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CHARACTERISTICS OF CLASSICAL NEAR-RING GROUP OF QUOTIENTS

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Abstract

In this paper, we present near-ring group structure of quotients of a near-ring group and It is seen that near-ring of quotients may appear as a particular case in some cases. We also try to explore how inheritance of so-called Goldie character plays an important role in the existence of such structures.

1. Introduction

Here we discuss some characteristics of near-ring group of quotients of a near-ring group with so-called unusual near-ring module structure as referred by Grainger [5]. Here we give some insight to some Goldie characters of such a structure leading to the existence of such quotients as well as some sort of inheritances of Goldie properties.

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We begin with the formal definition of such an unusual module structure. Suppose $(G, +)$ is a group and $(K, +, \cdot)$ is a right near-ring.

A *complementary representation* of K on G is a semigroup homomorphism $\theta : (K, \cdot) \to (\mathrm{End}(G), \circ)$.

Suppose that $(K, +, \cdot)$ is a right near-ring. Then an *unusual near-ring K module* is a pair $((G, +), *)$, where $(G, +)$ is a group and $* : G \times K \to G$ is a function which satisfies:

(i) $x * (a \cdot b) = (x * a) * b$ for all $x \in G$ and $a, b \in K$ and

(ii) $(x + y) * a = x * a + y * a$ for all $x, y \in G$ and $a \in K$.

Suppose $((G, +), *)$ is an unusual near-ring K-module. Then a function

$$\theta : K \to \mathrm{End}(G),$$

$$a \to a\theta$$

given by

$$a\theta : G \to G,$$

$$x \to x(a\theta) = x * a$$

is a semigroup homomorphism. Thus, '$*$' induces the complimentary representation.

Conversely, if $\theta : K \to \mathrm{End}(G)$ is a complimentary representation, then define a *right scalar multiplication*

$$* : G \times K \to G \text{ by } x * a = x(a\theta) \text{ for all } a \in K \text{ and } x \in G.$$

Thus, θ induces a right scalar multiplication that makes G an unusual near-ring K module. We consider such an unusual near-ring module structure and call it a *right near-ring group* of a right near-ring. (In contrast, we usually deal with the left near-ring group structure of a right near-ring.)

And as such the definition of an unusual near-ring group structure of an additive group (not necessarily abelian) over a right near-ring with identity 1 is as follows.

Let $(E, +)$ be a group and N be a near-ring with a map

$$\mu : E \times N \to E, \ \ (x, q) \to xq$$

such that for all $x, y \in E$ and $q, r \in N,$ we have

$$(x + y)q = xq + yq,$$

$$x(qr) = (xq)r,$$

$$x0 = 0$$

$$\text{and } x1 = x,$$

where zero in the left is the zero of N and the zero in the right is the zero in E. Then $E_N = (E, +, \mu)$ is called the *near-ring group*.

A subset A of a near-ring group E_N is a sub-near-ring group of E_N, if $x - y,\ xn \in A$ for all $x, y \in A, n \in N$.

The notion immediately leads us to the following.

If E and F are two such unusual near-ring N-groups, then a mapping $f : E \to F$ is an *N-homomorphism* if:

(i) f is a group homomorphism

(ii) $f(en) = f(e)n,$ for $e \in E$ and $n \in N$

in a usual way the notion of kernel of f follows.

The notion of an *N-map* follows when the condition (i) is absent.

In what follows, it contains the notion of essential as well as rational extensions together with some relevant results.

An N-subgroup $A(\neq 0)$ of E (i.e., $(A, +)$ is a subgroup of $(E, +)$, with $AN \subseteq A)$ is an *essential N-subgroup* of E or E is an essential extension of

A, if for every N-subgroup $X(\neq 0)$ of E, we have $A \cap X \neq 0$ and is denoted by $A \subseteq_e E$.

An N-subgroup D of E is a dense (or rational) N-subgroup of E or E is a *rational extension* of D, written $D \subseteq_r E$ if given $u, v \in E$ with $u \neq 0$ there exists $t \in N$ such that $vt \in D$ and $ut \neq 0$.

An N-subset D of L where N is a sub-near-ring of the near-ring L is $D \subseteq_r L$ if and only if given $k, l \in L$ with $l \neq 0$ there exists $x \in N$ such that $kx \in D$ and $lx \neq 0$.

An N-subgroup D of F where F is a near-ring group over L (where N is a sub-near-ring of near-ring L) is $D \subseteq_r F$ if and only if given $p, q \in L$ with $q \neq 0$ there exists $x \in D$ such that $px \in D$ and $qx \neq 0$.

If $D \subseteq F \subseteq E$, where E is a near-ring group, F is a sub-near-ring group and D is a normal sub-near-ring group of F such that $D \subseteq_r E$, then zero homomorphism is the only homomorphism from F/D to E (i.e., $\mathrm{Hom}(F/D, E) = (0)$). Clearly, a rational extension is an essential extension. Moreover, if $D \subseteq G \subseteq E$, where E is a near-ring group, G, an N-subgroup of E and D an N-subset of G, then $D \subseteq_r E$ implies $D \subseteq_r G \subseteq_r E$.

In this paper, we mainly present:

(i) the formal structure of near-ring group of quotients of a near-ring group E_N as mentioned above,

(ii) its relation with so-called *N-Ore condition* with respect to set S of non-zero divisors of N,

(iii) the sub-near-ring group character of so-called classical near-ring group of quotients,

(iv) the inheritance of finite independent family character of right $C(Q(N))$-subgroup of $C(Q(E))_{C(Q(N))}$ which arises from such a family of right N-subgroups of E_N.

Finally, we deal with some interesting results like necessary conditions of Goldie character of E_N if the near-ring group $C(Q(E))_{C(Q(N))}$ is so. Finally, we prove that in case of a semiprime Abelian Goldie near-ring group E_N with its non *N*-nilpotent elements being distributive with distributively closed essential *N*-subgroups possesses a classical near-ring group $C(Q(E))_{C(Q(N))}$ of right quotients having no *N*-nilpotent right $C(Q(N))$-subgroups.

Definitions and notation

We begin with the notion of a fraction of a near ring group E_N.

A *fraction of* E_N is an *N*-map $f : A_N \to E_N$, where A_N is a dense *N*-subgroup of E_N. It is easy to see that for given two fractions *f* and *g* of E_N with domains Dom *f* and Dom *g*, respectively, the maps:

(i) $f + g : \text{Dom } f \cap \text{Dom } g \to E_N$

$$x \to f(x) + g(x)$$

and (ii) for given two fractions *f* and $\frac{n}{1}$ $(n \in N)$ with domains Dom *f* and Dom $\frac{n}{1}$, and also $\frac{n}{1} : \text{Dom}\,\frac{n}{1} \to E_N$, $x \to nx$,

$$\left(f \cdot \frac{n}{1}\right) : \left(\frac{n}{1}\right)^{-1}(\text{Dom } f) \to E_N$$

$$\text{or } \left(f \cdot \frac{n}{1}\right) : \left(\frac{n}{1}\right)^{-1}(A_N) \to E_N$$

$$\left(A_N \subseteq E_N \Rightarrow \left(\frac{n}{1}\right)^{-1}(A_N) \subseteq \left(\frac{n}{1}\right)^{-1}(E_N) = \text{Dom}\left(\frac{n}{1}\right)\right)$$

$$x \to f\left(\left(\frac{n}{1}\right)(x)\right)$$

are also fractions of E_N.

Given two fractions f and g of E_N, f and g are '$\sim$' related (denoted '$f \sim g$') if and only if they agree on the common part of their domains.

It is to be noted that if f and g are fractions of E_N, then $f \sim g$ if and only if there exists a dense N-subgroup D of such that $f(x) = g(x)$ for all $x \in D$ and the relation '$\sim$' is an equivalence relation on the set of all fractions of E_N.

A near-ring group F_L with E_N as its sub-near-ring group is a *near-ring group of quotients of* E_N if $E_N \subseteq_r F_L$, where N is a sub-near-ring of near-ring L.

The near-ring group E_N satisfies the *N-Ore condition w.r.t a subset S of* N, if given $(x, r) \in E \times S$, there exists a common right multiple

$$xr' = rx'$$

such that $(x', r') \in E \times S$.

Let A be any sub-near-ring group of near-ring group E_N over near-ring N and $T \subseteq N$. Then

$$Ann_E(T) = \{a \in E \mid ax = 0 \text{ for all } x \in T\}$$

and $Ann_N(A) = \{x \in N \mid ax = 0 \text{ for all } a \in A\}$.

If $(G, +)$ is a group and N is a near-ring, then G is said to be *Goldie near-ring group* when

(i) G has no infinite independent family of non-zero N-subgroups, and

(ii) annihilators of subsets of G in N satisfy the ascending chain condition (under set inclusion).

Every Goldie near-ring is clearly a Goldie near-ring group.

The right singular N-subgroup of E_N is the right N-subgroup

$$Z_1(E_N) = \{x \in E_N \mid xL = 0, \text{ for some strictly essential}$$
$$N\text{-subgroup } L \text{ of } N\}.$$

The near-ring group E_N is called *non-singular* if $Z_1(E_N) = 0.$

A right N-subgroup A of E_N is '*N-nilpotent*' if we have some $n \in N$, $\alpha \in Z^+$, we have $An^{\alpha} = 0.$

An element '*a*' is said to be '*N-nilpotent*' if for some $n \in N$, $\alpha \in Z^+$, $an^{\alpha} = 0.$

We shall call a near-ring group E_N '*semiprime*' if E_N has no non-zero '*N-nilpotent*' N-subgroup of E_N.

2. Preliminaries

Now we present the important notion of what we are intending.

Proposition 2.1. *Let* F_L *be a near-ring group with* E_N *as a sub-near-ring group* (*where* N *is a sub-near-ring of near-ring* L). *Then* F_L *is a near-ring group of quotients of* E_N *if and only if for every* $q \in L$, $q \neq 0$, *we have* $q^{-1}E_N \subseteq_r E_N$, $q(q^{-1}E_N) \supset (0)$ *where* $q^{-1}E_N = \{x \in E_N \mid qx \in E_N\}.$

Proof. Suppose $E_N \subseteq_r F_L$. Given $z \in q^{-1}E_N$ and $n \in N$. Then we get

$$q(zn) \in E_N$$
$$\Rightarrow zn \in q^{-1}E_N.$$

Hence $q^{-1}E_N$ is a subset of F_L. $u, v \in F_L$, $v \neq 0$. Then $qu \in F_L$. Since $E_N \subseteq_r F_L$, there exists $t \in N$ such that $(qu)t \in E_N$ and $vt \neq 0.$

The first condition implies that $ut \in q^{-1}E_N$. Thus, given $u, v \in F_L$, $v \neq 0$, there exists $t \in N$ such that $ut \in q^{-1}E_N$ and $vt \neq 0$ leading there by $q^{-1}E_N \subseteq_r F_L$.

Since we have $q^{-1}E_N \subseteq E_N \subseteq F_L$ and $q^{-1}E_N \subseteq_r F_L$, it follows that $q^{-1}E_N \subseteq_r E_N$. Now $a \in E_N \cap qE_N$ gives $a \in E_N, qE_N$ and hence there exists $b \in E_N$ such that $a = qb$. Since $qb = a \in E_N$, we have $b = q^{-1}E_N$. Hence $a(= qb) \in q(q^{-1}E_N)$. Thus, $q(q^{-1}E_N) \supseteq E_N \cap qE_N$.

Recalling $E_N \subseteq_r F_L$ and $E_N \subseteq_e F_L$, we see that $E_N \cap qE_N \supset (0)$, which gives $q(q^{-1}E_N) \supset (0)$.

Suppose $q^{-1}E_N \subseteq_r E_N$ and $q(q^{-1}E_N) \supset (0)$ hold and $p, q \in L$ with $q \neq 0$. Now we show that there exists $x \in E_N$ for which $px \in E_N$ and $qx \neq 0$. As $q(q^{-1}E_N) \subseteq qE_N$ and

$$\begin{aligned} q(q^{-1}E_N) &= \{qx \mid x \in q^{-1}E_N\} \\ &= \{qx \mid qx \in E_N\} \\ &\subseteq E_N. \end{aligned}$$

We get $q(q^{-1}E_N) \subseteq E_N \cap qE_N$ which in turn gives $E_N \cap qE_N \supset (0)$.

Thus, there exists $b \in E_N$ such that $a = qb$. We note that $qb \neq 0$.

(1) Suppose $p = 0$ and $x = b$. Then we get $px(= 0b = 0) \in E_N$ and $qx(= qb) \neq 0$.

(2) Suppose $p \neq 0$. Then $q^{-1}E_N \subseteq_r E_N$ gives $p^{-1}E_N \subseteq_r E_N$.

And we have $b, qb \in E_N$ with $qb \neq 0$. Hence there exists $y \in N$ such that $by \in p^{-1}E_N$ and $(qb)y \neq 0$. Again $x = by$ gives $x \in E_N$ such that $px \in E_N$.

Thus, in both the cases $p = 0$ and $p \neq 0$, there exists $x \in E_N$ with $px \in E_N$ and $qx \neq 0$. Hence $E_N \subseteq_r F_L$.

We see in this section that the fractions of E_N yield a near-ring group of quotients of E_N.

Let $Q(E)$ be the set of all equivalence classes of $\bar{f}, \bar{g}, \ldots$ into which the fractions $f, g, \ldots$ of E_N are partitioned by the relation $\sim$, defined in $Q(E)$ by the rule $\bar{f} + \bar{g} = \overline{f + g}$.

We see that '+' in $Q(E)$ is justified. For, $f \sim h$ and $g \sim l$, let

$$x \in \text{Dom } f \cap \text{Dom } g \cap \text{Dom } h \cap \text{Dom } l.$$

Now

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ &= h(x) + l(x) \\ &= (h + l)(x).\end{aligned}$$

Hence $f + g \sim h + l$. Thus, if $\bar{f} = \bar{h}$ and $\bar{g} = \bar{l}$, we have $\overline{f + g} = \overline{h + l}$.

Again, let $Q(N)$ be the set of all equivalence classes $\overline{\frac{n}{1}}, \overline{\frac{\alpha}{1}}, \cdots$ into which the fractions $\frac{n}{1}, \frac{\alpha}{1}, \cdots$ of N are partitioned by the relation $\sim$. Then define in $Q(E)$ the rule as follows:

$$\bar{f}\, \overline{\frac{n}{1}} = \overline{f\, \frac{n}{1}}.$$

We see that it is well-defined in $Q(E)$. Suppose $f \sim g$ and $\frac{n}{1} \sim \frac{\alpha}{1}$.

Let $x \in \text{Dom}\frac{n}{1} \cap \text{Dom}\frac{\alpha}{1} \cap \left(\frac{\alpha}{1}\right)^{-1}(\text{Dom } f) \cap \left(\frac{\alpha}{1}\right)^{-1}\left(\text{Dom } \frac{n}{1}\right)$. Then

$$\begin{aligned}\left(f\, \frac{n}{1}\right)(x) &= f\left(\frac{n}{1}(x)\right) \\ &= f\left(\frac{\alpha}{1}(x)\right) \\ &= g\left(\frac{\alpha}{1}(x)\right) = \left(g\, \frac{\alpha}{1}\right)(x).\end{aligned}$$

Hence $f\, \frac{n}{1} \sim g\, \frac{\alpha}{1}$. Thus, if $\bar{f} = \bar{g}$ and $\overline{\frac{n}{1}} = \overline{\frac{\alpha}{1}}$, then we have $\overline{f\, \frac{n}{1}} = \overline{g\, \frac{\alpha}{1}}$.

Proposition 2.2. *The set* $Q(E)$ *all equivalence classes of* $\bar{f}, \bar{g}, \ldots$ *into which the fractions* $f, g, \ldots$ *of* E_N *are partitioned by the relation* $\sim$ *is a near-ring group defined by the rule* $\bar{f} + \bar{g} = \overline{f + g}$ *and* $\bar{f}\,\overline{\alpha} = \overline{f\alpha}$, *where* $\overline{\alpha} \in Q(N)$, *the near-ring of quotients.*

Proof. Let us define $\mu : Q(E) \times Q(N) \to Q(E)$, $(\bar{f}, \overline{\alpha}) = \bar{f}\,\overline{\alpha} = \overline{f\alpha}$, for all $\overline{\alpha} \in Q(N)$, $\bar{f} \in Q(E)$.

Now for all $\overline{\alpha_1}, \overline{\alpha_2} \in N$, $\bar{f}, \bar{g} \in Q(E)$, we have

$$
\begin{aligned}
(\bar{f} + \bar{g})(\overline{\alpha_1}) &= \overline{(f + g)\alpha_1} \\
&= \overline{f\alpha_1 + g\alpha_1} \\
&= \overline{f\alpha_1} + \overline{g\alpha_1} \\
&= \bar{f}\,\overline{\alpha_1} + \bar{g}\,\overline{\alpha_1},
\end{aligned}
$$

$$
\begin{aligned}
\bar{f}(\overline{\alpha_1}\;\overline{\alpha_2}) &= \bar{f}(\overline{\alpha_1\alpha_2}) \\
&= \overline{f(\alpha_1\alpha_2)} \\
&= ((\overline{f\alpha_1}))\overline{\alpha_2} \\
&= (\bar{f}\,\overline{\alpha_1})\overline{\alpha_2},
\end{aligned}
$$

$Q(E)$ has an additive identity $\overline{0}$ given by the fraction $0 : E_N \to E_N$, $x \to 0$.

For every $\bar{f} \in Q(E)$ we get a fraction

$$-f : \text{Dom } f \to E_N, \quad x \to -f(x).$$

Given $x \in \text{Dom } f$, we have

$$(f + (-f))(x) = f(x) - f(x) = 0,$$

$$0(x) = 0,$$

$$((-f) + (f))(x) = -f(x) + f(x) = 0.$$

Hence $\overline{f + (-f)} = \bar{0} = \overline{(-f) + f}$ or $\bar{f} + \overline{(-f)} = \bar{0} = \overline{(-f)} + \bar{f}.$

Thus, every $\bar{f} \in Q(E)$ has an inverse $\overline{(-f)} \in Q(E)$. And so we have

$$\bar{f}0 = 0.$$

Hence $Q(E) = (Q(E), +, \mu)$ is a near-ring group over $Q(N)$.

In particular, $Q(N) = (Q(N), +, \cdot)$ is a near ring and $Q(E)$ is a near-ring $Q(N)$ group.

Proposition 2.3. *E_N is embedded in the near-ring group $Q(E)_{Q(N)}$. (and so N is embedded in the near-ring $Q(N)$).*

Proof. For every $n \in N$, we get a left multiplication in the near-ring group of transformations of E_N,

$$\frac{n}{1} : E_N \to E_N, \quad x \to nx.$$

Given $x \in E_N$, $p \in N$, we see that

$$\left(\frac{n}{1}\right)(xp) = n(xp)$$

$$= (nx)p$$

$$= \left(\frac{n}{1}\right)(x)p.$$

Thus, the left multiplication $\left(\frac{n}{1}\right)$ is an N-map and hence a fraction of E_N.

Consider the map $\alpha : E_N \to Q(E)_{Q(N)}$, $x \to \frac{\bar{x}}{1}$, for $x \in E_N$. Then for $x, y \in E_N$, we have

$$\alpha(x + y) = \overline{\frac{x + y}{1}}$$

$$= \frac{\overline{x}}{1} + \frac{\overline{y}}{1}$$

$$= \alpha(x) + \alpha(y).$$

Again, for $n \in N,\ x \in E_N,$ we have

$$\alpha(xn) = \overline{\frac{xn}{1}} = \overline{\frac{x}{1}\frac{n}{1}}$$

$$= \overline{\frac{x}{1}n}$$

$$= \frac{\overline{x}}{1}n$$

$$= \alpha(x)n.$$

Thus, α is an N-homomorphism.

Now,

$$\text{kernel } \alpha = \left\{x \in E_N \mid \frac{\overline{x}}{1} = 0\right\}$$

$$= \{x \in E_N \mid x = 0\}$$

$$= (0).$$

Hence α is an N-monomorphism.

Note 2.4. Since the map $\alpha : E_N \to Q(E)_{Q(N)}$ is a monomorphism, we shall identify $\alpha(E_N)$ with E_N and $\frac{\overline{n}}{1}$ with n, for simplicity of notation.

Proposition 2.5. *If* $D \subseteq_r E_N$ *and* $q \in Q(N),\ q \neq 0,$ *then* $qD = (0)$ *implies* $q = 0$.

Proof. Let $q = \bar{t},$ where t is a fraction of N, and $n \in N.$ Then $qD = (0)$ implies that

$\overline{t(n/1)} = (0)$

or $t((n/1)(x)) = 0$ for every $x \in (n/1)^{-1}(\text{Domt}) \Rightarrow t(\text{Domt}) = 0.$

Thus, $q = \bar{t} = 0.$

Proposition 2.6. *If* $q \in Q(N)$, *t is a fraction of N, and f is a fraction of E such that* $q = \bar{t}$, *then* $Dom\ f \subseteq q^{-1}E_N$.

Proof. Let $r \in \text{Dom}\ f$. Then

$$qr = \bar{t}\,\frac{\bar{r}}{1} = t\overline{(r/1)} = \overline{t(r)}/1 \in E_N$$

$$\Rightarrow r \in q^{-1}E_N.$$

Hence $\text{Dom}\ f \subseteq q^{-1}E_N$.

As a corollary, we get

Corollary 2.7. *If* $q \in Q(E)_{Q(N)}$, *then* $q^{-1}E_N \subseteq_r E_N$, *where* $q^{-1}E_N = \{x \in E_N \mid qx \in E_N\}$.

Also,

Proposition 2.8. *If* $q \in Q(N)$, $q \neq 0$, *then*

$$q(q^{-1}E_N) \supset (0).$$

The proof immediately follows from Proposition 2.5.

Using the last two results and Proposition 2.1, we have

Theorem 2.9. $Q(E)_{Q(N)}$ *is a near-ring group of quotients of* E_N.

Proposition 2.10. *Let* E_N *satisfy the N-Ore condition with respect to a multiplicatively closed subset S of N and have N-homomorphisms*

$$\alpha : N \to Q(N),$$

$$\beta : E_N \to Q(E)_{Q(N)}$$

such that $r \in S$ *implies* $\alpha(r)^{-1} \in Q(N)$. *Then the subset*

$$ES^{-1} = \{\beta(a)\alpha(r)^{-1} \mid (a, r) \in E \times S\}$$

is a sub-near-ring group of $Q(E)_{Q(N)}$.

Proof. Let $\beta(a)\alpha(r)^{-1}, \beta(b)\alpha(s)^{-1} \in ES^{-1}$. Then we have $(a, r), (b, s) \in E \times S$. Since E_N satisfies the N-Ore condition w.r.t. S and $(a, r), (b, s) \in E \times S$, we get that there exists $(r', s'), (b', r'') \in E \times S$ such that $rs' = sr'$ and $br'' = rb'$. And hence

(i) $\beta(rs') = \beta(sr')$ and

(ii) $\beta(br'') = \beta(rb')$.

Again, as N satisfies the Ore condition w.r.t. S and $(r, s), (p, r) \in N \times S$, we therefore get that there exist $(r', s'), (p', r') \in N \times S$ such that $rs' = sr'$ and $pr'' = rp'$. And hence

(iii) $\alpha(rs') = \alpha(sr')$ and

(iv) $\alpha(pr'') = \alpha(rp')$.

Now

$$\beta(a)\alpha(r)^{-1} - \beta(b)\alpha(s)^{-1}$$

$$= \beta(a)(\alpha(s')\alpha(sr')^{-1} - \beta(b)(\alpha(r')\alpha(rs')^{-1})) \text{ (using (iii))}$$

$$= \beta(a)\alpha(s')\alpha(rs')^{-1} - \beta(b)\alpha(r')\alpha(rs')^{-1}$$

$$= (\beta(a)\alpha(s') - \beta(b)\alpha(r'))\alpha(rs')^{-1} \in ES^{-1}.$$

Again, let $\alpha(n)\alpha(r)^{-1} \in NS^{-1}$. Then

$$\beta(b)\alpha(s)^{-1}\alpha(n)\alpha(r)^{-1}$$

$$= \beta(b)\alpha(rs')\alpha(r')^{-1}\alpha(n)\alpha(r)^{-1} \quad \text{(using (iii))}$$

$$= \beta(b)\alpha(rs')\alpha(n')\alpha(r_1')^{-1},$$

where

$$\alpha(r')^{-1}\alpha(n) = \alpha(n')\alpha(r_1')^{-1}$$
$$= \beta(b)\alpha(rs'n')\alpha(rr_1')^{-1}$$
$$\in ES^{-1}.$$

Hence ES^{-1} is a sub-near-ring group of $Q(E)_{Q(N)}$.

Remark 2.11. We note that if in Proposition 2.10., $1 \in S$, then the near-ring group ES^{-1} has $\beta(1)$ as its identity.

Thus, we get

Proposition 2.12. *Let* E_N *be a near-ring group and* S *be a multiplicatively closed subset of* N *containing* 1, $\alpha : N \to Q(N)$, $\beta : E_N \to Q(E)_{Q(N)}$ *are homomorphisms satisfying the condition* $\alpha(r)^{-1} \in Q(N)$, *for* $r \in S$ *and the condition* $\beta(a) = 0$, *for* $a \in E_N$ *implies that there exists* $t \in S$ *with* $at = 0$.

Also, if the subset ES^{-1} *is a sub-near-ring group of* $Q(E)_{Q(N)}$, *then* E_N *satisfies the N-Ore condition with respect to S.*

Proof. Let $(a, r) \in E \times S$. Then $a \in E_N$ and $r \in S$. Since $r \in S$ implies $\alpha(r)^{-1} \in Q(N)$, $\alpha(r)^{-1}$ and $\alpha(1)^{-1}$ exist and as ES^{-1} is a sub-near-ring group, we get

$$\beta(1)\alpha(r)^{-1}\beta(a)\alpha(1)^{-1} \in ES^{-1} \Rightarrow \alpha(r)^{-1}\beta(a) \in ES^{-1}.$$

It follows from the definition of ES^{-1}, that for some $(b, s) \in E \times S$ such that

$$\alpha(r)^{-1}\beta(a) = \beta(b)\alpha(s)^{-1} \Rightarrow \beta(a\alpha(s)) = \beta(\alpha(r)b)$$
$$\Rightarrow \beta(a)\alpha(s) = \alpha(r)\beta(b)$$
$$\Rightarrow \beta((a\alpha(s)) - \alpha(r)b) = 0.$$

Thus, there exists $t \in S$ such that

$$(a(\alpha(s)) - \alpha(r)b)t = 0 \Rightarrow a\alpha(st) - \alpha(r)bt = 0.$$

Putting $bt = a^{/}$ and $st = r^{/}$, we see that given $(a, r) \in E \times S$, we have $(a^{/}, r^{/}) \in E \times S$ such that $\alpha(r)a^{/} = a\alpha(r^{/})$, i.e., E_N satisfies N-Ore condition with respect to S.

We now present our main result of the paper. In this part, we deal with the classical near-ring group of quotients of a near-ring group and with the Goldie character in such structures.

3. Main Results

It is easy to note that S is a multiplicatively closed subset of N, moreover every $s(\in S)$ is invertible in $Q(N)$.

The following is a criterion when the set described below may appear as a subgroup of $Q(E)_{Q(N)}$.

Proposition 3.1. *Let S be the set of non-zero divisors of N. If* E_N *satisfies the N-Ore condition with respect to S and* $s \in S$, *then* $sE_N \subseteq_r E_N$.

Proof. Clearly, sE_N is an N-subgroup of E_N. Let $a, b \in E_N$, $b \neq 0$. Then $(a, s) \in E \times S$ and hence there exists a common right multiple $as^{/} = sa^{/}$ such that $(a^{/}, s^{/}) \in E \times S$. Since $sa' \in sE_N$ and s' is a non-zero divisor and $b \neq 0$, we have $bs' \neq 0$. Thus, $sE_N \subseteq_r E_N$.

Because of Note 2.4, we regard E_N as a sub-near-ring group of $Q(E)_{Q(N)}$.

Following result gives how the N-ore condition is connected with classical near-ring group of quotients.

Proposition 3.2. *If* E_N *satisfies the N-Ore condition w.r.t. S, then the subset* $C(Q(E)) = \{xr^{-1} \in Q(E) | (x, r) \in E \times S\}$ *is a sub-near-ring group of* $Q(E)_{Q(N)}$.

$C(Q(E))$ is the *classical near-ring group of quotients of* E_N.

Proof. Let α and β be the embeddings

$$\alpha : N \to Q(N) \quad \text{and} \quad \beta : E_N \to Q(E)_{Q(N)}$$

$$r \to r \qquad\qquad x \to \frac{\bar{x}}{1}.$$

Also, for any $r \in S$, $\alpha(r)^{-1} \in Q(N)$. Thus,

$$C(Q(E)) = \{xr^{-1} \in Q(E) | (x, r) \in E \times S\}$$

$$= \{\beta(x)\alpha(r)^{-1} \in Q(E) | (x, r) \in E \times S\}$$

$$= ES^{-1}.$$

As ES^{-1} sub-near-ring group of $Q(E)_{Q(N)}$, $C(Q(E))_{c(Q(N))}$ is a sub-near-ring group of $Q(E)_{Q(N)}$ (where $C(Q(N))$ is the classical near-ring of quotients of near-ring *N*).

As in [2, Lemma 2.1.1], we have

Proposition 3.3. *Let* $C(Q(N))$ *be the complete near-ring of quotients of N. If* $s_1, s_2, ..., s_n \in S$, *then there exist* $x_1, x_2, ..., x_n \in N$ *and* $s \in S$ *such that* $s_i^{-1} = x_i s^{-1}$, $i = 1, 2, ..., n$.

Proposition 3.4. *If J be a right N-subgroup of* E_N, *then the subset* $JS^{-1} = \{xs^{-1} \in Q(E) | (x, s) \in J \times S\}$ *is a right* $C(Q(N))$ *subgroup of* $C(Q(E))_{C(Q(N))}$.

Proof. Let $p \in C(Q(E))_{C(Q(N))}$, $q \in C(Q(N))$. Then $p = as^{-1}$, $q = xt^{-1}$, where $a \in J$, $x \in N$, $s, t \in S$ and

$$pq = (as^{-1})(xt^{-1})$$

$$= a(s^{-1}xt^{-1}).$$

Since $xt^{-1} \in C(Q(N)),\ s^{-1} = 1\,s^{-1} \in NS^{-1} = C(Q(N)),$ we get

$$S^{-1}(xt^{-1}) \in C(Q(N)).$$

Let $s^{-1}(xt^{-1}) = bu^{-1},\ b \in N,\ u \in s.$ Then $ab \in J$ gives $pq \in JS^{-1}.$ Hence the result.

Proposition 3.5. *If* $\{J_1, J_2, ..., J_t\}$ *be an independent family of right* N*-subgroups of* E_N*, then* $\{J_1S^{-1}, J_2S^{-1}, ..., J_tS^{-1}\}$ *is an independent family of right* $C(Q(N))$*-subgroup of* $C(Q(E))_{C(Q(N))}$.

Proof. If possible, let $\{J_1S^{-1}, J_2S^{-1}, ..., J_tS^{-1}\}$ be not an independent family. Then there is an $m,\ 1 \le m \le t$ such that $J_mS^{-1} \cap \sum_{n \neq m} J_nS^{-1} \neq 0.$

Then there is a non-zero element, $j_ms_m^{-1} = j_1s_1^{-1} + \cdots + \widehat{j_ms_m} + \cdots + j_ts_t^{-1}$ ($\wedge$ stands for deletion of the term underneath) in the intersection.

By Proposition 3.3, for $s_1, ..., s_t \in S,$ we get $x_1, ..., x_t \in N$ and $s \in S$ such that

$$s_i^{-1} = x_1s^{-1}, \quad 1 \le i \le t.$$

Now,

$$j_mx_ms^{-1} = j_1x_1s^{-1} + \cdots + \widehat{j_mx_ms^{-1}} + \cdots + j_tx_ts^{-1}$$

$$= (\,j_1x_1 + \cdots + \widehat{j_mx_m} + \cdots + j_tx_t\,)\,s^{-1}.$$

And this gives

$$j_mx_m = j_1x_1 + \cdots + \widehat{j_mx_m} + \cdots + j_tx_t \neq 0.$$

So, $J_m \cap \left(\sum_{n \neq m} J_n\right) \neq (0)$ and is a contradiction, for $\{J_1, ..., J_t\}$ is an independent family of N-subgroup of E_N.

Therefore, $\{J_1S^{-1}, ..., J_tS^{-1}\}$ is an independent family of $C(Q(N))$-subgroup of $C(Q(E))_{C(Q(N))}$.

Proposition 3.6. *If J is an annihilator N-subgroup of near-ring group E_N (i.e., $J = Ann_E(T)$ for some $T \subseteq N$), then $JS^{-1} \cap E = J$.*

Proof. Here $x \in J$ gives $x = x1^{-1} \in JS^{-1}(1 \in S)$. So, $x \in JS^{-1} \cap E$ giving there by $J \subseteq JS^{-1} \cap E$.

Next, $y \in JS^{-1} \cap E$ gives $y = as^{-1} = x$, where $a \in J$, $s \in S$, $x \in E$. Now $as^{-1} = x \in E$, $xT = 0$ and we get $as^{-1} \in Ann_E(T)$ in E_N. Thus, $y(= as^{-1}) \in J$ which gives $JS^{-1} \cap E \subseteq J$.

Hence the result.

Proposition 3.7. *If $C(Q(E))_{C(Q(N))}$ is the classical near-ring group of quotients of E_N, $T \subseteq N$ and $J = Ann_E(T)$, then $JS^{-1} = Ann_{C(Q(E))}(T)$.*

Proof. Here $x \in Ann_{C(Q(E))}(T)$ gives $x \in C(Q(E))_{C(Q(N))}$ and $xT = 0$.

Since $x \in C(Q(E))_{C(Q(N))}$, $x = as^{-1}$ for some $a \in E_N$, $s \in S$. Then $as^{-1}T = (0)$. Therefore, $a \in Ann_E(T)$, or $a \in J$ and hence, $x = as^{-1} \in JS^{-1}$. It follows that $Ann_{C(Q(E))}(T) \subseteq JS^{-1}$.

To see the opposite inclusion, $y \in JS^{-1}$ gives $y = js^{-1}$, $j \in J$, $s \in S$. Since $j \in J = Ann_{C(Q(E))}(T)$, $j \in C(Q(E))_{C(Q(N))}$ and $jT = 0$ which gives $js^{-1}T = 0$ or $js^{-1} \in Ann_{C(Q(E))}(T)$. Thus, $y \in Ann_{C(Q(E))}(T)$, or $JS^{-1} \subseteq Ann_{C(Q(E))}(T)$.

Hence the result.

Proposition 3.8. *Let* $C(Q(E))_{C(Q(N))}$ *be the classical near-ring group of right quotients of* E_N. *If* E_N *is additively commutatively, then so also is* $C(Q(E))_{C(Q(N))}$.

Proof. Let $x, y \in C(Q(E))_{C(Q(N))}$. Then $x = ab^{-1}$, $y = cd^{-1}$ for some $a, c \in E_N$, $b, d \in S$.

Now from Proposition 3.3, there exist $t_1, t_2 \in N$, $s \in S$ such that

$$b^{-1} = t_1 s^{-1}, \quad d^{-1} = t_2 s^{-1}.$$

Therefore, $x = ab^{-1} = at_1 s^{-1} = us^{-1}$, where $u = at_1 \in E_N$ and $y = cd^{-1} = ct_2 s^{-1} = vs^{-1}$, where $v = ct_2 \in E_N$. Since $(E_N, +)$ is abelian, the result follows.

Proposition 3.9. *If the near-ring group* $C(Q(E))_{C(Q(N))}$ *is Goldie, then so is* E_N.

Proof. If possible, then let there be a strictly ascending chain of right annihilators of subsets of E_N,

$$J_1 \subset J_2 \subset \cdots, \text{ where } J_i = Ann_E(T_i), T_i \subseteq N.$$

Then by Proposition 3.7, the chain

$$J_1 S^{-1} \subseteq J_2 S^{-1} \subseteq \cdots$$

is an ascending chain of right annihilator of subsets of $C(Q(E))_{C(Q(N))}$. The near-ring group $C(Q(E))_{C(Q(N))}$ being Goldie, we therefore get $m \in Z^+$ such that

$$J_m S^{-1} = J_{m+1} S^{-1} = \cdots.$$

This gives $J_m S^{-1} \bigcap E = J_{m+1} S^{-1} \bigcap E = \cdots$.

And by Proposition 3.6, we get $J_m = J_{m+1} = \cdots$, a contradiction. Therefore, E_N cannot have an infinite strictly ascending chain of right annihilators of subsets of E_N.

Next, if $\{J_i\}$ is an infinite independent family of *N*-subgroups of E_N, then by Proposition 3.5, the set $\{J_i S^{-1}\}$ is an infinite independent family of right $C(Q(N))$-subgroup of $C(Q(E))_{C(Q(N))}$. And this is not possible because $C(Q(E))_{C(Q(N))}$ is Goldie. Therefore, E_N cannot have an infinite independent family of right *N*-subgroups of E_N. Therefore, E_N is Goldie.

Proposition 3.10. *Let A, B, C be right N-subgroups of* E_N *such that* $A \subseteq B \subseteq C \subseteq E_N$ *and A is N-essential in B, B is N-essential in C. Then A is N-essential in C.*

Proof. Let *D* be a (non-zero) *N*-subgroup of E_N and $D \subseteq C$. Since *B* is *N*-essential in *C*, we get $D \cap B \neq (0)$. And since *A* is *N*-essential in *B*, this gives

$$(D \cap B) \cap A \neq (0).$$

Now $D \cap A \supseteq D \cap B \cap A \neq (0)$. Thus, $D \cap A \neq (0)$. Hence *A* is *N*-essential in *C*.

Proposition 3.11. *If every essential N-subset of N contains a non-zero divisor, then* E_N *is non-singular.*

Proof. Let $x \in Z_l(E_N)$. Then there exists $L \subseteq_e N$ such that $xL = (0)$. Since $L \subseteq_e N$, by given condition, there exists a non-zero divisor $l \in L$. It follows that $xL = 0$. Since *L* is a non-zero divisor, we have $x = 0$. Thus, $Z_l(E_N) = 0$.

Lemma 3.12. *Let M and H be N-subgroups of a near-ring group* E_N

such that H is N-essential in M. If $a \in M,\ a \neq 0,$ *then there is an essential right N-subgroup L of N such that* $aL \neq 0,\ aL \subseteq H.$

Proof. Let $L = \{n \in N \mid an \in H\}.$ Then L is right N-subset of N and $aN \subseteq M$ (since M is an N-subgroup of E_N and $a \in M$). Also $aN \neq 0$ (for $l \in N$ implies $a \in aN$). Since H is N-essential in M, we get $aN \cap H \neq 0$ and $h = an(\neq 0) \in H$ gives $aL \neq 0.$

We now show that L is an essential right N-subset of N. Let $I(\neq 0)$ be a right N-subset of N. We claim that $I \cap L \neq 0.$ Now, $aI = 0$ gives $aI \subseteq H.$ So, $I \subseteq L$ giving thereby $I \cap L \neq 0.$ And if $aI \neq 0,$ then aI is an N-subgroup of E_N and $aI \subseteq M.$

Since H is N-essential in M, $aI \cap H \neq 0.$ Hence for some $x(\neq 0) \in I,$ $ax \in H.$ Thus, $x \in L.$ Therefore, $I \cap L \neq 0$ and this implies that L is an essential right N-subset of N.

And we get

Corollary 3.13. *For* $a \in M,$ $a^{-1}H = \{n \in N \mid an \in H\}$ *is an right essential N-subset of N.*

Lemma 3.14. *Let* E_N *be a Goldie near-ring group whose non N-nilpotent elements are distributive. If* $x \in E_N$ *is such that* $A(x) = 0,$ *then xN is an essential N-subgroup of* E_N.

Proof. Since $A(x) = 0,$ x is non N-nilpotent and thus it is distributive. Let M be an N-subgroup of E_N such that $M \cap xN = 0.$ Now, for a non-nilpotent $\alpha \in N$ and for a fix $s \in Z^+$ and for $t \leq s,$ let

$$y \in \left(\sum_{n \neq t} x\alpha^n N\right) \cap x\alpha^t N,\ (x\alpha^0 = x,\ n = 0, 1, ..., s).$$

Then $y = \sum_{n \neq t} x\alpha^n p_n = x\alpha^t p_t, \; p_n, \; p_t \in N,$ i.e.,

$$xp_0 = x\alpha^t p_t - x\alpha^s p_t - \cdots - x\alpha p_1$$

$$= x(\alpha^t p_t - \alpha^s p_t - \cdots - \alpha p_1) \text{ (since } x \text{ is distributive).}$$

Thus, $xp_0 \in M \cap xN$ which gives $xp_0 = 0$. It follows that

$$x\alpha^t p_t - x\alpha^s p_t - \cdots - x\alpha p_1 = 0, \text{ for } A(x) = 0.$$

Similarly, we get $x\alpha p_1 = x\alpha p_2 = \cdots = x\alpha p_t = 0$. Therefore,

$$\left(\sum_{n \neq t} x\alpha^n N\right) \cap x\alpha^t N = 0 \text{ for all } s \in Z^+ \text{ and } t \leq s.$$

Thus, the family $\{M, xN, x\alpha N, x\alpha^2 N, ...\}$ is an independent family. E_N being Goldie, there exists $u \in Z^+$ such that $x\alpha^{u+1}N = 0$. So, for any $m \in M$, $m\alpha^{u+1} = 0$ which gives $m = 0$. Therefore, $M = 0$. Thus, $M \cap xN = 0$ implies $M = 0$. Hence xN is an essential N-subgroup of E_N.

Lemma 3.15. *If N is a semiprime Goldie near-ring where non-nilpotent elements are distributive, then every distributively closed right essential N-subset of N contains a regular element.*

Proposition 3.16. *A semiprime abelian Goldie near-ring group E_N in which non N-nilpotent elements are distributive with distributively closed right essential N-subgroups has a classical near-ring group $C(Q(E))_{C(Q(N))}$ of right quotients which has no N-nilpotent right $C(Q(N))$-subgroups.*

Proof. Choose $x, y \in E_N$, $A(x) = 0$, E_N being Goldie, by Lemma 3.14, xN is an essential in E_N. Then, by Corollary 3.13, the set $\lambda = \{r \in N \mid xr \in yN\}$ is an essential in N. Therefore, by Lemma 3.15, λ contains a regular element, say r'. Thus, $xr' = rx'$ for some $x' \in E_N$. Thus the right N-Ore condition with respect to the set S of regular elements of N is satisfied in E_N. So by Proposition 3.2, E_N has a classical near-ring group of right quotients, say $C(Q(E))_{C(Q(N))}$.

Next, let J be a $C(Q(N))$-subgroup of $C(Q(E))_{C(Q(N))}$ such that $J\alpha^2 = 0$, $\alpha \in N$. Now $J \cap E$ is a right N-subgroup of E_N. Because of Proposition 3.4,

$$(J \cap E)C(Q(E)) = \{xc^{-1} \mid x \in JN,\ c \in S\}.$$

Now, $x \in J$, $x = ys^{-1}$, $y \in E$, $s \in S$. So, $xs = y \in JN$.

Thus,

$$x = (ys)s^{-1} \in (JN)C(Q(E)).$$

Conversely, if $ys^{-1} \in (J \cap E)C(Q(E))$, $y \in J \cap E$, $s \in S$, then $ys^{-1} \in J$. Hence $J = (J \cap E)C(Q(E))$. Again, $J\alpha^2 = 0$ gives $(J \cap E\alpha^2) \subseteq J\alpha^2 = 0$.

Therefore, $J \cap E$ is an N-nilpotent right N-subgroup of E_N and E_N is semiprime. Hence $J \cap E = 0$. Thus, it follows from what we have showed above that $J = 0$.

Example 1. $N = \{0, 1, 2, 3, 4, 5, 6\}$ is a near-ring under addition modulo 7 and multiplication defined by the following table:

•	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	4	4	2	1
2	0	2	4	1	1	4	2
3	0	3	6	5	5	6	3
4	0	4	1	2	2	1	4
5	0	5	3	6	6	3	5
6	0	6	5	3	3	5	6

It has no non-zero zero-divisors. Hence, every essential N-subset of this near-ring contains a non-zero-divisor. It follows from Proposition 3.11 that the near-ring group is non-singular.

Example 2. $N = \{0, a, b, c\}$ under the addition and multiplication defined by the following tables:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\bullet$	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
c	0	a	c	c

Its N-subsets are $\{0\}$, $\{0, a\}$ and $\{0, a, b, c\}$. Of these, the second and the third are essential. It is at once seen that the near-ring does not satisfy the condition of Proposition 3.11, but is non-singular.

Example 3. $N = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ is a near-ring under addition modulo 8 and multiplication as defined by the following table:

$\bullet$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	2	0	4	4	2
2	0	0	0	4	0	0	0	4
3	0	0	0	6	0	4	4	6
4	0	0	0	0	0	0	0	0
5	0	0	0	2	0	4	4	2
6	0	0	0	4	0	0	0	4
7	0	0	0	6	0	4	4	6

Here near-ring group ${}_N N$ has only two non-trivial N-subgroups $\{0, 4\}$ and $\{0, 2, 4, 6\}$ such that $\{0, 4\} \cap \{0, 2, 4, 6\} \neq 0.$ That is each of them has non-zero intersection with other N-subgroups of ${}_N N.$ Therefore,

$$\{0, 4\} \subseteq_e {}_N N, \{0, 2, 4, 6\} \subseteq_e {}_N N.$$

In this example, we see that $\{0, 4\} \subseteq_e \{0, 2, 4, 6\}.$ This shows the validity of Proposition 3.10.

Example 4. Consider the near-ring $N = \{0, 1, 2, 3, 4, 5, 6, 7\}$ under addition modulo 8 and multiplication defined by the following table:

$\bullet$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

Here $\{0, 4\}$ and $\{0, 2, 4, 6\}$ are ideals of ${}_N N$ such that $\{0, 4\} \cap \{0, 2, 4, 6\} \neq 0.$ Hence $\{0, 4\}$ and $\{0, 2, 4, 6\}$ are essential ideals of ${}_N N.$

Example 5. In Klein 4-group, the near-ring N without unity w.r.t. the operations addition defined in Table 1 and the multiplication defined in the Table 2 has the invariant subsets, $\{0, b\}, \{0, a, b\}.$ But for any invariant subsets, say $L(\neq 0)$ of N, we get $L^n \neq 0$ for any $n \in Z^+.$ Hence N has no non-zero nilpotent invariant subsets. In this sense, N is strongly semi-prime.

Table 1

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Table 2

$\bullet$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	b	a	b	c

Example 6. Consider a non-zero symmetric near-ring $N(= D_8)$ without unity w.r.t. the addition and multiplication defined by Tables 3 and 4, respectively.

Table 3

+	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
0	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
a	a	$2a$	$3a$	0	$a+b$	$2a+b$	$3a+b$	b
$2a$	$2a$	$3a$	0	a	$2a+b$	$3a+b$	b	$a+b$
$3a$	$3a$	0	a	$2a$	$3a+b$	b	$a+b$	$2a+b$
b	b	$3a+b$	$2a+b$	$a+b$	0	$3a$	$2a$	a
$a+b$	$a+b$	b	$3a+b$	$2a+b$	a	0	$3a$	$2a$
$2a+b$	$2a+b$	$a+b$	b	$3a+b$	$2a$	a	0	$3a$
$3a+b$	$3a+b$	$2a+b$	$a+b$	b	$3a$	$2a$	a	0

Table 4

$*$	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
0	0	0	0	0	0	0	0	0
a	0	a	$2a$	a	0	0	0	$2a$
$2a$	0	$2a$	0	$2a$	0	0	0	0
$3a$	0	$3a$	$2a$	$3a$	0	0	0	$2a$
b	0	b	b	b	b	b	b	b
$a+b$	0	$a+b$	$2a+b$	$a+b$	b	b	b	$2a+b$
$2a+b$	0	$2a+b$	b	$2a+b$	b	b	b	b
$3a+b$	0	$3a+b$	$2a+b$	$3a+b$	b	b	b	$2a+b$

The non-zero proper left N-subsets are $\{0, b\}$, $\{0, b, a+b\}$, $\{0, b, 2a+b\}$, $\{0, b, 2a, 2a+b\}$, $\{0, 2a, b, 2a+b, 3a+b\}$, etc.

Here we note for any subsets L of N, there exist no left N-subsets X of N such that $X^n L = 0$, for any $n \in Z^+$.

Example 7. Consider the near-ring $N(= Z_8)$ without unity w.r.t. addition modulo 8 and multiplication defined by the following table:

$\bullet$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	2	0	4	4	2
2	0	0	0	4	0	0	0	4
3	0	0	0	6	0	4	4	6
4	0	0	0	0	0	0	0	0
5	0	0	0	2	0	4	4	2
6	0	0	0	4	0	0	0	4
7	0	0	0	6	0	4	4	6

Here N has proper left N-subsets viz. $\{0, 1\}$, $\{0, 2\}$, $\{0, 4\}$, $\{0, 4, 5\}$, $\{0, 4, 6\}$, $\{0, 2, 4, 6\}$, $\{0, 4, 6, 7\}$, $\{0, 2, 3, 4, 6\}$, etc.

Now $\{0, 4, 5\}\{0, 5\} = \{0, 4\}(\neq 0)$ and $\{0, 4, 5\}^2\{0, 5\} = 0.$

It is easy to see that if $\{0, 5\}$ is replaced by $\{0, 4, 5\}^3 = 0$ and thereby $\{0, 4, 5\}$ is found as nilpotent subset of N.

Example 8. In the near-ring $N(= Z_8)$ without unity w.r.t. addition modulo 8 and multiplication defined by the following table, the only proper left N-subset is $\{0, 2\}$:

$\bullet$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	1	1	1	1	1
2	0	2	0	2	2	2	2	2
3	0	3	0	3	3	3	3	3
4	0	4	0	4	4	4	4	4
5	0	5	0	5	5	5	5	5
6	0	6	0	6	6	6	6	6
7	0	7	0	7	7	7	7	7

Hence for any subset $L(\neq 0)$ of it different from $\{0, 2\}$, we have $\{0, 2\}L \neq 0.$ But $\{0, 2\}^2 L = 0.$

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