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COMPLETION FOR EQUIPPED POSETS

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Abstract

A. G. Zavadskij called completion to one of the algorithms of differentiation introduced by him to classify equipped posets of finite growth. In this paper, we describe the categorical properties of such an algorithm.

1. Introduction

This paper is the third part of a series of works written by the first author concerning the investigation of categorical properties of some algorithms of differentiation for equipped posets [3, 4]. Such algorithms are functors whose main goal is to reduce the dimension of the initial category.

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The first algorithm of differentiation (with respect to maximal point) was introduced by Nazarova and Roiter in 1972 [7]. The categorical properties of such an algorithm were given by Gabriel in 1973 [5]. We note that, the algorithm with respect to a maximal point was used successfully by Kleiner in 1972 to classify ordinary posets of finite representation type [6]. Soon afterwards, Zavadskij defined the algorithm of differentiation with respect to a suitable pair of points. This algorithm allows to Nazarova and Zavadskij in 1982 to give a criterion for the classification of ordinary posets of finite growth [8, 12, 17].

In the 1980's, the main goal of the investigation of the poset representation theory was to classify posets with additional structures. For example, Bondarenko and Zavadskij gave criteria to classify posets with an equivalence relation. Actually, they gave criteria for tame and finite growth representation type for this kind of posets. Such criteria were obtained with the help of the algorithms of differentiation DII-DV introduced previously by Bondarenko and Zavadskij [1, 14].

At the end of the 1990's, Zabarilo and Zavadskij introduced equipped posets and gave criteria to classify equipped posets of one parameter giving a complete description of their indecomposables [11]. The reader is referred to [3, 4, 10] to more precise historical details of the investigation of the algorithms of differentiation.

We recall that the main problem in the theory of the algorithms of differentiation for posets ordinary or with additional structures consists of describing its categorical properties. Such descriptions allow to investigate in more efficient way the Gabriel quiver and the Auslander-Reiten quiver of the corresponding categories. For example, Zavadskij obtained in 1990 the categorical properties of his algorithm of differentiation with respect to a suitable pair of points. Those results allowed him to analyze the structure of the Auslander-Reiten quiver of the category of representations of posets of finite growth representation type [13, 17]. In the same line of work, the first author and Zavadskij in 2006 gave in [2] the categorical properties of the algorithm of differentiation II for posets with involution [2]. More recently, the first author gave the categorical properties of the algorithm of differentiation VII for equipped posets [3]. However, in order to describe the structure of the Auslander-Reiten quiver for equipped posets of finite growth, it is also necessary to obtain such properties for algorithms VIII, IX and completion introduced by Zavadskij in 2003 (recall that, Zavadskij defined differentiations VII-XVII to classify equipped posets of tame and of finite growth representation type) [15, 16]. For this reason, in this paper, we have chosen the completion to investigate its categorical properties.

This paper is organized as follows: Basic notation and definitions concerning some suitable category of representations of equipped posets are included in Section 2. Actually, since the main definitions, notation and properties of morphisms are given by the first author in [3, 4], in this paper, we only give definitions and notation concerning the completion for equipped posets. Finally, in Section 3, we give the definition of the completion for equipped posets and describe its categorical properties.

2. Preliminaries

In the present section, we introduce equipped posets and categories of representations of this kind of posets.

2.1. The category of representations of equipped posets

In this subsection, we define equipped posets and the category of representations of this kind of posets [3, 4, 9, 15, 16].

A poset $(\mathcal{P}, \leq)$ is called *equipped* if all the order relations between its points $x \leq y$ are separated into strong $(\text{denoted } x \trianglelefteq y)$ and weak $(\text{denoted } x \preceq y)$ in such a way that

$$x \leq y \trianglelefteq z \text{ or } x \trianglelefteq y \leq z \text{ implies } x \trianglelefteq z, \tag{1}$$

i.e., a composition of a strong relation with any other relation is strong.

In general, relations $\trianglelefteq$ and $\preceq$ are not order relations. These relations are antisymmetric but not reflexive. In particular, $\preceq$ is not reflexive (meanwhile $\trianglelefteq$ is transitive) [9].

We let $x \leq y$ denote an arbitrary relation in an equipped poset $(\mathcal{P}, \leq)$. The order $\leq$ on an equipped poset $\mathcal{P}$ gives raise to the relations $\prec$ and $\lhd$ of *strict inequality*: $x \prec y$ $(\text{respectively, } x \lhd y)$ in $\mathcal{P}$ if and only if $x \preceq y$ $(\text{respectively, } x \unlhd y)$ and $x \neq y$.

A point $x \in \mathcal{P}$ is called *strong* (*weak*) if $x \unlhd x$ $(\text{respectively, } x \preceq x)$. These points are denoted $\circ$ $(\text{respectively, } \otimes)$ in diagrams. We also denote $\mathcal{P}^{\circ} \subseteq \mathcal{P}$ $(\text{respectively, } \mathcal{P}^{\otimes} \subseteq \mathcal{P})$ the subset of strong points (respectively, weak points) of $\mathcal{P}$. If $\mathcal{P}^{\otimes} = \varnothing$, then the equipment is *trivial* and the poset $\mathcal{P}$ is ordinary.

Remark 1. Note that if $x \preceq y$ in an equipped poset $(\mathcal{P}, \leq)$ and there exists $t \in \mathcal{P}$ such that $x \leq t \leq y$, then $x, y \in \mathcal{P}^{\otimes}$, $x \preceq t$ and $t \preceq y$. Otherwise, if $x \unlhd t$ or $t \unlhd y$, then by definition, it is obtained the contradiction $x \unlhd y$.

If $\mathcal{P}$ is an equipped poset and $a \in \mathcal{P}$, then the subsets of $\mathcal{P}$ denoted $a^{\vee}$, $a_{\wedge}$, $a^{\triangledown}$, $a_{\vartriangle}$, $a^{\blacktriangledown}$, $a_{\blacktriangle}$, $a^{\curlyvee}$ and $a_{\curlywedge}$ are defined in such a way that:

$$a^{\vee} = \{x \in \mathcal{P} \mid a \leq x\}, \quad a_{\wedge} = \{x \in \mathcal{P} \mid x \leq a\},$$

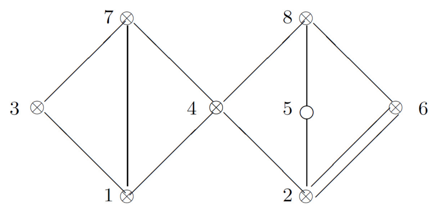
$$a^{\triangledown} = \{x \in \mathcal{P} \mid a \unlhd x\}, \quad a_{\vartriangle} = \{x \in \mathcal{P} \mid x \unlhd a\},$$

$$a^{\blacktriangledown} = a^{\vee} \setminus a, \quad a_{\blacktriangle} = a_{\wedge} \setminus a,$$

$$a^{\curlyvee} = \{x \in \mathcal{P} \mid a \preceq x\}, \quad a_{\curlywedge} = \{x \in \mathcal{P} \mid x \preceq a\}.$$

Subset $a^{\vee}$ $(\text{respectively, } a_{\wedge})$ is called the *ordinary upper* (respectively, *lower*) *cone*, associated to the point $a \in \mathcal{P}$ and subset $a^{\triangledown}(a_{\vartriangle})$ is called the *strong upper* (*lower*) *cone* associated to the point $a \in \mathcal{P}$, whereas subsets $a^{\blacktriangledown}$ and $a_{\blacktriangle}$ are called *truncated cones* (respectively, upper and lower) associated to the point $a \in \mathcal{P}$. In general, subsets $a^{\curlyvee}$ and $a_{\curlywedge}$ are not cones. Note that, if $x \in \mathcal{P}^{\circ}$, then $x^{\curlyvee} = x_{\curlywedge} = \varnothing$.

The diagram of an equipped poset $(\mathcal{P}, \leq)$ may be obtained via its Hasse diagram (with strong $(\circ)$ and weak points $(\otimes)$). In this case, a new line is added to the line connecting two points $x, y \in \mathcal{P}$ with $x \lhd y$ if and only if such relation cannot be deduced of any other relations in $\mathcal{P}$. In Figure 1, we show an example of this kind of diagrams:



$1^{\curlyvee} = \{1,3,4,8\}$ $\quad 1^{\triangledown} = \{7\}$
$2^{\curlyvee} = \{2,4,7\}$ $\quad 2^{\triangledown} = \{5,6,8\}$
$3^{\curlyvee} = \{3,7\}$ $\quad 3^{\triangledown} = \varnothing$
$4^{\curlyvee} = \{4,7,8\}$ $\quad 4^{\triangledown} = \varnothing$
$5^{\curlyvee} = \varnothing$ $\quad 5^{\triangledown} = \{5,8\}$
$6^{\curlyvee} = \{6,8\}$ $\quad 6^{\triangledown} = \varnothing$
$7^{\curlyvee} = \{7\}$ $\quad 7^{\triangledown} = \varnothing$
$8^{\curlyvee} = \{8\}$ $\quad 8^{\triangledown} = \varnothing$

Figure 1

For an equipped poset $(\mathcal{P}, \leq)$ and $A \subset \mathcal{P}$, we define the subsets, $A^{\triangledown}$, $A^{\curlyvee}$ and $A^{\vee}$ in such a way that

$$A^{\triangledown} = \bigcup_{a \in A} a^{\triangledown}, \quad A^{\curlyvee} = \bigcup_{a \in A} a^{\curlyvee}, \quad A^{\vee} = \bigcup_{a \in A} a^{\vee}.$$

Subsets $A_{\triangle}$, $A_{\curlywedge}$ and $A_{\wedge}$ are defined in the same way.

If $\mathcal{P}$ is an equipped poset, then a chain $C = \{c_i \in \mathcal{P} | 1 \leq i \leq n, c_{i-1} < c_i$ if $i \geq 2\} \subseteq \mathcal{P}$ is a *weak chain* if and only if $c_{i-1} \prec c_i$ for each $i \geq 2$. If $c_1 \prec c_n$, then we say that C is a *completely weak chain*. Moreover, a subset $X \subset \mathcal{P}$ is *completely weak* if $X = X^{\otimes}$ and weak relations are the only relations between points of X. Often, we let $\{c_1 \prec c_2 \prec \cdots \prec c_n\}$ denote a weak chain which consists of points $c_1, c_2, ..., c_n$. An ordinary chain C is denoted in the same way (by using the corresponding symbol $<$).

For an equipped poset $\mathcal{P}$ and $A, B \subset \mathcal{P}$, we write $A < B$ if $a < b$ for each $a \in A$ and $b \in B$. Notation $A \prec B$ and $A \lhd B$ are assumed in the same way.

Let $F \subset G$ be an arbitrary quadratic field extension with $G = F(\mathbf{u})$ for some fixed element $\mathbf{u} \in G$. Then each element $x \in G$ can be written uniquely in the form $\alpha + \mathbf{u}\beta$ with $\alpha, \beta \in F$ in this case (analogously to the case $(F, G) = (R, \mathbb{C}))\alpha$ is called the *real* part of *x* and β is the corresponding *imaginary* part of *x*.

The complexification of a real vector space can be generalized to the case (F, G), where $G = F(\mathbf{u})$ is a quadratic extension of *F*. In this case, we assume that $\mathbf{u}$ is a root of the minimal polynomial $t^2 + \mu t + \lambda, \ \ \lambda \neq 0$ $(\lambda, \mu \in F)$. In particular, if U_0 is an *F*-space, then the corresponding complexification is the *G*-vector space also denoted $U_0^2 = \widetilde{U}_0$. As in the case $(\mathbb{R}, \mathbb{C})$, we write $U_0^2 = U_0 + \mathbf{u}U_0 = \widetilde{U}_0$.

To each *G*-subspace *W* of $\widetilde{U}_0$, it is possible to associate the following *F*-subspaces of U_0,

$$W^+ = \operatorname{Re} W = \operatorname{Im} W \text{ and } W^- = gen\{\alpha \in U_0 \mid (\alpha, 0)^t \in W\} \subset W^+.$$

For a *G*-space *W*, we let $\widetilde{W}^+ = F(W)$ denote the *F-hull* of *W* such that $W \subset F(W)$.

If *Y* is an *F*-subspace of U_0 and $X = \widetilde{Y}$, then $X^+ = X^- = Y$. Therefore, *Y* is an *F*-form of *X*.

Remark 2. Any *G*-subspace *W* of $\widetilde{U}_0$ can be written as a direct sum of *G*-subspaces, $W = \widetilde{W}^- \oplus H$, where *H* is a complement of $\widetilde{W}^-$ in *W*. Therefore, $H^+ \simeq W^+/W^-$.

For each $x \in \mathcal{P}$, we let $\underline{U_x}$ denote the *radical subspace* of U_x such that $\underline{U_x} = \sum_{z \lhd x} F(U_z) + \sum_{z \prec x} U_z$.

If the field G is a quadratic extension of a field F, then a *representation of an equipped poset* over the pair (F, G) is a system of the form

$$U = (U_0; U_x \,|\, x \in \mathcal{P}), \tag{2}$$

where U_0 is a finite dimensional F-space and for each $x \in \mathcal{P}$, U_x is a G-subspace of $\tilde{U}_0$ such that

$$x \leq y \Rightarrow U_x \subset U_y,$$

$$x \trianglelefteq y \Rightarrow F(U_x) \subset U_y.$$

Remark 3. Note that, since $x \trianglelefteq x$ whenever $x \in \mathcal{P}^\circ$, $U_x \subset F(U_x) \subset U_x$. Therefore, if $x \in \mathcal{P}^\circ$, then $F(U_x) = U_x$.

We let rep $\mathcal{P}$ denote the category whose objects are the representations of an equipped poset $\mathcal{P}$ over a pair of fields (F, G). In this case, a morphism $\varphi : (U_0; U_x \,|\, x \in \mathcal{P}) \to (V_0; V_x \,|\, x \in \mathcal{P})$ between two representations U and V is an F-linear map $\varphi : U_0 \to V_0$ such that

$$\tilde{\varphi}(U_x) \subset V_x, \text{ for each } x \in \mathcal{P},$$

where $\tilde{\varphi} : \tilde{U}_0 \to \tilde{V}_0$ is the complexification of φ, i.e., the application G-linear induced by φ and defined in such a way that if $z = x + \mathbf{u}y \in \tilde{U}_0$, then $\tilde{\varphi}(z) = \varphi^2(z) = \varphi(x) + \mathbf{u}\varphi(y)$. The composition between morphisms of rep $\mathcal{P}$ is defined in a natural way and the sum $U \oplus V \in \text{rep}\,\mathcal{P}$ is defined as for ordinary posets. Therefore, rep $\mathcal{P}$ is a Krull-Schmidt category.

If $\mathcal{P}$ is an equipped poset and $U, V \in \text{rep}\,\mathcal{P}$, then U is a *sub-representation* of V if and only if spaces U_0, V_0, U_x and V_x satisfy the inclusions $U_0 \subset V_0$ and $U_x \subset V_x$ for each $x \in \mathcal{P}$.

Two representations $U, V \in \text{rep}\,\mathcal{P}$ are said to be *isomorphic* if and only if there exists an F-isomorphism $\varphi : U_0 \to V_0$ such that $\tilde{\varphi}(U_x) = V_x$, for each $x \in \mathcal{P}$.

The *main problem* dealing with equipped posets consists of classifying its indecomposable representations up to isomorphisms.

Each equipped poset $\mathcal{P}$ naturally defines a matrix problem of mixed type over the pair (F, G). Consider a rectangular matrix M separated into vertical stripes M_x, $x \in \mathcal{P}$, with M_x being over F (over G) if the point x is strong (weak):

$$\begin{array}{lcccc} & \multicolumn{2}{c}{x \to y} & & \\ M = & \otimes & \otimes & \bigcirc & \bigcirc \;, \\ \cline{2-5} & \multicolumn{1}{|c|}{G} & \multicolumn{1}{c|}{G} & \multicolumn{1}{c|}{F} & \multicolumn{1}{c|}{F} \\ \cline{2-5} \end{array}$$

such partitioned matrices M are called *matrix representations* of $\mathcal{P}$ over (F, G). Their *admissible transformations* are as follows:

(a) F-elementary row transformations of the whole matrix M;

(b) F-elementary (G-elementary) column transformations of a stripe M_x if the point x is strong (weak);

(c) In the case of a weak relation $x \prec y$, additions of columns of the stripe M_x to the columns of the stripe M_y with coefficients in G;

(d) In the case of a strong relation $x \lhd y$, independent additions both real and imaginary parts of columns of the stripe M_x to real and imaginary parts (in any combinations) of columns of the stripe M_y with coefficients in F (assuming that, for y strong, there are no additions to the zero imaginary part of M_y).

Two representations are said to be *equivalent* or *isomorphic* if they can be turned into each other with help of the admissible transformations. The corresponding *matrix problem* of mixed type over the pair (F, G) consists of classifying the indecomposable in the natural sense matrices M, up to equivalence.

Remark 4. The matrix problem for representations (a)-(d) occurs naturally in the classification of the objects $U \in \operatorname{rep} \mathcal{P}$ up to isomorphisms. In this case, it is associated to the representation U its matrix presentation $M_U = (M_x;\ x \in \mathcal{P})$ defined as follows:

If a point $x \in \mathcal{P}^0$ (respectively, $\mathcal{P}^{\otimes}$), then the columns of the stripe M_x consist of coordinates (with respect to a fixed ordered basis $\mathcal{B}$ of U_0) of a system of generators $\mathcal{G}$ of U_x^+ (respectively, G-subspace U_x) modulo its radical subspace $\underline{U_x^+}$ (respectively, $\underline{U_x}$). Problem (a)-(d) may be obtained by changing basis $\mathcal{B}$ and the system of generators $\mathcal{G}$.

If $X \subset \mathcal{P}$, $U \in \operatorname{rep} \mathcal{P}$, then the subspaces of U_0 denoted U_x, U_X^+, $\widehat{U}_X$ and $(\widehat{U}_X)^-$ are defined in such a way that:

$$U_X = \sum_{x \in X} U_x, \quad U_X^+ = \sum_{x \in X} U_x^+,$$

$$\widehat{U}_X = \bigcap_{x \in X} U_x, \quad (\widehat{U}_X)^- = \bigcap_{x \in X} U_x^-. \tag{3}$$

We also assume that

$$U_{\varnothing} = 0, \quad \widehat{U}_{\varnothing} = U_0. \tag{4}$$

The *dimension* of a representation $U \in \operatorname{rep} \mathcal{P}$ is a vector d such that $d = \underline{\dim}\, U = (d_0;\ d_x \,|\, x \in \mathcal{P})$, where $d_0 = \dim_F U_0$ and $d_x = \dim_G U_x / \underline{U_x}$. A representation $U \in \operatorname{rep} \mathcal{P}$ is *sincere* if $d_0 \neq 0$ and $d_x \neq 0$ for each $x \in \mathcal{P}$. In other words, the vector d of a sincere representation U has not null coordinates.

2.2. Some indecomposable objects

In this subsection, we give some examples of indecomposable objects in the category rep $\mathcal{P}$, where $\mathcal{P}$ is an equipped poset.

If $\mathcal{P}$ is an equipped poset and $A \subset \mathcal{P}$, then $P(A) = P(\min A) = (F;\, P_x \,|\, x \in \mathcal{P})$, $P_x = G$ if $x \in A^{\vee}$ and $P_x = 0$ otherwise. In particular, $P(\varnothing) = (F;\, 0, \ldots, 0)$.

If $a, b \in \mathcal{P}^{\otimes}$, then $T(a)$ and $T(a, b)$ denote indecomposable objects with matrix representation of the following form:

$$T(a) = \begin{array}{c} a \\ \boxed{\begin{array}{c} 1 \\ \mathbf{u} \end{array}} \end{array}, \; a \in \mathcal{P}^{\otimes}, \quad T(a, b) = \begin{array}{cc} a & b \\ \multicolumn{2}{c}{\boxed{\begin{array}{c|c} 1 & 0 \\ \mathbf{u} & 1 \end{array}}} \end{array}, \text{ with } a \prec b.$$

If we consider the notation (2) for objects in rep $\mathcal{P}$, then the object $T(a)$ may be described in such a way that $T(a) = (T_0;\, T_x \,|\, x \in \mathcal{P})$, where $T_0 = F^2$ and

$$T_x = \begin{cases} \tilde{T}_0 = G^2, & \text{if } x \in a^{\triangledown}, \\ G\{(1, \mathbf{u})^t\}, & \text{if } x \in a^{\curlyvee}, \\ 0, & \text{otherwise,} \end{cases}$$

where $(1, \mathbf{u})^t$ is the column of coordinates with respect to an ordered basis of T_0.

On the other hand, representation $T(a, b)$ may be described in such a way that $T(a, b) = (T_0;\, T_x \,|\, x \in \mathcal{P})$, where $T_0 = F^2$ and

$$T_x = \begin{cases} G\{(1, \mathbf{u})^t\}, & \text{if } a \preceq x \prec b, \\ \tilde{T}_0 = G^2, & \text{if } x \in a^{\triangledown} \bigcup b^{\vee}, \\ 0, & \text{otherwise.} \end{cases}$$

If $a \in \mathcal{P}^{\otimes}$ and $B \subset \mathcal{P}$ is a subset completely weak such that $a \prec B$, then we let $T(a, B)$ denote the representation of $\mathcal{P}$ which satisfies the following conditions with $T_0 = F^2$:

$$T_x = \begin{cases} G\{(1, \mathbf{u})^t\}, & \text{if } x \in a^{\curlyvee} \setminus B, \\ \tilde{T}_0 = G^2, & \text{if } x \in a^{\triangledown} + B^{\vee}, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $T(a, \varnothing) = T(a)$.

Remark 5. In [15], it is proved that $P(\varnothing)$, $P(c_i)$, $T(c_i)$ and $T(c_i, c_j)$, for $1 \le i < j \le n$ are the only indecomposable representations (up to isomorphisms) over the pair $(\mathbb{R}, \mathbb{C})$ of a completely weak chain $C = \{c_1 \prec \cdots \prec c_n\}$. In fact, if $U = (U_0; U_{c_i} \mid 1 \le i \le n)$ is a representation of C over $(\mathbb{R}, \mathbb{C})$, then in the corresponding matrix representation each block U_{c_i}, $1 \le i \le n$, may be reduced via admissible transformations to the following standard form:

$$U_{c_i} = \begin{array}{|c|c|} \hline I & \\ \hline & I \\ \hline & \mathbf{i}I \\ \hline & \\ \hline \end{array}$$

where the columns consist of generators of U_{c_i} modulo its radical subspace $\underline{U_{c_i}} = U_{c_{i-1}}$ with respect to a fixed basis of U_0 (in this case, empty cells indicate null coordinates). This result can be generalized in a natural way to the case (F, G) by using a suitable scalar $\mathbf{u} \in G$ instead of the constant $\mathbf{i} \in \mathbb{C}$ in the matrix presentation of U_{c_i} showed above.

If $X \subset U_0$, $Y \subset V_0$ are corresponding subspaces of the finite dimensional k-vector spaces U_0 and V_0, then $[X, Y]$ is a subspace of $\mathrm{Hom}_k(U_0, V_0)$ such that

$$\varphi \in [X, Y] \text{ if and only if } X \subset \mathrm{Ker}\,\varphi \text{ and } \mathrm{Im}\,\varphi \subset Y.$$

Note that if $X' \subset X$ and $Y \subset Y'$, then $[X, Y] \subset [X', Y']$.

The following results concerning linear maps and morphisms of categories of representations of equipped posets were proved in [2] and [3]. In this case, for a category $\mathcal{A}$, we let $\langle U_i \mid i \in I \rangle_{\mathcal{A}}$ denote the ideal consisting of all morphisms passed through finite direct sums of the objects U_i. That is, if $\varphi : U \to V \in \langle U_i \mid i \in I \rangle_{\mathcal{A}}$, then there exist morphisms $f, g \in \mathcal{A}$ such that $\varphi = U \xrightarrow{f} \bigoplus_i U_i^{m_i} \xrightarrow{g} V$ with $m_i = 0$ for almost all i.

Lemma 6. *If* $\varphi \in [X, Y]$ *and* $\varphi(X') \subset Y'$, *then*

$$\varphi \in [X + X', Y] + [X, Y \cap Y'].$$

Lemma 7. *Let U and V be two representations of an equipped poset* $\mathcal{P} = a^{\triangledown} + b_{\vartriangle} + \{a \prec X \prec c\}$, *where* $a, c \in \mathcal{P}^{\otimes}$, $b \in \mathcal{P}^{\circ}$ *is a strong point incomparable with a and c*, $\{a \prec X \prec c\}$ *is a completely weak set containing an arbitrary set X* (*eventually empty*). *Then for an F-linear map* $\varphi : U_0 \to V_0$, *we have the following equivalences*:

(a) $\varphi \in {}_U\langle T(a) \rangle_V \Leftrightarrow \varphi \in [(U_b + U_c)^-, V_a^+], \tilde{\varphi}(U_c) \subset V_a$,

(b) $\varphi \in {}_U\langle T(a, c) \rangle_V \Leftrightarrow \varphi \in [(U_b + U_{a+X})^-, V_a^+ \cap V_c^-], \quad \tilde{\varphi}(U_c) \subset \tilde{V}_a^+ \cap \tilde{V}_c^-, \ \tilde{\varphi}(U_{a+X}) \subset V_a \cap \tilde{V}_c^-$,

(c) $\varphi \in {}_U\langle P(a) \rangle_V \Leftrightarrow \varphi \in [U_b^+, V_a^-]$.

Corollary 8. *Let U and V be representations of an equipped poset* $\mathcal{P} = a^{\triangledown} + b_{\vartriangle} + \{a \prec c_1 \prec \cdots \prec c_n\}$, *where* $\{a \prec c_1 \prec \cdots \prec c_n\}$ *is a completely weak chain incomparable with the strong point b. Then for an F-linear map*, $\varphi : U_0 \to V_0$, *we have the following equivalences if* $1 \le i \le n$ $(U_{c_0} = U_a)$:

(a)

$$\varphi \in {}_U\langle T(a, c_i) \rangle_V \Leftrightarrow \varphi \in [(U_b + U_{c_{i-1}})^-, V_a^+ \cap V_{c_i}^-],$$

$$\tilde{\varphi}(U_{c_n}) \subset \tilde{V}_{c_i}^-, \tilde{\varphi}(U_{c_i^{\curlyvee}}) \subset \tilde{V}_a^+ \cap \tilde{V}_{c_i}^-, \tilde{\varphi}(U_{c_{i-1}}) \subset V_a \cap \tilde{V}_{c_i}^-,$$

(b) $\varphi \in {}_U\langle P(a)\rangle_V \Leftrightarrow \varphi \in [(U_b + U_{c_n})^-, V_a^+], \tilde{\varphi}(U_{c_n}) \subset V_a.$

Remark 9. Note that if $\varphi \in [(U_b + U_{c_{i-1}})^-, V_a^+ \cap V_{c_i}^-]$ in Corollary 8, item (a), then the condition $\tilde{\varphi}(U_{c_n}) \subset \tilde{V}_a^+ \cap \tilde{V}_{c_i}^- = V_a \cap \tilde{V}_{c_i}^-$, follows if $V_a = F(V_a)$. In the same way, the condition $\tilde{\varphi}(U_{c_n}) \subset V_a$ in item (b) follows if $V_a = F(V_a)$ and $\varphi \in [(U_b + U_{c_n})^-, V_a^+]$.

Remark 10. If a finite dimensional F-space $U_0 = \{e_t \,|\, t \in J\}$ and an F-subspace $K \subset U_0$ are such that for a fixed ordered basis subspace K has the form $K = F\{e_t \,|\, t \in I \subset J\}$, then we let $\alpha = \alpha_t K$ denote to a vector $\alpha \in K$ such that $\alpha = \sum_{t \in I} \alpha_t e_t$. Indices for scalar numbers α_t depend of the order given to the basis of the subspace K.

If a G-subspace $H \subset \tilde{U}_0$, is such that $H^- = 0$ and for a fixed ordered basis, we have that $H = G\{e_{i_t} + \mathbf{u}e_{j_t} \,|\, e_{i_t}, e_{j_t} \in U_0, t \in I, i \in I', j \in I''\}$, I, I' and I'' suitable sets of indices, then we write:

$$H^1 = F\{e_{i_t} \,|\, t \in I, i \in I'\},$$

$$H^2 = \{e_{j_t} \,|\, t \in I, j \in I''\}.$$

Therefore, $F(H) = \tilde{H}^1 \oplus \tilde{H}^2$. Thus, if a vector $\alpha \in F(H)$, then $\alpha = \alpha_{i_t}\tilde{H}^1 + \alpha_{j_t}\tilde{H}^2$, where $\alpha_{i_t}, \alpha_{j_t} \in G$. In particular, if $\alpha_{i_t} = \alpha_{j_t}$, for each $t \in I$, then the vector $\beta = \alpha_{i_t}\tilde{H}^1 + \mathbf{u}\alpha_{j_t}\tilde{H}^2 \in H$, may be also written in the forms $\beta = \alpha_{i_t} H = \alpha_{i_t}(\tilde{H}^1 + \mathbf{u}\tilde{H}^2)$.

3. The Completion

In this section, we give the categorical properties of the completion for equipped posets which is a special differentiation introduced by Zavadskij in order to classify equipped posets of finite growth [15, 16].

If $\mathcal{P}$ is an equipped poset, then a pair of comparable weak points (a, b) with $a \prec b$ is *special* provided $\mathcal{P}$ can be written in the form $\mathcal{P} = a^{\triangledown} + b_{\vartriangle} + \Sigma$, where Σ is the *interior* completely weak of the interval $[a, b] = \{x \in \mathcal{P} \mid a \leq x \leq b\} \subseteq \mathcal{P}$.

The completion of poset $\mathcal{P}$ with respect to the special pair (a, b) is the transition from $\mathcal{P}$ to an equipped poset $\overline{\mathcal{P}}_{(a,b)} = \overline{\mathcal{P}}$ obtained from $\mathcal{P}$ by strengthening relation $a \prec b$ in $\mathcal{P}$. In such a case, it is obtained a new strong relation of the form $a \lhd b$. Figure 2 shows a diagram for this differentiation.

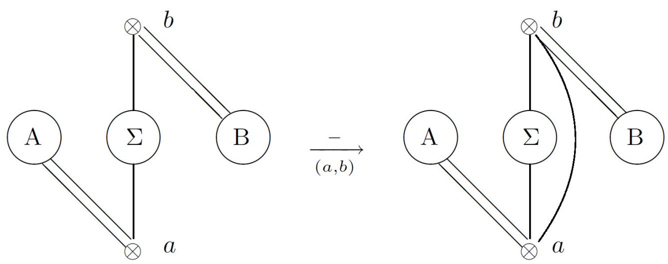


Figure 2

Note that, if U is a representation of an equipped poset $\mathcal{P}$ (over the pair of fields (F, G)), then the corresponding subspace U_a can be written in the form $U_a = \tilde{U}_a^- \oplus R_a$ and $R_a = M_a \oplus N_a$, where $U_a^- \oplus M_a^+ = U_a^+ \bigcap U_b^-$.

We let $C^{(a,b)}$ denote the first version of the *completion functor* defined in such a way that $C^{(a,b)} : \operatorname{rep}\mathcal{P} \to \operatorname{rep}\overline{\mathcal{P}}$ in this case, each object $U \in \operatorname{rep}\mathcal{P}$ is applied to the object $\overline{U} \in \operatorname{rep}\overline{\mathcal{P}}$ such that

$$\overline{U}_0 = U_0,$$

$$\overline{U}_b = U_b + \tilde{N}_a^+ = U_b + F(U_a),$$

$\overline{U}_x = U_x$ for remaining points $x \in \overline{\mathcal{P}}$,

$\overline{\varphi} = \varphi,$ for a linear map-morphism $\varphi : U_0 \to V_0.$ (5)

For example, $\overline{T(a, b)} = \overline{T(a)} = T(a).$

The following lemma was proved by Zavadskij in [15].

Lemma 11. *Category* $rep\,\overline{\mathcal{P}}_{(a,b)}$ *is a full subcategory of the category* $rep\,\mathcal{P}$ *which consists of objects without direct summands in the class* $[T(a)]$ *of the indecomposable* $T(a)$, *therefore*

$$Ind\,\overline{\mathcal{P}} = Ind\,\mathcal{P} \backslash [T(a)].$$

The following theorem allows to obtain an equivalence between quotient categories of $\operatorname{rep}\mathcal{P}$ and $\operatorname{rep}\overline{\mathcal{P}}$.

Theorem 12. *The completion functor* $C^{(a,b)}$ *induces the following equivalence between quotient categories*

$$rep\,\mathcal{P}/\langle T(a), T(a, b)\rangle \xrightarrow{\sim} rep\,\overline{\mathcal{P}}/\langle T(a)\rangle.$$

Proof. We let R, $\overline{R}$ denote categories $\operatorname{rep}\mathcal{P}$ and $\operatorname{rep}\overline{\mathcal{P}}$, respectively. In the same way, we let $\Theta = \langle T(a), T(a, b)\rangle_R$, $\overline{\Theta} = \langle T(a)\rangle_{\overline{R}}$ denote the ideals consisting of morphisms of the corresponding categories passing through direct sums of indecomposable $T(a)$, $T(a, b)$ and $T(a)$, respectively. Therefore, for each pair of objects $U, V \in R$, the following inclusions hold, $\Theta(U, V) \subset \overline{\Theta}(\overline{U}, \overline{V}) \subset \overline{R}(\overline{U}, \overline{V})$, $\Theta(U, V) \subset R(U, V) \subset \overline{R}(\overline{U}, \overline{V})$. Thus, in order to obtain the result, it is enough to prove the following identities:

$$R(U, V) \cap \overline{\Theta}(\overline{U}, \overline{V}) = \Theta(U, V) \quad \text{and} \quad R(U, V) + \overline{\Theta}(\overline{U}, \overline{V}) = \overline{R}(U, V).$$

Note that, Lemma 7 (applied to poset $\mathcal{P}$) allows to conclude that for each pair of morphisms $f, g \in R(U, V)$ and $h \in \overline{R}(\overline{U}, \overline{V})$, it holds:

$$f \in [U_b^-, V_a^+], \quad \tilde{f}(U_b) \subset V_a \Leftrightarrow f \in \langle T(a) \rangle_R,$$

$$g \in [\underline{U_b^-}, V_a^+ \cap V_b^-], \quad \tilde{g}(U_b) \subset \tilde{V}_a^+ \cap \tilde{V}_b^-,$$

$$\tilde{g}(U_{a^\curlyvee \backslash b}) \subset V_a \cap \tilde{V}_b^- \Leftrightarrow g \in \langle T(a, b) \rangle_R,$$

$$h \in [\underline{U_b^-}, V_a^+], \quad \tilde{h}(U_b) \subset \tilde{V}_a^+, \quad \tilde{h}(U_{a^\curlyvee}) \subset V_a \Leftrightarrow h \in \langle T(a) \rangle_{\overline{R}}.$$

On the other hand, if $\varphi \in R(U, V) \cap \overline{\Theta}(\overline{U}, \overline{V})$, then $\tilde{\varphi}(U_x) \subset V_x$ if $x \in \mathcal{P}$, in particular $\varphi(U_b^-) \subset V_a^+ \cap V_b^-$. If we apply Lemma 6 to morphism φ, then $\varphi \in [\underline{U_b^-}, V_a^+ \cap V_b^-] + [\underline{U_b^-} + U_b^-, V_a^+]$. Since $[\underline{U_b^-} + U_b^-, V_a^+] = [U_b^-, V_a^+]$, $\varphi \in [\underline{U_b^-}, V_a^+ \cap V_b^-] + [U_b^-, V_a^+]$. Therefore, $\varphi \in \Theta(U, V)$. Because, it is possible to write φ in the form $\varphi = \varphi_1 + \varphi_2$, with $\varphi_1, \varphi_2 \in \Theta(U, V)$.

If $\psi \in \overline{R}(\overline{U}, \overline{V})$, then $\tilde{\psi}(U_x) \subset V_x$ for all $x \in \mathcal{P} \backslash b$ and $\tilde{\psi}(U_b) \subset V_b + \tilde{N}_a^+$. Furthermore, $\tilde{\psi}(U_b \cap \underline{\tilde{U}_b^-}) \subset \tilde{\psi}(\underline{\tilde{U}_b^-}) \subset \underline{\tilde{V}_b^-} \subset V_b$. Since the inclusion $\tilde{\psi}(U_x) \subset V_x$ for $x \in b_\curlywedge \cap \mathcal{P}^\otimes$ does not hold in general, $\psi \in R(U, V)$ is not always true.

The following notation and definitions are necessary to define a morphism $w \in \overline{\Theta}(\overline{U}, \overline{V})$ such that $\psi - w \in R$.

For each $x \in b_\curlywedge \cap \mathcal{P}^\otimes$, it holds $U_x = \underline{U_x} \oplus D_x$, where D_x is a complementary of $\underline{U_x}$ in U_x. Furthermore, $D_x = \tilde{L}_x^- \oplus H_x$, $H_x = M_x \oplus N_x$, with $M_x = \sum_{y \in x^\curlyvee} M_{xy} \subset \tilde{U}_b^-$. Subspaces $M_{xy} \subset M_x$ are such that $M_{xy} = H_x \cap \tilde{U}_y^-$, for some $y \in x^\curlyvee$. Moreover, if $Z_{xy} = \{z \in x^\curlyvee \mid M_{xy} \subset \tilde{U}_z^-\}$, then $y \in \min Z_{xy}$. Actually, $U_b^- \cap U_x^+ = U_x^- \oplus M_x^+$.

If $U_b = \widetilde{U}_b^- \oplus T_b$, then for each $x \in b_{\curlywedge}$, we have $N_x \subset T_b = \sum_{x \in b_\blacktriangle \cap \mathcal{P}^{\otimes}} N_x \oplus N_b$ and $N_x^+ \cap \underline{U_b^-} = 0$.

$$U_x^- = \left(U_x^- \cap U_{b_\triangle}^+ + \sum_{y \in x_{\curlywedge} \setminus x} U_y^- + Z_x \right) \oplus Y_x,$$

where a vector $\alpha \in Z_x$ if $\alpha \in M_{yx}^+$, for some $y \in x_{\curlywedge} \setminus x$ and Y_x is a complementary in U_x^-. Therefore, U_b^- may be written in the form; $U_b^- = (\underline{U_b^-} + Z_b) \oplus Y_b$, with $Z_b = Z_b \cap \underline{U_b^-} \oplus Z_{b_*}$. Same notation, we use in representation V, by replacing for each $x \in \mathcal{P}^{\otimes}$, the symbol H_x instead of G_x. Also, by replacing notation M', N', T', Y', Z' instead of M, N, T, Y, Z, respectively.

We write $V_b = V_a \oplus L_{b_*}$, with $V_b^+ = V_a^+ + L_{b_*}^+$. In fact, we suppose $L_{b_*} = \widetilde{L}_{b_*^a}^- \oplus \widetilde{L}_{b_*^0}^- \oplus G_{b_*}$, $\widetilde{L}_{b_*}^- = \widetilde{L}_{b_*^a}^- \oplus \widetilde{L}_{b_*^0}^-$, $L_{b_*^a}^+ = V_a^+ \cap L_{b_*}^+$, further $L_{b_*^0}^+ \cap V_a^+ = G_{b_*}^+ \cap V_a^+ = 0$. Therefore, $V_b^+ = N_a'^+ \oplus G_{b_*}'^+$, with $G_{b_*}'^+ = V_a^- \oplus M_a'^+ \oplus L_{b_*^0} \oplus G_{b_*}^+$.

We consider $V_0 = V_b^+ \oplus X_0$ and subspace $W_0' = L_{b_*}^+ \oplus X_0$, for a complementary X_0, in such a way that the pair of subspaces, $(N_a'^+, W_0')$ is a $(V_a^+, L_{b_*}^+)$-cleaving in V_0. Furthermore, since $T_b^+ = N_{b_\blacktriangle}^+ \oplus N_b^+$, $N_{b_\blacktriangle}^+ \cap M_x^+ = 0$, for all $x \in b_\blacktriangle$. If $U_0 = U_b^+ \oplus X_0$, with X_0 a complementary, $U_b^+ = U_{b_\blacktriangle}^+ \oplus Y_b \oplus N_b^+$ and $U_{b_\blacktriangle}^+ = Z_{b_*} \oplus X_b$, X_b a complementary. Thus, $U_b^+ = Z_{b_*} \oplus X_b \oplus Y_b \oplus N_b^+$ and $U_0 = Z_{b_*} \oplus X_b \oplus Y_b \oplus N_b^+ \oplus X_0$.

If $X_1 = Z_{b_*}$, $X_2 = Y_b$, $X_3 = N_b^+$ and $e_s \in X_i$ is a basic vector for some $i \in \{1, 2, 3\}$, then by using adequate notation for linear combinations in subspaces N'_a, G'_{b_*} (see Remark 10), we write

$$\psi(e_s) = \alpha_{a_n} N_a'^{+} + \alpha_{g_m} G_{b_*}'^{+}.$$

If the basic vectors $e_i, e_j \in T_b^+$, $\alpha = e_i + \mathbf{u}e_j \in N_b$ and

$$\psi(e_i) = \alpha_{a_n} N_a'^{+} + \alpha_{g_m} G_{b_*}'^{+},$$

$$\psi(e_j) = \alpha'_{a_n} N_a'^{+} + \alpha'_{g_m} G_{b_*}'^{+},$$

then $(\alpha_{g_m} + \mathbf{u}\alpha'_{g_m})\tilde{G}_{b_*}'^{+} \in V_b$. On the other hand, if $\alpha \in M_x \cap \tilde{U}_b^-$, for some $x \in b_{\curlywedge}$, then $(\alpha_{g_m} + \mathbf{u}\alpha'_{g_m})\tilde{G}_{b_*}'^{+} \in \tilde{V}_b^-$.

If $w_t : U_0 \to V_0$ is the F-linear map such that $w_t(e_s) = \alpha_{a_n} N_a'^{+}$, if $e_s \in X_t$, $w_t = 0$ otherwise for each $t \in \{1, 2, 3\}$ and we write $w = w_1 + w_2 + w_3$, then $w \in [\underline{U_b^-}, V_a^+]$. In fact, $w \in \langle T(a)\rangle_{\overline{R}}$.

Furthermore,

$$(\tilde{\psi} - \tilde{w})(\tilde{U}_b^-) \subset \tilde{V}_b^-,$$

$$(\tilde{\psi} - \tilde{w})(U_x \cap \tilde{X}_b) = \tilde{\psi}(U_x \cap \tilde{X}_b) \subset \tilde{\psi}(U_x) \subset V_x, \text{ if } x \in b_{\blacktriangle} \cap b_{\curlywedge},$$

in this case, if $\alpha = e_i + \mathbf{u}e_j \in M_{xb}$, with $e_i, e_j \in X_1$, then $(\tilde{\psi} - \tilde{w})(\alpha) = (\alpha_{g_m} + \mathbf{u}\alpha'_{g_m})\tilde{G}_{b_*}'^{+} \in \tilde{V}_b^- \cap V_x \subset V_x$. Therefore, $(\tilde{\psi} - \tilde{w})(U_x) \subset V_x$.

If $x \in a^{\triangledown}$, then we have $(\tilde{\psi} - \tilde{w})(U_x) \subset V_x + F(V_a) \subset V_x$. If $x \in b_{\triangle}$, then $(\tilde{\psi} - \tilde{w})(U_x) = \tilde{\psi}(U_x) \subset V_x$.

If $e_i, e_j \in X_3$ and $\alpha = e_i + \mathbf{u}e_j \in N_b$, then $(\tilde{\psi} - \tilde{w})(\alpha) = (\alpha_{g_m} + \mathbf{u}\alpha'_{g_m})\tilde{G}'^{+}_{b_*} \in V_b$. Thus, $(\tilde{\psi} - \tilde{w})(U_b) \subset V_b$ and $\psi - w \in R(U, V)$.

Since morphism $w \in \overline{\Theta}(\overline{U}, \overline{V})$, we conclude $\psi \in R(U, V) + \overline{\Theta}(\overline{U}, \overline{V})$ and with this fact, we are done. □

Corollary 13. *If* $\Gamma(R)$ *and* $\Gamma(\overline{R})$ *are Gabriel quivers of categories* $R = \operatorname{rep}\mathcal{P}$ *and* $\overline{R} = \operatorname{rep}\overline{\mathcal{P}}$, *respectively, then* $\Gamma(R)\backslash[T(a), T(a, b)] \simeq \Gamma(\overline{R})\backslash[T(a)]$.

The *second version of the completion functor* $C_{(a,b)}$ for an equipped poset with a special pair of points (a, b) is defined in the following form:

$C_{(a,b)} : \operatorname{rep}\mathcal{P} \to \operatorname{rep}\overline{\mathcal{P}}$ assigns to each object $U \in \operatorname{rep}\mathcal{P}$, the object $\overline{U} \in \operatorname{rep}\overline{\mathcal{P}}$ such that

$$\overline{U}_0 = U_0,$$

$$\overline{U}_a = \tilde{U}_b^- \cap U_a = \tilde{U}_a^- \oplus M_a,$$

$$\overline{U}_x = U_x \text{ for remaining points } x \in \operatorname{rep}\overline{\mathcal{P}},$$

$$\overline{\varphi} = \varphi, \text{ for a linear map-morphism } \varphi : U_0 \to V_0. \tag{6}$$

For example, $\overline{T(a)} = \overline{T(a^{\blacktriangledown})} = T(a^{\blacktriangledown})$, where $T(a^{\blacktriangledown})$ is the representation of $\mathcal{P}$ (over (F, G)) such that:

$$T(a^{\blacktriangledown}) = (T_0, T_x \mid x \in \mathcal{P}), T_0 = F^2, T_x = G\{(1, \mathbf{u})^t\}$$

if $x \in a^{\curlyvee} \backslash a$ and $(1, \mathbf{u})^t$ is a vector of coordinates with respect to an ordered fixed basis of T_0, $T_x = \tilde{T}_0 = G^2$, if $x \in a^{\triangledown}$, $T_x = 0$ otherwise.

Lemma 11 and the following theorem allow us to obtain a relationship between categories $\operatorname{rep}\mathcal{P}$ and $\operatorname{rep}\overline{\mathcal{P}}$ via the functor $C_{(a,b)}$:

Theorem 14. *The functor* $C_{(a,b)}$ *induces the following equivalence between quotient categories*:

$$\operatorname{rep}\mathcal{P}/\langle T(a), T(a^{\blacktriangledown})\rangle \xrightarrow{\sim} \operatorname{rep}\overline{\mathcal{P}}/\langle T(a^{\blacktriangledown})\rangle.$$

Proof. Let $R = \operatorname{rep}\mathcal{P}$, $\overline{R} = \operatorname{rep}\overline{\mathcal{P}}$, $\Delta = \langle T(a), T(a^{\blacktriangledown})\rangle_R$ and $\overline{\Delta} = \langle T(a^{\blacktriangledown})\rangle_{\overline{R}}$. Thus, for each pair of objects $U, V \in \operatorname{rep}\mathcal{P}$, the following inclusions hold, $\Delta(U, V) \subset \overline{\Delta}(\overline{U}, \overline{V}) \subset \overline{R}(\overline{U}, \overline{V})$, $\Delta(U, V) \subset R(U, V) \subset \overline{R}(\overline{U}, \overline{V})$. Therefore, it is enough to prove:

$$R(U, V) \cap \overline{\Delta}(\overline{U}, \overline{V}) = \Delta(U, V) \text{ and } R(U, V) + \overline{\Delta}(\overline{U}, \overline{V}) = \overline{R}(U, V).$$

Lemma 7 allows to conclude the following identities for linear maps $f, g \in R(U, V)$, $h \in \overline{R}(\overline{U}, \overline{V})$ and $D^{+}_{V_a} = \left(\bigcap_{x \in a^{\curlyvee} \setminus a} V^{+}_{x}\right) \cap \left(\bigcap_{y \in a^{\triangledown}} V^{-}_{y}\right)$:

$$f \in \langle T(a)\rangle_R \Leftrightarrow f \in [U^{-}_{b}, V^{+}_{a}], \quad \tilde{f}(U_b) \subset V_a,$$

$$g \in \langle T(a^{\blacktriangledown})\rangle_R \Leftrightarrow g \in [U^{+}_{a} + U^{-}_{b}, D^{+}_{V_a}], \tilde{g}(U_b) \subset \bigcap_{x \in a^{\curlyvee} \setminus a} V_x,$$

$$h \in \langle T(a^{\blacktriangledown})\rangle_{\overline{R}} \Leftrightarrow h \in [U^{-}_{b}, D^{+}_{V_a}], \quad \tilde{h}(U_b) \subset \bigcap_{x \in a^{\curlyvee} \setminus a} V_x.$$

If $\varphi \in R(U, V) \cap \overline{\Delta}(\overline{U}, \overline{V})$, then $\tilde{\varphi}(U_x) \subset V_x$ if $x \in \mathcal{P}$. In particular, $\varphi(U^{+}_{a}) \subset V^{+}_{a}$.

Lemma 6 allows us to write $\varphi \in [U^{-}_{b} + U^{+}_{a}, D^{+}_{V_a}] + [U^{-}_{b}, D^{+}_{V_a} \cap V^{+}_{a}]$. Since $[U^{-}_{b}, D^{+}_{V_a} \cap V^{+}_{a}] = [U^{-}_{b}, V^{+}_{a}]$, $\varphi \in [U^{-}_{b}, V^{+}_{a}] + [U^{-}_{b} + U^{+}_{a}, D^{+}_{V_a}]$. Since there exist morphisms $\varphi_1, \varphi_2 \in \Delta(U, V)$ such that $\varphi = \varphi_1 + \varphi_2$, we conclude $\varphi \in \Delta(U, V)$.

If $\psi \in \overline{R}(\overline{U}, \overline{V})$, then $\tilde{\psi}(U_x) \subset V_x$, for all $x \in \mathcal{P} \backslash a$. Furthermore, $\tilde{\psi}(\tilde{U}_b^- \cap U_a) \subset \tilde{V}_b^- \cap V_a \subset V_a$, since in general, $\tilde{\psi}(U_a) \not\subset V_a$ not necessarily $\psi \in R(U, V)$.

For each $x \in a^{\curlyvee}$, we can write $U_x = \underline{U_x} \oplus K_x$, $K_x = \tilde{K}_x^- \oplus H_x$, $H_x = M_x \oplus N_x$, where $\alpha \in M_x$ if $\alpha \in \tilde{U}_b^-$, $U_b^- \cap U_x^+ = U_x^- \oplus M_x^+$, $N_x^+ \cap U_{b_\vartriangle}^+ = 0$.

$U_x^- = \underline{U_x^-} \oplus P_x$ with $P_x = Y_x \oplus Z_x$ and $\alpha \in Z_x$ if $\alpha \in M_y^+ \cap U_x^-$, for some $y \in x_\curlywedge \backslash x$, we write $U_0 = U_a^+ + U_b^- \oplus Y_0$ and $W_0 = U_b^- \oplus Y_0$.

For the representation V, we have $\left(\bigcap\limits_{x \in a^{\curlyvee} \backslash a} V_x \right) \cap \left(\bigcap\limits_{y \in a^{\triangledown}} \tilde{V}_y^- \right) =$ $V_a \oplus X_1$, $V_a = \tilde{V}_a^- \oplus M_a \oplus N_a$, $X_1 = \tilde{X}_a^- \oplus \tilde{X}_0^- \oplus G_{x_0}$. With $\tilde{X}_1^- = \tilde{X}_a^- \oplus \tilde{X}_0^-$, $X_a^+ \subset X_1^+ \cap V_a^+$, $X_0^+ \cap V_a^+ = G_{x_0}^+ \cap V_a^+ = 0$ and $X_0 = \tilde{X}_0^- \oplus G_{x_0}$, $V_0 = (X_1^+ + V_a^+) \oplus Y_0'$ and $W_0' = V_a^+ \oplus Y_0'$, Y_0 and Y_0' are corresponding complementary subspaces in U_0 and V_0.

Let (N_a^+, W_0) be a (U_a^+, U_b^-)-cleaving in U_0 and (X_0^+, W_0') be the (X_1^+, V_a^+)-cleaving in V_0.

If $e_s \in N_a^+$ is a basic vector, then by definition

$$\psi(e_s) = \alpha_{l_i^a} V_a^+ + \alpha_{q_j^b} X_0^+.$$

If e_i, e_j are basic vectors of N_a^+, with $\alpha = e_i + \mathbf{u} e_j \in N_a$ and

$$\psi(e_i) = \alpha_{l_i^a} V_a^+ + \alpha_{q_j^b} X_0^+, \quad \psi(e_j) = \alpha'_{l_i^a} V_a^+ + \alpha'_{q_j^b} X_0^+,$$

then

$$\widetilde{\psi}(\alpha) = (\alpha_{l_i^a} + \mathbf{u}\alpha'_{l_i^a})V_a + (\alpha_{q_j^b} + \mathbf{u}\alpha'_{q_j^b})X_0 \in V_a \oplus X_0.$$

Therefore, if $w : U_0 \to V_0$ is the F-linear map such that $w(W_0) = 0$ and for $e_s \in N_a^+$, $w(e_s) = \alpha_{q_j^b} X_0^+$, according with the notation for the image under ψ of $N(a)$. Thus, $w \in [U_b^-, D_{V_a}^+]$ because $X_1^+ + V_a^+ \subset D_{V_a}^+$. In fact, $w \in \langle T(a^{\blacktriangle})\rangle_{\overline{R}}$.

It holds that if $\alpha = e_i + \mathbf{u}e_j \in N_a$, then $(\widetilde{\psi} - \widetilde{w})(\alpha) = (\alpha_{l_i^a} + \mathbf{u}\alpha'_{l_i^a})V_a \in V_a$.

If $x \in a^{\triangledown}$, then $(\widetilde{\psi} - \widetilde{w})(U_x) \subset \widetilde{D}_{V_a}^+ + V_x \subset V_x$, in case $x \in b_{\vartriangle}$, we have $(\widetilde{\psi} - \widetilde{w})(U_x) = \widetilde{\psi}(U_x) \subset V_x$, by definition $(\widetilde{\psi} - \widetilde{w})(\widetilde{U}_b^- \cap U_a) \subset \widetilde{V}_b^- \cap V_a \subset V_a$.

If $x \in (a^{\curlyvee} \setminus a)$, then we can write U_x in the form $U_x = U_a \oplus U_{x_*}$, con $U_{x_*}^+ \cap N_a^+ = 0$, thus $(\widetilde{\psi} - \widetilde{w})(U_x) \subset V_a + V_x \subset V_x$, because $\widetilde{w}(U_{x_*}) = 0$, since U_{x_*} is a complementary of U_a in U_x. Since $\varphi = \psi - w \in R(U, V)$, we conclude $\psi = \varphi + w \in R(U, V) + \overline{\Delta}(\overline{U}, \overline{V})$ and with this fact, we are done. □

Remark 15. Morphism w in the proof of Theorem 14 may be constructed in such a way that if $e_{N_a^+} \in \mathrm{End}_F U_0 = \mathrm{End}_F(N_a^+ \oplus W_0)$, $e_{X_0^+} \in \mathrm{End}_F V_0 = \mathrm{End}_F(X_0^+ \oplus W_0')$ are the corresponding splitting idempotents of summands N_a^+ of U_0 and X_0^+ of V_0. Then $w = e_{X_0^+}\psi e_{N_a^+}$.

As in the case for Theorem 12, we have the following corollary for the second version of the completion functor:

Corollary 16. *If* $\Gamma(R)$ *and* $\Gamma(\overline{R})$ *are Gabriel quivers of categories* $R = \operatorname{rep}\mathcal{P}$ *and* $\overline{R} = \operatorname{rep}\overline{\mathcal{P}}$, *respectively, then* $\Gamma(R)\backslash[T(a), T(a^{\blacktriangledown})] \simeq \Gamma(\overline{R})\backslash[T(a^{\blacktriangledown})]$.

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