion [14] is valid, we require [11] that $\hat{\hat{u}}(\xi, \tau) \in L^2[[0, \infty) \times [0, \infty)]$ and

$$\int_0^\infty\int_0^\infty [xu_x^2(x, t) + tu_t^2(x, t)]e^{(x+t)}dxdt < \infty. \tag{2.13}$$

In Appendix A, we prove the following theorem, which adapts some of the results of [11] to the present context:

Theorem 1. Define $u_N(x, t) = e^{\alpha_1 x}e^{\alpha_2 t}w_N(e^{-x}, e^{-t})$. *If* $u(x, t)$ *verifies*

$$u(x, t)e^{\frac{1}{2}x}e^{-\frac{1}{2}t} \in L^2(\mathbb{R}_+^2),$$

$$u_x(x, t)e^{\frac{1}{2}x}e^{-\frac{1}{2}t} \in L^2(\mathbb{R}_+^2),$$

$$u(x, t)e^{-\frac{1}{2}x}e^{\frac{1}{2}t} \in L^2(\mathbb{R}_+^2),$$

$$u_t(x, t)e^{-\frac{1}{2}x}e^{\frac{1}{2}t} \in L^2(\mathbb{R}_+^2)$$

and the norm $\| f(x, t)\|_w^2$ *is defined as*

$$\| f(x, t)\|_w^2 \equiv \int_0^\infty \int_0^\infty | f(x, t)|^2 e^{-(1+2\alpha_1)x} e^{-(1+2\alpha_2)t} dxdt,$$

then

$$\| U_N(x, t) - u(x, t)\|_w^2 \leq \frac{1}{4(N+1)^2}(I_1 + I_2), \tag{2.14}$$

where

$$I_\gamma = \int_0^\infty \int_0^\infty \alpha_\gamma^2 u(x, t)^2 e^{-2\left(\alpha_1 + \frac{(-1)^\gamma}{2}\right)x} e^{-2\left(\alpha_2 - \frac{(-1)^\gamma}{2}\right)t} dxdt$$

$$+ \int_0^\infty \int_0^\infty [u_x(x, t)^{2-\gamma} + u_t(x, t)^{\gamma-1}]^2 e^{-2\left(\alpha_1 + \frac{(-1)^\gamma}{2}\right)x} e^{-2\left(\alpha_2 - \frac{(-1)^\gamma}{2}\right)t} dxdt \tag{2.15}$$

with $\gamma = 1, 2$. *Moreover, if the moments* $\mu_{mn} \equiv \hat{\hat{u}}(m + 1 + \alpha_1, n + 1 + \alpha_2)$ *have an error such that* $Tr(\mu\mu^T) = \sum_{m=1}^N \sum_{n=1}^N \mu_{mn} \leq \varepsilon^2$, *then*

$$\| u_N(x, t) - u(x, t)\|_w^2 \leq \frac{1}{4(N+1)^2}(I_1 + I_2) + \varepsilon^2 c^2 \tag{2.16}$$

with $c = (2N + 1)(N + 1)^2 2^{6N} \dfrac{2^8}{2^6 - 1}$.

Example 1. Here we consider an example of the linear K-G equation as given by equation (1.2), with initial and boundary conditions given by (1.3) and (1.4), respectively,

$$a = 1;\ b = 1;\ f(x,\ t) = 2e^{-(x+t)}[t(x-1)\cos(x) - (x-t)\sin(x)];$$

$$\varphi_1(x) = 0;\ \varphi_2(x) = x\sin(x)e^{-x};\ \varphi_3(t) = 0;\ \varphi_4(t) = 0.$$

The known exact solution is: $u(x,\ t) = xt\sin(x)e^{-(x+t)}$.

By Laplace transforming, we calculate

$$\tilde{f}(\xi,\ t) = \frac{2e^{-t}(1+\xi)(2+t\xi^2)}{(2+2\xi+\xi^2)^2},$$

so equation (2.3) gives

$$G(\xi,\ t) = \tilde{f}(\xi,\ t).$$

We propose

$$\tilde{u}_p(\xi,\ t) = e^{-t}(A+Bt) \text{ with } A = 0;\ B = \frac{2(1+\xi)}{(2+2\xi+\xi^2)^2}$$

as a particular solution of equation (2.3), introduce this expression into the general solution (2.4) and determine the constants c_1 and c_2 by solving the system (2.4) with

$$\tilde{\varphi}_1(\xi) = 0; \quad \tilde{\varphi}_2(\xi) = \frac{2(1+\xi)}{(2+2\xi+\xi^2)^2}.$$

We find $c_1 = c_2 = 0$, so the general solution is

$$\tilde{u}(\xi,\ t) = 2te^{-t}(1+\xi)/(2+2\xi+\xi^2)^2.$$

By Laplace transforming $\tilde{u}(\xi,\ t)$ with respect to the variable *t*, we have

$$\hat{\tilde{u}}(\xi,\ \tau) = \frac{2(1+\xi)}{(2+2\xi+\xi^2)^2(1+\tau)^2}$$

and the moments are

$$\mu_{mn} \equiv \hat{\hat{u}}(m+1+\alpha_1,\ n+1+\alpha_2) = \frac{2(2+m)}{(5+4m+m^2)^2(2+n)^2}.$$

We observe that for $\alpha_1 = \alpha_2 = 0,$ the moments are already well defined, and it is enough to solve the finite Hausdorff moment problem

$$\int_0^1\int_0^1 z_1^m z_2^n w(z_1,\ z_2) dz_1 dz_2 = \mu_{mn}$$

with the moments

$$\mu_{mn} = 2(2+m)[(5+4m+m^2)^2(2+n)^2]^{-1} \quad (m,\ n = 0, 1,\ 2,\ ...,\ N).$$

Also, it should be remarked that we have verified that the function $u(x,\ t) = xt\sin(x)e^{-(x+t)}$ fulfills condition (2.13).

In Figure 1, we show the solution we have obtained for $u(x,\ t) = w(e^{-x},\ e^{-t})$ with $w(z_1,\ z_2)$ approximate by $w_N(z_1,\ z_2)$ as given by equations (2.10)-(2.12) for $N = 5.$ We also compare our numerical solution with the known exact solution.

III. Nonlinear K-G Equation

In this section, we consider the nonlinear K-G defined in general by equation (1.1) with the initial conditions (1.3) and Cauchy boundary conditions (1.4).

A. Green identity

Let us take the auxiliary function

$$h(x,\ t;\ r,\ s) = e^{-x(r+1)}e^{-t(s+1)} \tag{3.1}$$

that verifies

$$h_{rr}(x,\ t;\ r,\ s) - h_{ss}(x,\ t;\ r,\ s) = (x^2 - t^2)h(x,\ t;\ r,\ s). \tag{3.2}$$

In the region $D = [0, M] \times [0, M]$, we apply the planar Green identity

$$\iint_D u\nabla^2 h dA + \iint_D (\nabla u \cdot \nabla h) dA = \oint_{\partial D} u \nabla h \cdot \check{n}\, d\ell, \tag{3.3}$$

where ∂D is the contour of the region D.

Replacing here expression (3.1) for h and using equation (3.2) together with equation (1.1), we obtain

$$\int_0^M \int_0^M [(x^2 - t^2) h(x, t; r, s) u(r, s) - ag(u(r, s)) h(x, t; r, s)] drds = \phi(x, t), \tag{3.4}$$

where

$$\begin{aligned}\phi(x, t) \equiv & \int_0^M h(x, t; r, M)[u_s(r, M) + tu(r, M)]dr \\ & - \int_0^M h(x, t; M, s)[u_r(M, s) + xu(M, s)]ds \\ & - \int_0^M h(x, t; r, 0)[u_s(r, 0) + tu(r, 0)]dr \\ & + \int_0^M h(x, t; 0, s)[u_r(0, s) + xu(0, s)]ds \\ & + b\int_0^M \int_0^M f(r, s) drds.\end{aligned} \tag{3.5}$$

B. Generalized moment problem

From here on we take $M \to \infty$ in such a way that

$$\lim_{M \to \infty} \int_0^M h(x, t; M, s)[u_r(M, s) + xu(M, s)]ds \to 0,$$

$$\lim_{M \to \infty} \int_0^M h(x, t; r, M)[u_s(r, M) + tu(r, M)]dr \to 0.$$

If in equations (3.4)-(3.5), we let $x = t$, then we have

$$\int_0^\infty \int_0^\infty g(u(r, s))K(t; r, s)drds = -\frac{1}{a}\phi(t, t) \tag{3.6}$$

with

$$K(t; r, s) = e^{-(r+s+2)t}. \tag{3.7}$$

Using a basis $\{\Psi_m(t)\}_{m=0}^\infty$, we transform this bi-dimensional Fredholm integral equation of the first kind into a bi-dimensional generalized moment problem of the type, we study in [12]:

$$\int_0^\infty \int_0^\infty g(u(r, s))K_m(r, s)drds = \mu_m \quad (m = 0, 1, 2, ...), \tag{3.8}$$

where

$$K_m(r, s) = \int_0^\infty K(t; r, s)\Psi_m(t)dt \tag{3.9}$$

and the moments μ_m are

$$\mu_m = -\frac{1}{a}\int_0^\infty \phi(t, t)\Psi_m(t)dt. \tag{3.10}$$

If the functions $\{K_m(r, s)\}_m$ are linearly independent, then the problem of generalized moments defined by equations (3.8)-(3.10) can be solved as we did in [12]: finding the solution $\psi(r, s) = g(u(r, s))$ to the corresponding finite problem, say with $m = 0, 1, 2, ..., N$ $(N \in \mathbb{N})$. Thus, if $g(u)$ has continuous inverse, then $g^{-1}[\psi(r, s)]$ will be an estimation of $u(r, s)$.

Let us consider the basis $\{\varphi_i(r, s)\}_{i=0}^\infty$ obtained by applying the Gram-Schmidt orthonormalization process on $\{K_m(r, s)\}_{m=0}^N$ and then adding to the resulting set the necessary functions until an orthonormal basis is achieved. Thus

$$\langle \varphi_i(r, s) | \varphi_j(r, s) \rangle = \int_0^{\infty} \int_0^{\infty} \varphi_i(r, s) \varphi_j(r, s) drds = \delta_{ij} \quad (i, j = 0, 1, 2, \dots)$$

and the solution $\psi(r, s)$ can be expanded:

$$\psi(r, s) = \sum_{i=0}^{\infty} \lambda_i \varphi_i(r, s),$$

but we approximate it by truncating the expansion [12]:

$$\psi(r, s) \approx \psi_N(r, s) = \sum_{i=0}^{N} \lambda_i \varphi_i(r, s), \tag{3.11}$$

where

$$\lambda_i = \sum_{j=0}^{i} C_{ij} \mu_j \quad (i = 0, 1, 2, \dots, N) \tag{3.12}$$

with the coefficients C_{ij} verifying the linear system

$$C_{ij} = \left(\sum_{k=j}^{i-1} (-1) \frac{\langle K_i(r, s) | \varphi_k(r, s) \rangle}{\| \varphi_k(r, s) \|^2} C_{kj} \right)$$

$$\cdot \| \varphi_i(r, s) \|^{-1} \quad (1 < i \leq N; 1 \leq j < i). \tag{3.13}$$

The diagonal terms are $C_{ii} = \| \varphi_i(x) \|^{-1}$ $(i = 0, 1, \dots, N)$ and $\langle u(r, s) | v(r, s) \rangle$ denotes the inner product in the Hilbert space.

In Appendix B, we extend to the bi-dimensional case the arguments used in [12] to demonstrate the

Theorem 2. *Let the set of real numbers* $\{\mu_k\}_{k=0}^{N}$ *and let* ε *and* E *be two positive numbers such that*

$$\sum_{k=0}^{N} \left| \int_0^{\infty} \int_0^{\infty} K_k(r, s) \psi(r, s) drds - \mu_k \right|^2 \leq \varepsilon^2 \tag{3.14}$$

and

$$\int_0^\infty \int_0^\infty (r\psi_r^2 + s\psi_s^2) e^r e^s drds \leq E^2, \tag{3.15}$$

if besides

$$r^k \psi(r, s) \to 0 \text{ for } r \to \infty, \quad \forall s, k \in \mathbb{N},$$

$$s^k \psi(r, s) \to 0 \text{ for } s \to \infty, \quad \forall r, k \in \mathbb{N},$$

then

$$\int_0^\infty \int_0^\infty |\psi(r, s)|^2 drds$$

$$\leq \min_n \left\{ \| \mathbf{C}^\dagger \mathbf{C} \|^2 \varepsilon^2 + \frac{1}{2(n+1)} E^2; \, n = 0, 1, \ldots, N \right\}, \tag{3.16}$$

where **C** *is the lower triangular matrix with elements* C_{ij} $(1 < i \leq N; 1 \leq j < i)$ (*equation* (3.13)) *and* $\mathbf{C}^\dagger$ *is its transpose*. *Moreover*, *the truncated solution* $\psi_N(r, s)$ *given by equation* (3.11) *verifies*

$$\int_0^\infty \int_0^\infty |\psi_N(r, s) - \psi(r, s)|^2 drds \leq \| \mathbf{C}^\dagger \mathbf{C} \|^2 \varepsilon^2 + \frac{1}{2(N+1)} E^2. \tag{3.17}$$

If $g^{-1}(x)$ is Lipschitz in $\mathbb{R}^2$, say if $\| g^{-1}(x) - g^{-1}(x') \| \leq \lambda \| x - x' \|$ for some λ and $\forall x, x' \in \mathbb{R}^2$, then according to the previous theorem,

$$\int_0^\infty \int_0^\infty |u_N(r, s) - u(r, s)|^2 drds \leq \lambda \left\{ \| \mathbf{C}^\dagger \mathbf{C} \|^2 \varepsilon^2 + \frac{1}{2(N+1)} E^2 \right\}.$$

C. The linear K-G again

If in equation (3.4), we take $g(u(r, s)) = u(r, s)$ (assuming $M \to \infty$), then we have

$$\int_0^\infty \int_0^\infty (x^2 - t^2 - a) h(x, t; r, s) u(r, s) drds = \phi(x, t). \tag{3.18}$$

Defining

$$K(x,\ t;\ r,\ s) = (x^2 - t^2 - a)h(x,\ t;\ r,\ s)$$

and setting $x = m$ and $t = n$ $(m,\ n \in \mathbb{N})$, we obtain using expression (3.1) for $h(x,\ t;\ r,\ s)$,

$$\int_0^\infty \int_0^\infty e^{-mr} e^{-ns} u(r,\ s)\,drds = \mu_{mn}, \quad m,\ n = 0,\ 1,\ 2,\ \ldots, \tag{3.19}$$

where

$$\mu_{mn} = \frac{\phi(m,\ n)e^{m+n}}{(m^2 - n^2 - a)}. \tag{3.20}$$

This can be viewed as a bi-dimensional generalized Stieltjes moment problem. By changing variables $(r,\ s) \to (z_1,\ z_2)$, where $z_1 = e^{-r};\ z_2 = e^{-s}$, we recover the Hausdorff moment problem given by equation (2.8) with $w(z_1,\ z_2)$ defined by (2.9). As before, the parameters α_1 and α_2 are chosen so that $\phi(m,\ n)e^{n+m}/(m^2 - n^2 - a)$ will be well defined.

D. Examples

1. Linear K-G

Example 2. We consider again Example 1; say, equations (1.2), (1.3) and (1.4) with $a = 1;\ b = 1;$

$$f(x,\ t) = 2e^{-(x+t)}[t(x-1)\cos(x) - (x-t)\sin(x)];$$

$\varphi_1(x) = 0;\ \varphi_2(x) = x\sin(x)e^{-x};\ \varphi_3(t) = 0;\ \varphi_4(t) = 0$ whose exact solution, we know, is $u(x,\ t) = xt\sin(x)e^{-(x+t)}$, but now we will transform it into the bi-dimensional generalized Stieltjes moment problem given by equations (3.19)-(3.20). Equation (3.5) gives with $M \to \infty$ and conditions (1.3) and (1.4):

$$\phi(x,\ t) = \frac{2(1+t)(x^2 - t^2 + 1)}{(2 + 2x + x^2)^2(1+t)^2}$$

so the moments (equation (3.20)) are

$$\mu_{mn} = \frac{2(1+n)(m^2 - n^2 + 1)e^{m+n}}{(2 + 2m + m^2)^2(1+m)^2(m^2 - n^2 - 1)}; \quad m,\ n = 0, 1, 2, \ldots, N.$$

Thus we orthonormalize the basis $\{r^m s^n e^{-(r+s)}\}_{mn}$ and follow the procedure of [11] to obtain $u_N(r,\ s)$. In particular, in Figure 2, we show the result for $N = 5$. There we also display the exact solution $u(r,\ s)$ for comparison. The accuracy, estimated by

$$\left[\int_0^\infty \int_0^\infty |u_N(r,\ s) - u(r,\ s)|^2 drds\right]^{1/2}, \tag{3.21}$$

gives 0.000553422 in the present case.

Example 3. We numerically solve equations (1.2), (1.3) and (1.4) with $a = 1$; $b = 1$; $f(x,\ t) = -(x+t)e^{-(x+t)}$; $\varphi_1(x) = xe^{-x}$; $\varphi_2(x) = e^{-x}(1-x)$; $\varphi_3(t) = te^{-t}$; $\varphi_4(t) = e^{-t}(1-t)$. The exact closed solution is $u(x,\ t) = (x+t)e^{-(x+t)}$. The function $\phi(x,\ t)$ is

$$\phi(x,\ t) = \frac{x+t}{(1+x)^2} - \frac{2+x+t}{(1+x)^2(1+t)^2} - \frac{x+t}{(1+t)^2}$$

and the moments to be considered in the Stieltjes problem are

$$\mu_{mn} = \frac{1}{m^2 - n^2 - 1}\left[\frac{m+n}{(1+m)^2} - \frac{2+m+n}{(1+m)^2(1+n)^2} - \frac{m+n}{(1+n)^2}\right]e^{(m+n)};$$

$$m,\ n = 0, 1, 2, \ldots, N.$$

Using the same basis and also $N = 5$, we obtain the curve shown in Figure 3. The estimated accuracy is 0.00157485.

2. Nonlinear K-G

Here we give two examples of the solution of equation (1.1) with conditions (1.3) and (1.4) using the procedure outlined in Section III.

Example 4.

$$a = 1;\ \ b = 1;\ \ f(x, t) = e^{-x}e^{-(1+t)^2};\ \ g[u(x, t)] = (4t^2 + 8t + 1)u(x, t);$$

$$\varphi_1(x) = (1 + x)e^{-(1+x)};\ \ \varphi_2(x) = -2(1 + x)e^{-(1+x)};\ \ \varphi_3(t) = e^{-(1+t)^2};$$

$$\varphi_4(t) = 0.$$

The exact closed solution is $u(x, t) = (1 + x)e^{-x}e^{-(1+t)^2}$. In equation (3.6), we have

$$\phi(t, t) = \frac{1}{2}e^{\frac{1}{4}(-4+t)t}\sqrt{\pi}\,erfc\left(1 + \frac{t}{2}\right)\left[\frac{t^2 + t - 2}{2(1 + t)}\right] + \frac{4 - t^2}{(1 + t)^2}e^{-(1+2t)}$$

with *erfc* the complementary error function. From equations (3.7) and (3.9), using the basis $\{\Psi_m(t)\}_m = \{t^m e^{-t}\}_m$, we obtain

$$K_m(r, s) = \frac{(m - 1)!}{(3 + r + s)^m}, \quad m = 1, 2, ..., N \tag{3.22}$$

and from equation (3.10), the corresponding moments μ_m. Here we take $N = 5$. Then according to equations (3.11)-(3.13), we have the truncated solution $\psi_N(r, s)$ and the estimation of $u(r, s)$: $(4s^2 + 8s + 1)^{-1}\psi_N(r, s)$. In Figure 4, we compare our results with the exact solution. In this case, the accuracy (expression (3.21)) is 0.0603017.

Example 5. $a = 1;\ \ b = 1;\ \ f(x, t) = e^{-\frac{1}{2}(x+t)};\ \ g[u(x, t)] = \sqrt{u(x, t)};$ $\varphi_1(x) = e^{-x};\ \ \varphi_2(x) = -e^{-x};\ \ \varphi_3(t) = e^{-t};\ \ \varphi_4(t) = -e^{-t}$. The exact closed solution is $u(x, t) = e^{-(x+t)}$. We calculate

$$\phi(t, t) = \frac{4e^{-2t}}{(1+2t)^2}$$

and the moments are as given by equation (3.10). Using the previous algorithm, with $K_m(r, s)$ as in equation (3.22), we have $\psi_N(r, s)$ in the form of the truncated expansion (3.11) and so $u_N(r, s) = [\psi_N(r, s)]^2$. Figure 5 displays $u_N(r, s)$ for $N = 5$ and also the exact solution $u(x, t) = e^{-(x+t)}$ for comparison. The calculated accuracy is 0.0521479.

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Appendix A: Proof of Theorem 1

Taking into account the definitions of $u_N(x, t)$ and of the norm $\| \bullet \|_w^2$, we have

$$\| u_N(x, t) - u(x, t) \|_w^2 = \| w_N(z_1, z_2) - w(z_1, z_2) \|^2,$$

where

$$\| f(x, y) \|^2 \equiv \int_0^1 \int_0^1 | f(x, y) |^2 dxdt.$$

But it is proved [11] that

$$\| w_N(z_1, z_2) - w(z_1, z_2) \|^2 \leq \frac{1}{4(N+1)^2} (I_1 + I_2)$$

with

$$I_\gamma \equiv \| w_{z_\gamma}(z_1, z_2) \|^2 = \int_0^1 \int_0^1 w_{z_\gamma}(z_1, z_2)^2 dz_1 dz_2 \quad (\gamma = 1, 2).$$

Derivating $w(z_1, z_2)$ in equation (2.9) with respect to z_γ $(\gamma = 1, 2)$ and affecting the double integral, we obtain for I_γ expression (2.15) of the text. Besides, if noise is considered such that $Tr(\mu\mu^T) \leq \varepsilon^2$, since in this case is [11],

$$\| w_N(z_1, z_2) - w(z_1, z_2) \|^2 \leq \frac{1}{4(N+1)^2}(I_1 + I_2) + c^2 Tr(\mu\mu^T)$$

with $c = (2N+1)(N+1)^2 2^{6N} \dfrac{2^8}{2^6 - 1}$, then equation (2.16) is recovered. □

Appendix B: Proof of Theorem 2

We closely follow the demonstration given in [12] for the one-dimensional moment problem which in turn is based in Talenti work [14] for the Hausdorff problem. Here we just introduce the necessary modifications for the general bi-dimensional problem.

Without lost of generality, we take $\mu_k = 0$ $(k = 0, 1, 2, ..., N)$ in equation (3.14). Let us write $\psi(r, s)$ in the form

$$\psi(r, s) = h_N(r, s) + t_N(r, s),$$

where $h_N(r, s)$ is the orthogonal projection of $\psi(r, s)$ on the linear space generated by the set $\{K_m(r, s)\}_{m=0}^N$ and $t_N(r, s) = \psi(r, s) - h_N(r, s)$ the orthogonal projection of $\psi(r, s)$ onto the orthogonal complement. The functions $h_N(r, s)$ and $t_N(r, s)$ can be expanded in the basis $\{\varphi_i(r, s)\}_{i=0}^\infty$:

$$h_N(r, s) = \sum_{i=0}^{N} \lambda_i \varphi_i(r, s); \quad t_N(r, s) = \sum_{i=N+1}^{\infty} \lambda_i \varphi_i(r, s)$$

with

$$\lambda_i = \int_0^\infty \int_0^\infty \varphi_i(r, s)\psi(r, s)drds \quad (i = 1, 2, ...).$$

The relation between the coefficients λ_i and the moments

$$\mu_i = \int_0^\infty \int_0^\infty K_i(r, s)\psi(r, s)drds \quad (i = 1, 2, ...)$$

reads

$$\lambda_i = \sum_{j=0}^{i} C_{ij}\mu_j \quad (i = 0, 1, 2, ...),$$

where the matrix components C_{ij} are given by equation (3.13) in the text. Thus we have

$$\sum_{j=0}^{i} C_{ij} g_j(r, s) = \varphi_i(r, s) \quad (i = 0, 1, 2, ...)$$

or, in matricial form, $\lambda = \mathbf{C}_\mu$, where

$$\lambda = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_N \end{bmatrix}$$

and

$$\mathbf{C} = \begin{bmatrix} C_{00} & & & \\ C_{10} & C_{11} & & \\ \vdots & \vdots & & \\ C_{N0} & C_{N1} & \cdots & C_{NN} \end{bmatrix}.$$

Taking into account the previous equations, the orthonormalization condition of the set $\{\varphi_i(r, s)\}_{i=0}^{\infty}$ and the condition given by equation (3.14), we have

$$\int_0^\infty \int_0^\infty | h_N(r, s)|^2 drds = \langle \lambda | \lambda \rangle = \langle \mathbf{C}^\dagger \mathbf{C} \mu | \mu \rangle$$

$$\leq \| \mathbf{C}^\dagger \mathbf{C} \| \cdot \| \mu \|^2 = \| \mathbf{C}^\dagger \mathbf{C} \|^2 \varepsilon^2.$$

In order to estimate the norm of $t_N(r, s)$, we observe that each element of the orthonormal set $\{\varphi_i(r, s)\}_{i=0}^\infty$ can in turn be expanded in terms of the elements of another orthonormal basis, in particular, the set $\{P_{kl}(r, s)\}_0^\infty$, with $P_{kl}(r, s) = e^{-r} L_k(r) e^{-s} L_l(s)$, where $L_k(r)$ denotes the Laguerre polynomials:

$$\varphi_i(r, s) = \sum_{k=0}^\infty \sum_{l=0}^\infty \gamma_{i;\, kl} P_{kl}(r, s).$$

Then, defining $\lambda_{kl} = \sum_{i=N+1}^\infty \lambda_i \gamma_{i;\, kl}$, it follows

$$\int_0^\infty \int_0^\infty | t_N(r, s)|^2 drds \leq \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(k+1)}{N+1} \lambda_{kl}^2$$

and also

$$\int_0^\infty \int_0^\infty | t_N(r, s)|^2 drds \leq \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(l+1)}{N+1} \lambda_{kl}^2.$$

Therefore,

$$\int_0^\infty \int_0^\infty | t_N(r, s)|^2 drds \leq \frac{1}{2(N+1)} \int_0^\infty \int_0^\infty (r\psi_r^2 + s\psi_s^2) e^r e^s drds.$$

By adding these expressions for the two norms $\| h_N(r, s) \|^2$ and $\| t_N(r, s) \|^2$, the result (3.16) in the text is obtained. In a similar way, inequality (3.17) is proved. □

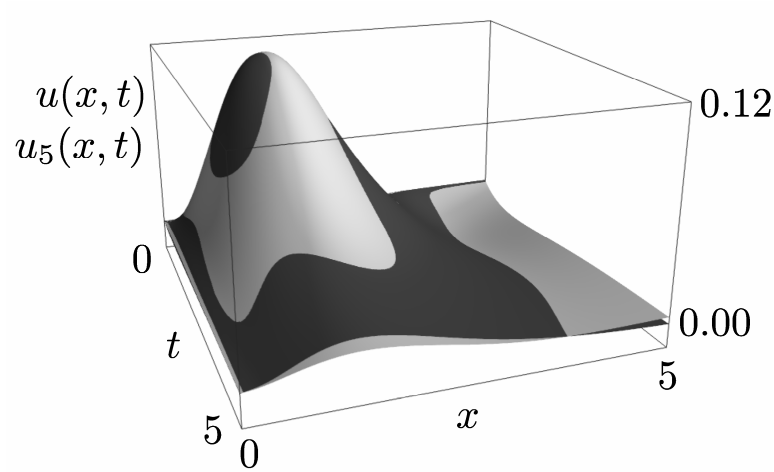

Figure 1. Comparison of the exact solution $u(x, t)$ with the estimate $u_5(x, t)$ for Example 1. The function $u_5(x, t)$ is a truncated solution to the bi-dimensional finite Hausdorff moment problem obtained by Laplace transforming the linear K-G. Dark grey: exact; light grey: estimation.

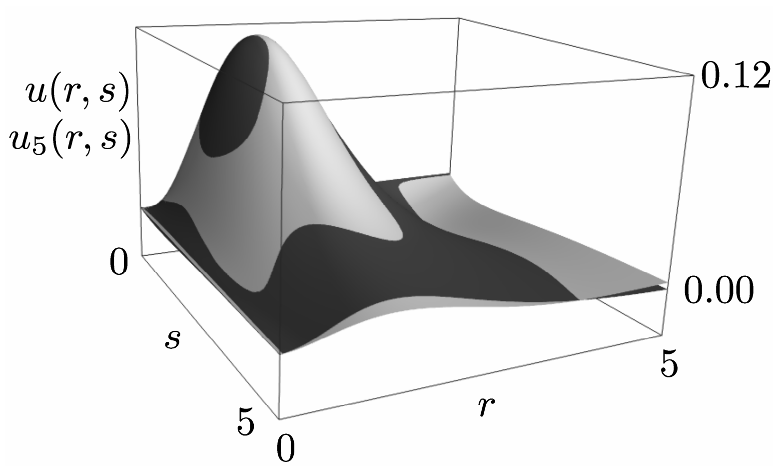

Figure 2. Comparison of the exact solution $u(r, s)$ with the estimate $u_5(r, s)$ for the example as in Figure 1. The function $u_5(x, t)$ is a truncated solution to the bi-dimensional finite generalized Stieltjes moment problem obtained by transforming the linear K-G by means of Green identity. Dark grey: exact; light grey: estimation.

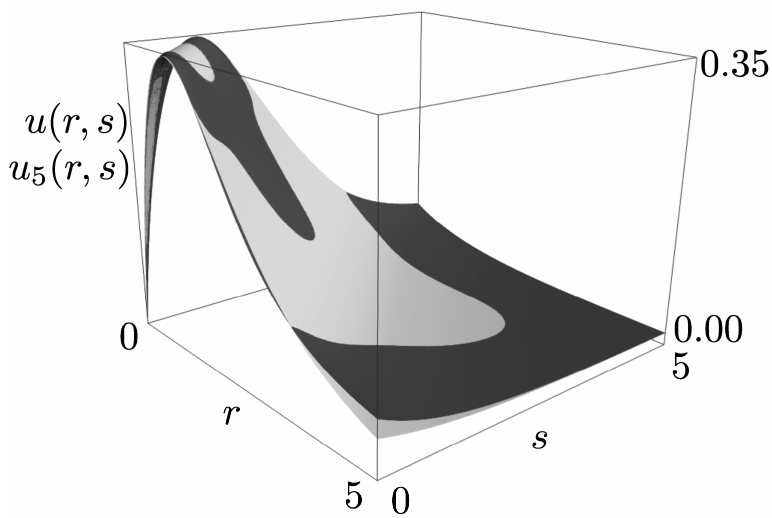

Figure 3. Comparison of the exact solution $u(r, s)$ with the estimate $u_5(r, s)$ for Example 3. The function $u_5(x, t)$ is a truncated solution to the bi-dimensional finite generalized Stieltjes moment problem obtained by transforming the linear K-G by means of Green identity. Dark grey: exact; light grey: estimation.

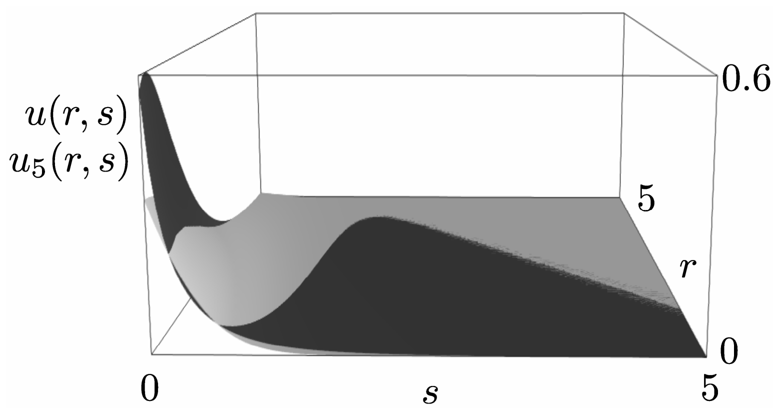

Figure 4. Comparison of the exact solution $u(r, s)$ with the estimate $u_5(r, s)$ for Example 4. The function $u_5(x, t)$ is a truncated solution to the bi-dimensional finite generalized Stieltjes moment problem obtained by transforming the nonlinear K-G by means of Green identity. Dark grey: exact; light grey: estimation.

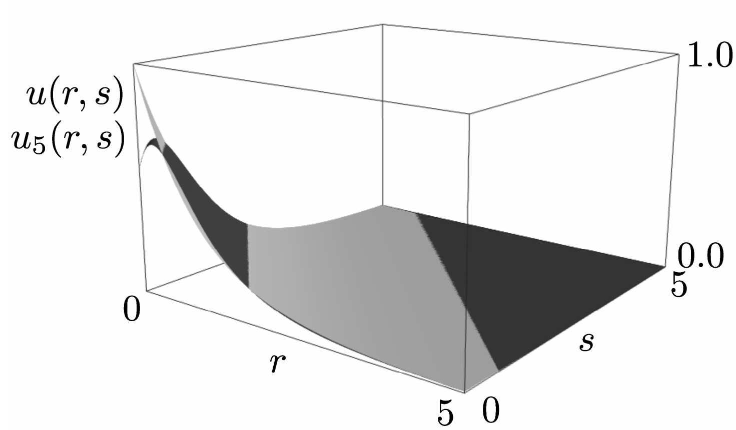

Figure 5. Comparison of the exact solution $u(r, s)$ with the estimate $u_5(r, s)$ for Example 5. The function $u_5(x, t)$ is a truncated solution to the bi-dimensional finite generalized Stieltjes moment problem obtained by transforming the nonlinear K-G by means of Green identity. Dark grey: exact; light grey: estimation.

References

[1] S. M. El-Sayed, The decomposition method for studying the Klein-Gordon equation, Chaos Solitons Fractals 18 (2003), 1025-1030.

[2] D. Kaya and S. M. El-Sayed, A numerical solution of the Klein-Gordon equation and convergence of the decomposition method, Appl. Math. Comput. 156 (2004), 341-353.

[3] F. Shaken and M. Dehghan, Numerical solution of the Klein-Gordon equation via He’s variational iteration method, Nonlinear Dynam. 51 (2008), 89-97.

[4] M. Lynch, Large amplitude instability in finite difference approximation to the Klein-Gordon equation, Appl. Numer. Math. 31 (1999), 173-182.

[5] B.-Y. Guo, X. Li and L. Vázquez, A Legendre spectral method for solving the nonlinear Klein-Gordon equation, Comput. Appl. Math. 15 (1996), 19-36.

[6] X. Li and B.-Y. Guo, A Legendre spectral method for solving nonlinear Klein-Gordon equation, J. Comput. Math. 15 (1997), 105-126.

[7] M. Dehghan and A. Shokri, Numerical solution of the nonlinear Klein-Gordon equation using radial basis functions, J. Comput. Appl. Math. 230 (2009), 400-410.

[8] A. G. Bratsos, On the numerical solution of the Klein-Gordon equation, Numer. Methods Partial Difference Equations 25 (2009), 939-951.

[9] S. Jiménez and L. Vázquez, Analysis of four numerical schemes for a nonlinear Klein-Gordon equation, Appl. Math. Comput. 35 (1990), 61-94.

[10] D. D. Ang, R. Gorenflo, V. K. Le and D. D. Trong, Moment theory and some inverse problems in potential theory and heat conduction, Lectures Notes in Mathematics, Springer-Verlag, Berlin, 2002.

[11] M. B. Pintarelli and F. Vericat, Bi-dimensional inverse moment problems, Far East J. Math. Sci. (FJMS) 54(1) (2011), 1-23.

[12] M. B. Pintarelli and F. Vericat, Stability theorem and inversion algorithm for a generalized moment problem, Far East J. Math. Sci. (FJMS) 30(2) (2008), 253-274.

[13] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1972.

[14] G. Talenti, Recovering a function from a finite number of moments, Inverse Problems 3 (1987), 501-517.