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BAYESIAN INDEPENDENT TESTING IN A BIVARIATE WEIBULL MODEL

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Abstract

In this paper, we consider a two-components system in which the lifetimes have a bivariate Weibull distribution. We propose a Bayesian testing procedure for independence based on Bayes factor. We use a noninformative prior such as an improper prior for the parameters so that the prior is defined only up to arbitrary constant which affects the values of Bayes factors. Also, we compute the fractional Bayes factor (FBF) proposed by O'Hagan [13] to compensate for that arbitrariness. We compute the FBFs and select the highest posterior probabilities for the hypotheses, respectively. Additionally, we give a numerical example to illustrate our procedure. According to the results, FBF methodology for Bayesian independent testing is reasonable.

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1. Introduction

In many of the aforementioned studies of system component data, the component lifetimes were assumed to be statistically independent for the sake of simplicity in mathematical treatment. However, the assumption of independence is unrealistic because many systems of the component life lengths have a well-defined dependence structure. The reason is that a common cause failure or a similar environmental factor might cause statistical dependence between lifetimes of components (Esary and Proschan [7]). The usefulness of a bivariate Weibull (BVW) distribution can be visualized in many contexts, such as the times to the first and second failures of a repairable device, the breakdown times of dual generators in a power plant, or the survival times of the organs in a two-organ system (such as lungs or kidneys in the human body). Lu and Bhattacharyya [10, 11] considered some new construction of BVW distributions. Lu [8] derived Weibull extensions of the Freund and Marshall-Olkin bivariate exponential models. Lu [9] suggested Bayes parameter estimation for the BVW distribution and Cho et al. [5] derived independent test statistics for the BVW model.

In this paper, we assume that lifetimes of two-components system have a BVW distribution. We focus only on independent testing of the BVW model based on Bayes factor but Bayesian testing depends rather strongly on the prior distributions. Many statistical analyses are often required to appear objective, so the research on noninformative priors has grown enormously over recent years. However, noninformative priors are typically improper, so that such priors are defined only up to arbitrary constants which affect the values of Bayes factors. San Martini and Spezzaferri [14] and Berger and Pericchi [1] have made efforts to compensate for that arbitrariness. In particular, O’Hagan [13] proposed the FBF for the Bayesian testing problem. FBF approaches for Bayesian testing were studied by many authors. Cho [2] proposed a multiple comparisons procedure based on FBF for negative binomial populations. Cho [4] and Cho and Joe [6] studied Bayesian testing and multiple comparisons procedure for a bivariate exponential model based on censored data, respectively.

The goal in this paper is to propose a Bayesian testing procedure for independence in BVW model based on FBF. We use a noninformative prior such as an improper prior for the parameters so that such prior is defined only up to an arbitrary constant, which affects the values of Bayes factors. We compute the FBF to compensate for that arbitrariness, also we compute the posterior probabilities for the hypotheses, respectively. Finally, we give a numerical example to illustrate our procedure.

2. Preliminaries

Let random variable (X, Y) be lifetimes of two components that follow a BVW distribution with parameter $\Theta = (\xi_1, \xi_2, \xi_3, \psi)$. Then the joint probability density function of (X, Y) is given as

$$f(x, y : \xi_1, \xi_2, \xi_3, \psi)$$

$$= \begin{cases} \xi_1(\xi_2 + \xi_3)\psi^2 x^{\psi-1} y^{\psi-1} \exp[-\xi_1 x^{\psi} - (\xi_2 + \xi_3) y^{\psi}], \; 0 < x < y < \infty, \\ \xi_2(\xi_1 + \xi_3)\psi^2 x^{\psi-1} y^{\psi-1} \exp[-(\xi_1 + \xi_3) x^{\psi} - \xi_2 y^{\psi}], \; 0 < y < x < \infty, \\ \xi_2 \psi x^{\psi-1} \exp[-\xi x^{\psi}], \; 0 < x = y < \infty, \end{cases} \tag{1}$$

where $\xi_1, \xi_2, \xi_3, \psi > 0$ and $\xi = \xi_1 + \xi_2 + \xi_3$. Then random variables X and Y are independent if and only if $\xi_3 = 0$. Also, X and Y are symmetrically distributed if and only if $\xi_1 = \xi_2$.

Suppose that there are n two-component units under study and ith pair of the components have lifetimes (x_i, y_i) which have (1). Let $(x, y) = ((x_1, y_1), \ldots, (x_n, y_n))$ be observation of sample size n. Let $I(\cdot)$ be an indicator function and we define n_j $(j = 1, 2, 3)$ as follows:

$$n_1 = \sum_{i=1}^{n} I(x_i < y_i), \;\; n_2 = \sum_{i=1}^{n} I(y_i < x_i), \;\; n_3 = \sum_{i=1}^{n} I(x_i = y_i).$$

Then the likelihood function of the sample of size n is given by:

$$L(\Theta \mid (x, y)) = \xi_1^{n_1} \cdot \xi_2^{n_2} \cdot \xi_3^{n_3} \cdot (\xi_1 + \xi_3)^{n_2} \cdot (\xi_2 + \xi_3)^{n_1}$$

$$\cdot\, \psi^{2(n_1+n_2)+n_3} \cdot \left[\prod_{i=1}^{n} (x_i \cdot y_i)\right]^{(\psi-1)}$$

$$\cdot \exp\left[-\xi_1 \sum_{i=1}^{n} x_i^{\psi} - \xi_2 \sum_{i=1}^{n} y_i^{\psi} - \xi_3 \sum_{i=1}^{n} \max(x_i, y_i)^{\psi}\right]. \quad (2)$$

On the other hand, let $\pi_i^N(\Theta_i)$ be an improper prior distribution under H_i, $i = 1, 2$ usually written as $\pi_i^N(\Theta_i) \propto h_i(\Theta_i)$, where h_i is a function whose integral over the parameter space under H_i diverges. Formally, we can write $\pi_i^N(\Theta_i) = c_i h_i(\Theta_i)$, although the normalizing constant c_i does not exist, but treating it as an unspecified constant. The posterior probability that H_i is true is given as:

$$P(H_i \mid (x, y)) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} B_{ji}^N\right)^{-1}, \quad (3)$$

where p_i is the prior probability of H_i being true and B_{ji}^N the Bayes factor of H_j to H_i, is defined by

$$B_{ji}^N = \frac{\int_{\Theta_j} L(\Theta_j \mid (x, y)) \pi_j^N(\Theta_j) d\Theta_j}{\int_{\Theta_i} L(\Theta_i \mid (x, y)) \pi_i^N(\Theta_i) d\Theta_i}, \quad (4)$$

where $L(\Theta_i \mid (x, y))$ is the likelihood function under H_i, $i = 1, 2$. The posterior probabilities in (3) are then used to select the most plausible hypothesis.

Hence, the corresponding Bayes factor B_{ji}^{N} is indeterminate. To solve this problem, O'Hagan [13] proposed the FBF for Bayesian testing problem as follows, the FBF of model H_j to model H_i is:

$$B_{ji}^{F} = \frac{q_j(b, x, y)}{q_i(b, x, y)}, \tag{5}$$

where

$$q_i(b, x, y) = \frac{\int_{\Theta} L(\Theta_i \mid (x, y)) \cdot \pi_i^N(\Theta_i) d\Theta_i}{\int_{\Theta_i} L^b(\Theta_i \mid (x, y)) \cdot \pi_i^N(\Theta_i) d\Theta_i}$$

and b specifies a fraction of likelihood which is to be used as a prior density.

3. Bayesian Hypothesis Test

The goal here is to propose a Bayesian testing procedure for independence based on FBF. In this paper, we assume that ξ_1 is equal to ξ_2, that is, $\xi_1 = \xi_2 (\equiv \xi_0)$ so that the lifetimes of two components have equal failure rates. In addition, we assume that ψ is fixed. Hence, we set the hypothesis $H_1 : \xi_3 = 0$ vs. $H_2 : \text{not } H_1$. Here, let $\Theta_1 = \xi_0$ and $\Theta_2 = (\xi_0, \xi_3)$.

In this paper, we set the noninformative priors for $H_1 : \xi_3 = 0$ vs. $H_2 : \text{not } H_1$ by $\pi_1^N(\Theta_1) = \frac{1}{\xi_0}$ and $\pi_2^N(\Theta_2) = \frac{1}{\xi_0 \xi_3}$. To test the hypothesis of independence based on FBF, we need to compute (5). The likelihood function under $H_1 : \xi_3 = 0$ is given by (6):

$$L(\Theta_1 \mid (x, y)) = \xi_0^{2(n_1+n_2)} \cdot \psi^{2(n_1+n_2)} \cdot \left[\prod_{i=1}^{n} (x_i \cdot y_i)\right]^{(\psi-1)}$$
$$\cdot \exp\left[-\xi_0 \sum_{i=1}^{n} (x_i^{\psi} + y_i^{\psi})\right]. \tag{6}$$

Then $q_1(b, x, y)$ under $H_1 : \xi_3 = 0$ is given by (7):

$$q_1(b, x, y) = \frac{\int_{\Theta_1} L(\Theta_1 | (x, y))\pi_1^N(\Theta_1) d\Theta_1}{\int_{\Theta_1} L^b(\Theta_1 | (x, y))\pi_1^N(\Theta_1) d\Theta_1}$$

$$= \psi^{2(n_1+n_2)(1-b)} \cdot \left[\prod_{i=1}^{n}(x_i \cdot y_i)\right]^{(\psi-1)(1-b)}$$

$$\cdot \frac{\Gamma(2(n_1 + n_2)) \cdot b^{2b(n_1+n_2)}\left[\sum_{i=1}^{n}(x_i^{\psi} + y_i^{\psi})\right]^{2(n_1+n_2)(b-1)}}{\Gamma(2b(n_1 + n_2))}. \quad (7)$$

On the other side, the likelihood function under H_2 : not H_1 is given by (8):

$$L(\Theta_2 | (x, y)) = \xi_0^{n_1+n_2} \cdot \xi_3^{n_3} \cdot (\xi_0 + \xi_3)^{n_1+n_2}$$

$$\cdot \psi^{2(n_1+n_2)+n_3} \cdot \left[\prod_{i=1}^{n}(x_i \cdot y_i)\right]^{(\psi-1)}$$

$$\cdot \exp\left[-\xi_0 \sum_{i=1}^{n}(x_i^{\psi} + y_i^{\psi}) - \xi_3 \sum_{i=1}^{n} \max(x_i, y_i)^{\psi}\right]. \quad (8)$$

Then $q_2(b, x, y)$ under H_2 : not H_1 is given by (9):

$$q_2(b, x, y) = \frac{\int_{\Theta_2} L(\Theta_2 | (x, y))\pi_2^N(\Theta_2) d\Theta_2}{\int_{\Theta_2} L^b(\Theta_2 | (x, y))\pi_2^N(\Theta_2) d\Theta_2}$$

$$= \psi^{[2(n_1+n_2)+n_3](1-b)} \cdot \left[\prod_{i=1}^{n}(x_i \cdot y_i)\right]^{(\psi-1)(1-b)} \cdot \frac{A_1}{A_2}, \quad (9)$$

where

$$A_1 = \sum_{i=0}^{n_1+n_2} \frac{(n_1 + n_2)!}{i!(n_1 + n_2 - i)!} \cdot \frac{\Gamma(n_1 + n_2 + i)}{\left[\sum_{i=1}^{n} (x_i^{\psi} + y_i^{\psi})\right]^{n_1+n_2+i}}$$

$$\cdot \frac{\Gamma(n_1 + n_2 + n_3 - i)}{\left[\sum_{i=1}^{n} \max(x_i^{\psi} + y_i^{\psi})\right]^{n_1+n_2+n_3-i}}$$

and

$$A_2 = \int_0^{\infty}\int_0^{\infty} \xi_0^{b(n_1+n_2)-1}\xi_3^{bn_3-1}(\xi_0 + \xi_3)^{b(n_1+n_2)}$$

$$\cdot \exp\left(-\xi_0 b\sum_{i=1}^{n}(x_i^{\psi} + y_i^{\psi}) - \xi_3 b\sum_{i=1}^{n}\max(x_i^{\psi},\ y_i^{\psi})\right)d\xi_0 d\xi_3.$$

Therefore, the FBF of H_2 to H_1 is given by (10):

$$B_{21}^F = \psi^{n_3(1-b)} \cdot \frac{\Gamma(2b(n_1 + n_2)) \cdot \left[\sum_{i=1}^{n} (x_i^{\psi} + y_i^{\psi})\right]^{2(n_1+n_2)(1-b)}}{\Gamma(2(n_1 + n_2)) \cdot b^{2b(n_1+n_2)}} \cdot \frac{A_1}{A_2}. \quad (10)$$

Using FBF by (10), the posterior probability for hypothesis H_i, $i = 1, 2$ is computed by $P(H_i \mid (x,\ y)) = \left(\sum_{j=1}^{2} \frac{p_j}{p_i} B_{ji}^F\right)^{-1}$. Thus, we can select the hypothesis with highest posterior probability based on FBF by (3).

4. A Numerical Example and Concluding Remarks

In this section, we present a numerical example to illustrate for the proposed Bayesian procedure for independence testing $H_1 : \xi_2 = 0$ vs.

H_2 : not H_1 based on FBF. We take the prior probability of H_i being true, $p_i = 0.5,\ i = 1,\ 2.$

The sample of size 15 is simulated from the bivariate Weibull model with the parameters $\xi_0 = 3.0,\ \xi_3 = 2.0$ and $\psi = 1.$ Also, we set $b = 2/15.$ Then we note that the true hypothesis may be $H_2.$ The generated BVW data is given as Table 1:

Table 1. The generated data

i	x_i	y_i	i	x_i	y_i	i	x_i	y_i
1	0.1505	0.1505	6	0.3641	0.3122	11	0.2365	0.2365
2	0.1113	0.0125	7	0.0483	0.0483	12	0.2380	0.3842
3	0.2108	0.4129	8	0.0781	0.0781	13	0.0225	0.0393
4	0.0108	0.1677	9	0.8387	0.3974	14	0.0235	0.1141
5	0.1483	0.1483	10	0.0604	0.0604	15	0.0727	0.0008

For the above generated data, we can compute the FBF $B_{21}^F = 5.3538$ by (10). Also, we can obtain the posterior probability $P(H_2 \mid (x,\ y)) = 0.8426$ by (3). That is, there is strong evidence for H_2 in terms of the posterior probabilities based on FBF. Hence, we reject H_1 that the lifetimes of two-components system are independent.

Until now, we suggested Bayesian testing procedure for independence of a bivariate Weibull model based on FBF. According to the results, FBF methodology for Bayesian independent testing is reasonable. Also, an extension of the method to Bayesian testing problems for the other model would be accomplished straightforwardly. The research topics pertaining to the examination of its performance are worthy of study and are left as a future subject of research.

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