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REPRESENTING THICK INDIFFERENCE IN SPATIAL MODELS

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Abstract

The fuzzy set approach to modeling thick indifference can accommodate highly irregular shaped indifference curves, even those that are concave or multi-modal. We show that its ability to do so owes to a homomorphism that permits a region of interest (spatial model) to be mapped to a simpler region with a suitable and natural partial ordering where the results are determined and then faithfully transferred back to the original region of interest. We then prove that in all but a limited number of cases, spatial models of individual preferences of thick indifference result in an empty majority rule maximal set if and only if the Pareto set contains a union of cycles.

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1. Introduction

It has long been known that the probability of a majority rule maximal set increases when actors possess thick indifference over individual preferences [12, 13, 11, 5, 6, 1, 3, 4]. Many of the studies in this genre make use of the concept of an epsilon-core (ε-core) [13], a threshold distance in Euclidean space that must be exceeded before players distinguish between alternatives (see [3, 6]). Unless an alternative lies outside of the region defined by the ε-core, a player is indifferent between it and the core's center. Actors are essentially indifferent to alternatives in close proximity. Unfortunately, applying the approach in empirical analyses is hampered by the complexity of calculating the existence of a majority rule maximal set. It is even more problematic when thick indifference introduces irregularly shaped preference curves.

We showed in [7] that the fuzzy set approach to modeling thick indifference can accommodate highly irregular shaped indifference curves, even those that are concave or multi-modal. Its ability to do so owes to a homomorphism that permits a region of interest (spatial model) to be mapped to a simpler region with a suitable and natural partial ordering where the results are determined and then faithfully transferred back to the original region of interest. These results in [7] were stated without proof. In this paper, we present a formal proof of these results. We then prove that in all but a limited number of cases, spatial models of individual preferences of thick indifference result in an empty majority rule maximal set if and only if the Pareto set contains a union of cycles, Theorem 3.14.

Before proceeding, we provide some definitions and set forth notation needed for the paper. Let N denote a finite set of players and X denote a set of alternatives. If R is a binary relation on X, we let P denote the strict preference relation associated with R, i.e., $P = \{(x, y) \in R \mid (y, x) \notin R\}$. Let $\mathcal{R}$ denote the set of all binary relations on X that are reflexive, complete, and transitive. Let $\mathcal{R}^n = \{\boldsymbol{\rho} \mid \boldsymbol{\rho} = (R_1, ..., R_n), R_i \in R, i = 1, ..., n\}$, where $n =$

$|N|$. Let $\boldsymbol{\rho} \in R^n$. Then the Pareto set of $\boldsymbol{\rho}$ is defined to be $PS_N(\boldsymbol{\rho}) = \{x \in X \mid \forall y \in Y(\exists i \in N),\ yP_i x \Rightarrow \exists j \in N,\ xP_j y\}$. If $R \in \mathcal{R}$, the maximal set of R with respect to a subset S of X is defined to be $M(R, S) = \{x \in S \mid \forall y \in S,\ xRy\}$. Define the binary relation R on X by $\forall x, y \in X$, $(x, y) \in R$ if and only if $|\{i \in N \mid xR_i y\}| \geq n/2$. Let $R(x, y; \boldsymbol{\rho}) = \{i \in N \mid xR_i y\}$ and $P(x, y; \boldsymbol{\rho}) = \{i \in N \mid xP_i y\}$. Then $(x, y) \in P$ if and only if $|P(x, y; \boldsymbol{\rho})| > n/2$. For this R, $M(R, X)$ is called the *majority rule maximal set*. An *aggregation rule* is a function from $\mathcal{R}^n$ into $\mathcal{B}$, where $\mathcal{B}$ is the set of all binary relations on X which are reflexive and complete.

2. Relation Spaces and Majority Rule

Let R be a binary relation on a set X, i.e., R is a subset of $X \times X$. The pair (X, R) is called a *relation space*. If $(\mathcal{A}, \widetilde{R})$ and (X, R) are relation spaces, we give conditions when results from (X, R) can be faithfully carried back from (X, R) to $(\mathcal{A}, \widetilde{R})$ by the preimage of a homomorphism of $(\mathcal{A}, R)$ onto (X, R). In particular, we give conditions involving the maximal sets and the Pareto sets of $(A, \widetilde{R})$ and (X, R), Theorems 2.4, 2.6 and 2.7.

Definition 2.1. Let $(\mathcal{A}, \widetilde{R})$ and (X, R) be relation spaces. Let f^* be a function of $\mathcal{A}$ into X. Then f^* is called a *homomorphism* of $(\mathcal{A}, \widetilde{R})$ into (X, R) if $\forall a, b \in \mathcal{A}$, $(a, b) \in \widetilde{R}$ if and only if $(f^*(a), f^*(b)) \in R$. If f^* maps $\mathcal{A}$ onto X, we say f^* maps $(\mathcal{A}, \widetilde{R})$ onto (X, R). For all $(a, b) \in \widetilde{R}$, we write $f^*((a, b)) = (f^*(a), f^*(b))$ and $f^*(\widetilde{R}) = \{f^*((a, b)) | (a, b) \in \widetilde{R}\}$.

Let f^* be a *homomorphism* of $(\mathcal{A}, \widetilde{R})$ into (X, R). Then $\forall a, b \in \mathcal{A}$, $(a, b) \in \widetilde{R}$ if and only if $(f^*(a), f^*(b)) \in R$. Thus if $a, a', b, b' \in \mathcal{A}$ and $f^*(a) = f^*(a')$, $f^*(b) = f^*(b')$, it is not possible that $(a, b) \in \widetilde{R}$ and $(a', b') \notin \widetilde{R}$.

Proposition 2.2. *Let* f^* *be a homomorphism of* $(\mathcal{A}, \widetilde{R})$ *onto* (X, R). *Then* $f^*(\widetilde{R}) = R$.

Proof. Clearly, $f^*(\widetilde{R}) \subseteq R$. Let $(x, y) \in R$. Since f^* maps $\mathcal{A}$ onto X, there exists $a, b \in \mathcal{A}$ such that $f^*(a) = x$ and $f^*(b) = y$. Thus $(x, y) = (f^*(a), f^*(b)) = f^*((a, b)) \in f^*(\widetilde{R})$.

Proposition 2.3. *Let* f^* *be a homomorphism of* $(\mathcal{A}, \widetilde{R})$ *onto* (X, R). *Then* $\forall a, b \in \mathcal{A}, (a, b) \in \widetilde{P}$ *if and only if* $(f^*(a), f^*(b)) \in P$.

Proof. Let $a, b \in \mathcal{A}$. Then $(a, b) \in \widetilde{P} \Leftrightarrow (a, b) \in \widetilde{R}, \; (b, a) \notin \widetilde{R} \Leftrightarrow (f^*(a), f^*(b)) \in R, (f^*(a), f^*(b)) \notin R \Leftrightarrow (f^*(a), f^*(b)) \in P$.

Theorem 2.4. *Let* f^* *be a homomorphism of* $(\mathcal{A}, \widetilde{R})$ *onto* (X, R). *Then* $f^*(M(\widetilde{R}, \mathcal{A})) = M(R, X)$. *Furthermore,* $f^{*-1}(M(R, X)) = M(\widetilde{R}, \mathcal{A})$.

Proof. $a \in M(\widetilde{R}, \mathcal{A}) \Leftrightarrow \forall b \in \mathcal{A}, a\widetilde{R}b \Leftrightarrow \forall f^*(b) \in X, \; f^*(a)Rf^*(b) \Leftrightarrow f^*(a) \in M(R, X)$, where the latter equivalence holds since f^* maps $\mathcal{A}$ onto X. Thus if $f^*(a) \in f^*(M(\widetilde{R}, \mathcal{A}))$, then $a \in M(\widetilde{R}, \mathcal{A})$. Hence $f^*(a) \in M(R, X)$. Thus $f^*(M(\widetilde{R}, \mathcal{A})) \subseteq M(R, X)$. Let $x \in M(R, X)$. Then $\forall y \in X, xRy$. Let $a \in \mathcal{A}$ be such that $f^*(a) = x$. Let $b \in \mathcal{A}$. Then $f^*(a)Rf^*(b)$ since $x = f^*(a)$ and $x \in M(R, X)$. Hence $a\widetilde{R}b$ by Definition 2.1. Thus $a \in M(\widetilde{R}, \mathcal{A})$ and so $x = f^*(a) \in f^*(M(\widetilde{R}, \mathcal{A}))$. Thus $M(R, X) \subseteq f^*(M(\widetilde{R}, \mathcal{A}))$.

Clearly, $f^{*-1}(M(R, X)) \supseteq M(\widetilde{R}, \mathcal{A})$. Let $a \in f^{*-1}(M(R, X))$. Suppose $\exists b \in \mathcal{A}$ such that $(a, b) \notin \widetilde{R}$. Then $(f^*(a), f^*(b)) \notin R$ since f^* is a homomorphism. Thus $f^*(a) \notin M(R, X)$, a contradiction of $a \in$

$f^{*-1}(M(R, X))$. Hence $(a, b) \in \widetilde{R}$, $\forall b \in \mathcal{A}$. Thus $a \in M(\widetilde{R}, \mathcal{A})$. Hence $f^{*-1}(M(R, X)) \subseteq M(\widetilde{R}, \mathcal{A})$.

Let $(\mathcal{A}, \widetilde{R}_i)$ be a relation space, $i = 1, ..., n$. Let f_i^* be a homomorphism of $(\mathcal{A}, \widetilde{R}_i)$ onto (X, R_i), $i = 1, ..., n$. Then $R_i = f_i^*(\widetilde{R}_i)$, $i = 1, ..., n$ by Proposition 2.2.

Definition 2.5. Let $\widetilde{f}$ be an aggregation rule on $(\mathcal{A}, (\widetilde{R}_1, ..., \widetilde{R}_n))$ and let f be an aggregation rule on $(X, (R_1, ..., R_n))$. Let f_i^* be a homomorphism of $(\mathcal{A}, \widetilde{R}_i)$ onto (X, R_i), $i = 1, ..., n$. Let f^* be a homomorphism of $(\mathcal{A}, \widetilde{f}((\widetilde{R}_1, ..., \widetilde{R}_n)))$ onto $(X, f((R_1, ..., R_n)))$. Then f^* is said to *preserve the pair* $(\widetilde{f}, f)$ *with respect to* $(f_1^*, ..., f_n^*)$ if $f^*(\widetilde{f}(\widetilde{R}_1, ..., \widetilde{R}_n)) = f((R_1, ..., R_n))$.

Theorem 2.6. *Let* $\widetilde{f}$ *be an aggregation rule on* $(\mathcal{A}, (\widetilde{R}_1, ..., \widetilde{R}_n))$ *and let f be an aggregation rule on* $(X, (R_1, ..., R_n))$. *Let* f_i^* *be a homomorphism of* $(\mathcal{A}, \widetilde{R}_i)$ *onto* (X, R_i), $i = 1, ..., n$. *Let* f^* *be a homomorphism of* $(\mathcal{A}, \widetilde{f}((\widetilde{R}_1, ..., \widetilde{R}_n)))$ *onto* $(X, f((R_1, ..., R_n)))$ *such that* f^* *preserves* $(\widetilde{f}, f)$ *w.r.t.* $(f_1^*, ..., f_n^*)$. *Then* $f^*(PS_N(\widetilde{R})) = PS_N(R)$, *where* $\widetilde{R} = \widetilde{f}((\widetilde{R}_1, ..., \widetilde{R}_n))$ *and* $R = f((R_1, ..., R_n))$. *Furthermore*, $f^{*-1}(PS_N(\widetilde{R})) = PS_N(R)$.

Proof. $a \in PS_N(\widetilde{R}) \Leftrightarrow \forall b \in \mathcal{A}$ $(\exists i \in N, b\widetilde{P}a \Rightarrow \exists j \in N, a\widetilde{P}b)$. Thus if $f^*(a) \in f^*(PS_N(\widetilde{R}))$, then $a \in PS_N(\widetilde{R})$. Hence $f^*(a) \in PS_N(R)$. Thus $f^*(PS_N(\widetilde{R})) \subseteq PS_N(R)$. Let $x \in PS_N(R)$. Let $y \in X$. If $\exists i \in N$ such that yP_ix, then $\forall j \in N$ such that xR_jy. Let $a \in \mathcal{A}$ be such that $f^*(a) = x$. Let $b \in \mathcal{A}$. Then $f^*(b)P_i f^*(a) \Leftrightarrow b\widetilde{P}_ia$ and $f^*(a)P_j f^*(b) \Leftrightarrow a\widetilde{P}_jb$. Thus if $\exists i \in N$ such that $b\widetilde{P}_ia$, then $\exists j \in N$ such that $a\widetilde{P}_jb$.

Thus $a \in PS_N(\widetilde{R})$ and so $x = f^*(a) \in f^*(PS_N(\widetilde{R}))$. Hence $PS_N(R) \subseteq f^*(PS_N(\widetilde{R}))$.

Clearly, $f^{*-1}(PS_N(R)) \supseteq PS_N(\widetilde{R})$. Let $a \in f^{*-1}(PS_N(R))$. Suppose $a \notin PS_N(\widetilde{R})$. Then it is not the case that $\forall b \in A$, $\exists i \in N$, $b\widetilde{R}_i a \Rightarrow \exists j \in N$, $a\widetilde{P}_j b$. Thus $\exists b \in \mathcal{A}$ such that it is not the case that $\exists i \in N$, $b\widetilde{R}_i a \Rightarrow \exists j \in N$, $a\widetilde{P}_j b$. Thus if $\exists i \in N$ such that $b\widetilde{P}_i a$, then it is not the case that $\exists j \in N$ such that $a\widetilde{P}_j b$ and so $b\widetilde{R}_j a$, $\forall j \in N$. Hence $(b, a) \in \widetilde{R}_i$, $\forall i \in N$ and so $(f^*(a), f^*(b)) \in R_i$, $\forall i \in N$. Thus $(b, a) \in \widetilde{R}_i$, $\forall i \in N$ and so $(f^*(b), f^*(a)) \in R_i$, $\forall i \in N$. Hence $(f^*)(a) \notin PS_N(R)$ which contradicts the fact that $a \in f^{*-1}(PS_N(R))$. Thus $a \in PS_N(\widetilde{R})$ and so $f^{*-1}(PS_N(\widetilde{R})) \subseteq PS_N(R)$.

Theorem 2.7. *Let $\widetilde{f}$ be an aggregation rule on $(\mathcal{A}, (\widetilde{R}_1, ..., \widetilde{R}_n))$ and let f be an aggregation rule on $(X, (R_1, ..., R_n))$. Let f_i^* be a homomorphism of $(\mathcal{A}, \widetilde{R}_i)$ onto (X, R_i), $i = 1, ..., n$. Let f^* be a homomorphism of $(\mathcal{A}, \widetilde{f}((\widetilde{R}_1, ..., \widetilde{R}_n)))$ onto $(X, f((R_1, ..., R_n)))$ such that f^* preserves $(\widetilde{f}, f)$ w.r.t. $(f_1^*, ..., f_n^*)$. Then $\widetilde{f}$ is a simple majority rule if and only if f is a simple majority rule.*

Proof. Since by Proposition 2.3, $\forall a, b \in \mathcal{A}$, $(a, b) \in \widetilde{P}_i$ if and only if $(f^*(a), f^*(b)) \in P_i$, $i = 1, ..., n$, it follows that $|\widetilde{P}(a, b; \widetilde{f}((\widetilde{R}_1, ..., \widetilde{R}_n)))| = |P(f^*(a), f^*(b)); f((R_1, ..., R_n))|$. The desired result now follows.

3. Majority Rule Maximal Sets

We now consider conditions under which a majority rule maximal set exists.

Let N denote the set of players and X denote the set of alternatives. We assume that X is a subset of a universe U of interest. Let $\mathcal{R}$ denote the set of all binary relations on X which are reflexive, complete and transitive. Let $\mathcal{R}^n = \{\boldsymbol{\rho} \mid \boldsymbol{\rho} = (R_1, ..., R_n), R_i \in \mathcal{R}, i = 1, ..., n\}$, where $|N| = n$. Let $\leq$ be a partial order on U. Suppose that $\leq$ satisfies the following properties:

(1) $\forall x, y \in U, x \leq y$ implies $\forall i \in N, yR_i x$;

(2) $\forall x, y, z \in U, \forall i \in N, x \leq y$ and $xR_i z$ implies $yR_i z$;

(3) $\forall x, y, z \in U, \forall i \in N, x \leq y$ and $xP_i z$ implies $yP_i z$;

(4) $\forall x, y \in U, x < y$ implies $\exists i \in N$ such that $yP_i x$;

(4′) $\forall x, y, z \in U, \forall i \in N, x \leq y$ and $zR_i y$ implies $zR_i x$;

(5) $\forall x, y \in U, x$ and y incomparable under $\leq$ implies $\exists i \in N$ such that $xP_i y$ implies $\exists j \in N$ such that $yP_j x$.

Let $\boldsymbol{\rho} \in R^n$. Let f be an aggregation rule.

Definition 3.1. Define the binary relation R on X by $\forall x, y \in X, (x, y) \in R$ if and only if $|\{i \in N \mid xR_i y\}| \geq n/2$. Define $P \subseteq X \times X$ by $\forall x, y \in X$, $(x, y) \in P$ if and only if $(x, y) \in R$ and $(y, x) \notin R$. Let $R(x, y; \boldsymbol{\rho}) = \{i \in N \mid xR_i y\}$ and $P(x, y; \boldsymbol{\rho}) = \{i \in N \mid xP_i y\}$.

Proposition 3.2. *Let* $x, y \in X$. *Then* $(x, y) \in P$ *if and only if* $|P(x, y; \boldsymbol{\rho})| > n/2$.

Proof. $xPy \Leftrightarrow xRy$ and not $yRx \Leftrightarrow |\{i \in N \mid xR_i y\}| \geq n/2$ and $|\{j \in N \mid yR_j x\}| < n/2$. Since each R_i is complete, R is complete. Hence $xPy \Leftrightarrow |\{i \in N \mid xP_i y\}| > n/2$ by a simple counting procedure. Thus $xPy \Leftrightarrow |P(x, y; \boldsymbol{\rho})| > n/2$.

Definition 3.3.

$$PS_N(R) = \{x \in X \mid \forall y \in X (\exists i \in N, yP_i x \Rightarrow \exists j \in N, xP_j y)\}.$$

Definition 3.4. $M(R, X) = \{x \in X \mid \forall y \in X, xRy\}$.

Definition 3.5. $M_R = \{x \in X \mid \nexists y \in X,\ x < y\}$.

Proposition 3.6. $M_R = PS_N(R)$.

Proof. Suppose $x \in M_R$. Let $y \in X$. Suppose $\exists i \in N$ such that yP_ix. Now there does not exist $y \in X$ such that $x < y$. Thus $\forall y \in X$, either $y \leq x$ or x and y are not comparable. Since yP_ix, $y \leq x$ is impossible else xR_iy, $\forall i \in N$ by (1). Hence x and y are incomparable under $\leq$. Thus $\exists j \in N$ such that xP_jy by (5). Hence $x \in PS_N(R)$. Thus $M_R \subseteq PS_N(R)$.

Suppose $x \in PS_N(R)$. Suppose there exists $y \in X$ such that $x < y$. Then $\exists i \in N$ such that yP_ix. Since $x \in PS_N(R)$, there exists $j \in N$ such that xP_jy. Thus $x < y$ is impossible. Hence $x \in M_R$. Therefore, $PS_N(R) \subseteq M_R$.

Corollary 3.7. *Let* $x \in X$.

(1) *Suppose* $\forall y \in X$, $x \leq y$ *implies* $x = y$. *Then* $x \in PS_N(R)$.

(2) *If* $x \notin PS_N(R)$, *then there exists* $y \in PS_N(R)$ *such that* $x < y$.

Proof. (1) Clearly, $x \in M_R$, but $M_R = PS_N(R)$.

(2) Since $x \notin PS_N(R)$, $x \notin M_R$. Thus there exists $y \in X$ such that $x < y$. Let y be the largest such element. Then $y \in M_R = RS_N(R)$.

Definition 3.8. Define $\langle\ \rangle : \mathcal{P}(U) \to \mathcal{P}(U)$ by $\forall S \in \mathcal{P}(U)$, $\langle S \rangle = \{x \in U \mid \exists s \in S,\ x \leq S\}$.

We note that the next result has been used in the study of automata theory and graph theory, [10].

Proposition 3.9. *Let* $\langle\ \rangle : \mathcal{P}(U) \to \mathcal{P}(U)$ *be defined as above. Then the following conditions hold*:

(1) $\forall S \in \mathcal{P}(U)$, $S \subseteq \langle S \rangle$;

(2) $\forall S_1, S_2 \in \mathcal{P}(U)$, $S_1 \subseteq S_2$ *implies* $\langle S_1 \rangle \subseteq \langle S_2 \rangle$;

(3) $\forall S \in \mathcal{P}(U)$, $\langle S \rangle = \langle\langle S \rangle\rangle$;

(4) $\forall S \in \mathcal{P}(U)$, $\langle S \rangle = \bigcup_{s \in S} \langle \{s\} \rangle$;

(5) $\forall S \in \mathcal{P}(U)$, $\forall x, y \in X$, $x \in \langle S \cup \{y\} \rangle$ *and* $x \notin \langle S \rangle$ *implies* $x \in \langle \{y\} \rangle$.

Proof. (1) Let $s \in S$. Then $s \leq s$ and so $s \in \langle S \rangle$. Thus $S \subseteq \langle S \rangle$.

(2) Let $x \in \langle S_1 \rangle$. Then there exists $s \in S_1$ such that $x \leq s$. Since $s \in S_2$, $x \in \langle S_2 \rangle$.

(3) By (1), $\langle S \rangle \subseteq \langle\langle S \rangle\rangle$. Let $x \in \langle\langle S \rangle\rangle$. Then there exists $y \in \langle S \rangle$ such that $x \leq y$. There exists $s \in S$ such that $y \leq s$. Since $\leq$ is transitive, $x \leq s$. Thus $x \in \langle S \rangle$. Hence $\langle\langle S \rangle\rangle \subseteq \langle S \rangle$.

(4) For all $s \in S$, $\langle \{s\} \rangle \subseteq \langle S \rangle$ by (2). Thus $\bigcup_{s \in S} \langle \{s\} \rangle \subseteq \langle S \rangle$. Let $x \in \langle S \rangle$. Then there exists $s \in S$ such that $x \leq s$. Thus $x \in \langle \{s\} \rangle$ and so $x \in \bigcup_{s \in S} \langle \{s\} \rangle$. Hence $\langle S \rangle \subseteq \bigcup_{s \in S} \langle \{s\} \rangle$.

(5) Suppose $x \in \langle S \cup \{y\} \rangle$ and $x \notin \langle S \rangle$. Then there does not exist $s \in S$ such that $x \leq s$. Hence $x \leq y$. Thus $x \in \langle \{y\} \rangle$.

Theorem 3.10. $\langle X \rangle = \langle PS_N(R) \rangle$.

Proof. Clearly, $PS_N(R) \subseteq X$. Thus $\langle PS_N(R) \rangle \subseteq \langle X \rangle$. Let $x \in X$. If $x \notin \langle PS_N(R) \rangle$, then $x \notin PS_N(R)$ and so by (2) of Corollary 3.7, there exists $y \in PS_N(R)$ such that $x < y$. Thus $x \in \langle \{y\} \rangle \subseteq \langle PS_N(R) \rangle$. Hence $X \subseteq \langle PS_N(R) \rangle$ and so $\langle X \rangle \subseteq \langle PS_N(R) \rangle$.

Lemma 3.11. $M(R, X) \cap PS_N(R) = \varnothing$ *if and only if* $M(R, X) = \varnothing$.

Proof. Suppose $M(R, X) \neq \varnothing$. Let $x \in M(R, X)$. By Theorem 3.10, there exists $y \in PS_N(R)$ such that $y \geq x$. Since $x \in M(R, X)$, xRz for all

$z \in X$ by (3). Since $y \geq x$, yRz for all $z \in X$. Thus $y \in M(R, X)$. Hence $M(R, X) \cap PS_N(R) \neq \emptyset$.

Lemma 3.12. *Let* $s \in PS_N(R)$. *Then there does not exist* $c \in PS_N(R)$ *such that cPs if and only if* $s \in M(R, X)$.

Proof. Since R is complete and not cPs for all $c \in PS_N(R)$, it follows that sRc for all $c \in PS_N(R)$. Let $x \in X$. By Theorem 3.10, there exists $c \in PS_N(R)$ such that $c \geq x$. Thus sRx by $(4')$. Hence $s \in M(R, X)$. The converse is immediate.

Lemma 3.13. (1) *Let* $s \in PS_N(R)$. *If there exists* $x \in X$ *such that xPs, then there exists* $c \in PS_N(R)$ *such that cPs.*

(2) $M(R, X) = \emptyset$ *if and only* $\forall s \in PS_N(R)$, *there exists* $c \in PS_N(R)$ *such that cPs.*

Proof. (1) By Theorem 3.10, there exists $c \in PS_N(R)$ such that $c \geq x$. Hence cPs by (3).

(2) Suppose $M(R, X) = \emptyset$. Then the result holds by Lemma 3.12. Conversely, suppose $M(R, X) \neq \emptyset$. By Lemma 3.11, $M(R, X) \cap PS_N(R) \neq \emptyset$ and so there exists $s \in M(R, X) \cap PS_N(R)$. Hence there does not exist $c \in PS_N(R)$ such that cPs.

Let $V = \{v \in U \mid v$ is not in a cycle$\}$. Let $N_1 = V \backslash N_2$, where $N_2 = \{w \in V \mid \forall R \in \mathcal{R}, w \in PS_N(R) \Rightarrow M(R, X) \neq \emptyset\}$. Let $M_1 = \{w \in V \mid \forall R \in \mathcal{R}^n, w \notin PS_N(R)\}$. Assume $M_1 \subseteq N_1$. Let $N_1' = N_1 \backslash M_1$. Suppose N_1 is such that none of its elements are strictly preferred to one of $U \backslash V$.

Theorem 3.14. $M(R, X) = \emptyset$ *if and only if* $PS_N(R) = (\bigcup_{k=1}^{n} C_k) \cup (\bigcup_{j=1}^{m} C_j') \cup N_1''$, *where* $N_1'' \subseteq N_1'$, C_k *are cycles,* $k = 1, ..., n$, C_j' *are subsets of cycles which are not themselves cycles,* $j = 1, ..., m$, *and*

(1) $\forall s \in \bigcup_{j=1}^{m} C'_j$, *there exists* $c \in (\bigcup_{k=1}^{n} C_k) \cup (\bigcup_{j=1}^{m} C'_j)$ *such that* *cPs*,

(2) $\forall s \in N''_1$, *there exists* $c \in (\bigcup_{k=1}^{n} C_k) \cup (\bigcup_{j=1}^{m} C'_j)$ *such that cPs*.

Proof. It follows that $PS_N(R) \subseteq (\bigcup_{k=1}^{n} C_k) \cup (\bigcup_{j=1}^{m} C'_j) \cup V$. Since no element of M_1 can be in $PS_N(R)$, $PS_N(R) \subseteq (\bigcup_{k=1}^{n} C_k) \cup (\bigcup_{j=1}^{m} C'_j) \cup (N_1 \backslash M_1) \cup N_2$. Hence it follows that $PS_N(R) = (\bigcup_{k=1}^{n} C_k) \cup (\bigcup_{j=1}^{m} C'_j) \cup N''_1 \cup N'_2$ for certain cycles C_k, $k = 1, ..., n$, C'_j subsets of cycles which are not themselves cycles, $j = 1, ..., m$, and for some $N''_1 \subseteq N'_1$, and $N'_2 \subseteq N_2$.

Suppose $M(R, X) = \varnothing$. Since $N_2 \cap PS_N(R) \neq \varnothing$ implies $M(R, X) \neq \varnothing$, $PS_N(R) = (\bigcup_{k=1}^{n} C_k) \cup (\bigcup_{j=1}^{m} C'_j) \cup N''_1$, i.e., $N'_2 = \varnothing$. Since no element of $N_1 \backslash M_1$ is preferred to one of $U \backslash V$, no element of $N_1 \backslash M_1$ is preferred to one of $PS_N(R)$. Hence $\forall s \in \bigcup_{j=1}^{m} C'_j$, $\exists c \in (\bigcup_{k=1}^{n} C_k) \cup (\bigcup_{j=1}^{m} C'_j)$ such that *cPs* by Lemma 3.12, else $M(R, X) \neq \varnothing$. By Lemma 3.12, $\forall s \in N''_1$, there exists $c \in (\bigcup_{k=1}^{n} C_k) \cup (\bigcup_{j=1}^{m} C'_j)$ such that *cPs*.

For the converse, the conditions imply $\forall s \in PS_N(R)$, $\exists c \in PS_N(R)$ such that *cPs*. Hence no element of $PS_N(R)$ is in $M(R, X)$. Thus by Lemma 3.11, $M(R, X) = \varnothing$.

Theorem 3.15. $M(R, X) \subseteq \langle M(R, X) \cap PS_N(R) \rangle$.

Proof. If $M(R, X) = \varnothing$, then the result is immediate since $\langle \varnothing \rangle = \varnothing$. Suppose $M(R, X) \neq \varnothing$. Let $s \in M(R, X)$. Suppose $s \notin PS_N(R)$. Since $\langle X \rangle = \langle PS_N(R) \rangle$, $s \in \langle PS_N(R) \rangle$. Hence there exists $c \in PS_N(R)$ such that $s < c$. Since sRx, $\forall x \in X$, cRx, $\forall x \in X$. Thus $c \in M(R, X)$. Hence $c \in M(R, X) \cap PS_N(R)$. Thus $s \in \langle M(R, X) \cap PS_N(R) \rangle$. Therefore, $M(R, X) \subseteq \langle M(R, X) \cap PS_N(R) \rangle$.

Examples can be found in [7], where $M(R, X) \nsubseteq PS_N(R)$ and where $M(R, X) \subseteq PS_N(R)$.

4. Conclusions

The application of fuzzy set theory to represent thick indifference preferences in spatial models carries with it two interesting substantive differences from conventional approaches. First, it is well known that under single-peaked, Euclidean preferences in one-dimensional space, the ideal policy position of the median voter is the only undefeated alternative and is the predicted outcome under majority rule [2]. This is not necessarily the case with fuzzy preferences [7]. A second substantive difference concerns the Pareto set. In conventional two-dimensional models, the Pareto set is determined by drawing a convex hull around the player's ideal points. The interpretation does not hold for all fuzzy preference profiles [7].

We also highlight the ability of the approach considered in this paper to deal with highly irregular preferences, which standard mathematical approaches can only tackle with substantial difficulty. Our main theorem, Theorem 3.14, makes it clear that it is not the shape of players' preferences that matters. It is the intersections of the players' preferences that matters.

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