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ELECTRIC FORCE BETWEEN TWO CHARGED COAXIAL RINGS

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Abstract

Using differential element method, the electric potential on a coaxial circle established by a charged ring is expressed in terms of complete elliptic integral. The electric force between two charged coaxial rings is then achieved. The characteristics of the force, symbolic symmetry and geometric symmetry, and the physical significance are analyzed and illustrated. The force magnitude has its maximum, which reason is explained and numerical solution is sought. The force between the ring and a point charge on the central is treated as a special case. Two ways, making use of field intensity and the principle of virtual work, are present to calculate that force. Their consistency is demonstrated.

1. Introduction

Electric force, as a general phenomenon, exists in electromagnetic devices. It is one of interesting topics studied in recent years. In [1] and [2], the electric force on the walls of some transmission lines was studied. Charged ring is extensively used in electronic equipment. The electric force

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between charged rings would affect the electromagnetic properties of such devices [3]. Thus, it is worth to investigate the electric force between two charged circular rings.

The aim of this paper is focused on this problem. With the aid of complete elliptic integral, the electric potential on coaxial circle set up by a charged ring is calculated first. After finding the field intensity, the force between two charged coaxial rings is determined. Two kinds of symmetries as well as their physical significance are discussed. It stands out that the force magnitude has its maximum, which is illustrated and sought by using numerical method. The expression of the force between the ring and a point charge on the central axis, treated as a special case, is drawn from the general result. Two ways are employed to determine the force. Their consistency is analyzed.

2. Electric Potential

A system, which consists of two parallel coaxial circular rings of radius r_1 and r_2 with separation l, is sketched in Figure 1. A cylindrical coordinate system is established.

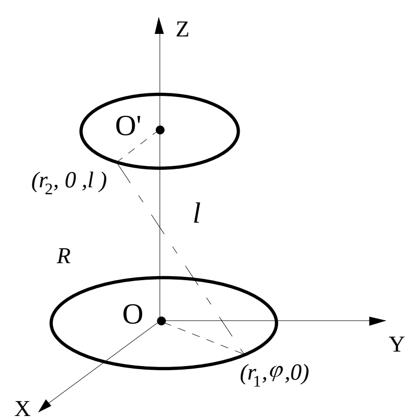

Figure 1. Two charged coaxial rings.

Suppose the lower ring is charged uniformly. The linear charge density on that ring is λ_1. For a point on the upper ring, say $(r_2, 0, l)$, the electric potential produced by a segment, $r_1 d\varphi$, at $(r_1, \varphi, 0)$ on lower ring is

$$dU = \frac{\lambda_1 r_1}{4\pi\varepsilon_0\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\varphi + l^2}}\, d\varphi. \tag{1}$$

Therefore, the potential of upper ring set up by lower ring is calculated as

$$U = \int dU = \frac{\lambda_1 r_1}{4\pi\varepsilon_0}\int_0^{2\pi} \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\varphi + l^2}}\, d\varphi. \tag{2}$$

Introducing a transformation

$$\varphi = \pi - 2\theta. \tag{3}$$

Equation (2) then becomes

$$U = \frac{\lambda_1 k}{2\pi\varepsilon_0}\sqrt{\frac{r_1}{r_2}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2\theta}}\, d\theta = \frac{\lambda_1 k}{2\pi\varepsilon_0}\sqrt{\frac{r_1}{r_2}}K(k), \tag{4}$$

where $K(k)$ is the complete elliptic integral of the first kind. The modulus k is

$$k = 2\sqrt{\frac{r_1 r_2}{(r_1 + r_2)^2 + l^2}} < 1. \tag{5}$$

3. Force between Two Charged Rings

If the upper ring is also uniformly charged with linear density λ_2, now we compute the electric force between two rings.

Due to symmetry, the electric force between two rings must be along z axis. Our attention is focused on the field component in z direction. At the point $(r_2, 0, l)$ on the upper ring, the z component of electric field produced by lower ring is

$$E_z = -\frac{dU}{dl}. \tag{6}$$

Substituting Equation (4) in previous equation yields

$$E_z = -\frac{\lambda_1}{2\pi\varepsilon_0}\sqrt{\frac{r_1}{r_2}}\frac{d}{dl}(kK(k)) = -\frac{\lambda_1}{2\pi\varepsilon_0}\sqrt{\frac{r_1}{r_2}}\frac{dk}{dl}\frac{d}{dk}(kK(k))$$

$$= \frac{\lambda_1 lk^3}{8\pi\varepsilon_0 r_2\sqrt{r_1 r_2}}\frac{d}{dk}(kK(k)). \tag{7}$$

Therefore, the electric force between two rings is

$$f = q_2 E_z = 2\pi r_2 \lambda_2 E_z = \frac{\lambda_1\lambda_2 lk^3}{4\varepsilon_0\sqrt{r_1 r_2}}\frac{d}{dk}(kK(k)). \tag{8}$$

Note [4]

$$\frac{d}{dk}(kK(k)) = K(k) + k\frac{dK(k)}{dk} = K(k) + \frac{E(k)}{2(1-k^2)} - \frac{K(k)}{2}$$

$$= \frac{1}{2}\left[K(k) + \frac{E(k)}{1-k^2}\right], \tag{9}$$

where $E(k)$ is the complete elliptic integral of the second kind. Thus, Equation (8) becomes

$$f = \frac{\lambda_1\lambda_2 lk^3}{8\varepsilon_0\sqrt{r_1 r_2}}\left[K(k) + \frac{E(k)}{1-k^2}\right]. \tag{10}$$

4. Maximum Magnitude of the Force

Equation (10) demonstrates the quantitative relation of the force, *f*, versus the separation, *l* and the radii, r_1 and r_2. Introduce dimensionless force

$$\tilde{f} = \frac{f}{\frac{\lambda_1\lambda_2}{8\varepsilon_0}} = \frac{lk^3}{\sqrt{r_1 r_2}}\left[K(k) + \frac{E(k)}{1-k^2}\right]. \tag{11}$$

The graph of $\tilde{f}$ versus the relative quantities, l/r_2 and r_1/r_2, is plotted in Figure 2.

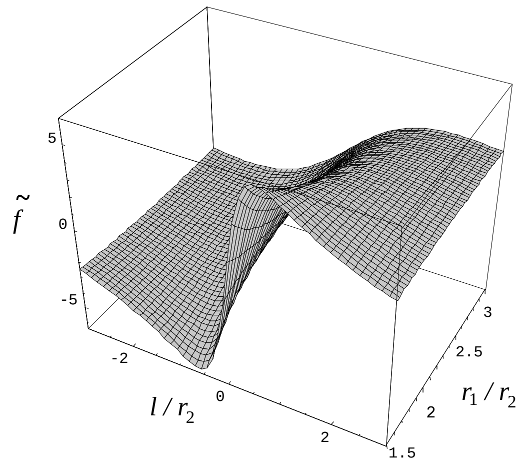

Figure 2. The graph of dimensionless force versus relative separation and radius.

Figure 2 shows there is maximum magnitude of the force. That can be explained qualitatively. On the one hand, due to the symmetry, the force must be zero when $l = 0$, which is the case of two rings in the same plane. On the other hand, the force is zero as *l* approaches infinity. So, the magnitude of *f*, as a continuous nonnegative function of *l*, should reach its maximum somewhere [5].

To determine that maximum of the force quantitatively, differentiating Equation (11) with respect to *l* and letting it be zero

$$\frac{d\tilde{f}}{dl} = 0 \tag{12}$$

which is the necessary condition for the position where the force reaches the maximum magnitude. Although we cannot obtain analytic solution for Equation (12), we can find its numerical solution with the aid of some mathematical software, such as Mathematica [6]. The operation program is as follows:

```
Clear[df,f,k,l,n,r1,r2];
n=Input[n];
k=2*Sqrt[r1*r2/((r1+r2)^2+l^2)];
f=l*k^3/Sqrt[r1*r2] *(EllipticK[k^2]+EllipticE[k^2]/(1-k^2));
df=D[f,l];
r2=1; r1=n;
FindRoot[df==0,{l,Sqrt[n]}];
l=l/.%;
f;
Print["l=",l,"r2"]
Print["fmax=",f]
```

In the program, according to the rules of Mathematica, the modulus in elliptic integral should be written as k^2 instead of *k*.

Take an example. Under the condition of $r_1 = 2r_2$, inputting $n = 2$ from the keyboard, we approximately obtain a numerical solution $l = 1.16911r_2$ and $\tilde{f}_{\max} = 4.17007$. This situation can be clearly seen in Figure 3.

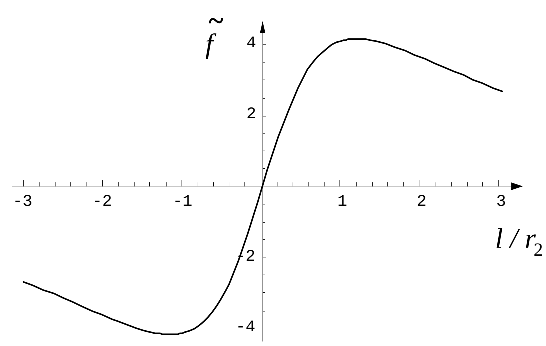

Figure 3. The function curve of dimensionless force versus relative separation $(r_1 = 2r_2)$.

5. Two Kinds of Symmetry

5.1. Symbolic symmetry

If we exchange the indexes 1 and 2 in Equation (10), then the result keeps the same. That means there is a symbolic symmetry in the expression of the force. The physical significance is the electric force exerted on one ring has the same magnitude on another. This is the inevitable deduction from Newton's third law.

5.2. Geometric symmetry

Equation (11) implies that $\tilde{f}$ is the odd function of l. If we replace l with $-l$, then $\tilde{f}$ has the same magnitude and opposite direction. Therefore, the graph of $\tilde{f}$ has geometric symmetry about the origin O. Figures 2 and 3 reflect that central symmetry.

6. Special Case

In term of the charge on a ring, $q = 2\pi r\lambda$, Equation (10) becomes

$$f = \frac{q_1 q_2 l}{4\pi^2 \varepsilon_0 (\sqrt{(r_1 + r_2)^2 + l^2})^3} \left[K(k) + \frac{E(k)}{1 - k^2} \right]. \tag{13}$$

Suppose the upper ring diminishes to a point charge q_2. Letting $r_2 = 0$ in Equation (5) leads to $k = 0$. Note

$$K(0) = E(0) = \pi/2. \tag{14}$$

From Equation (13), we derive

$$f = \frac{q_1 q_2 l}{4\pi \varepsilon_0 (\sqrt{r_1^2 + l^2})^3}. \tag{15}$$

Since the electric field for the points on the central axis of a charged ring is well known by us

$$E = E_z = \frac{q_1 l}{4\pi \varepsilon_0 (\sqrt{r_1^2 + l^2})^3} \tag{16}$$

using $f = q_2 E$ we also get Equation (15). That supports the correctness of our general result in this special case.

7. Principle of Virtual Work

The potential energy of the two charged rings is

$$W = q_2 U. \tag{17}$$

Take l as general coordinate. Employing the principle of virtual work, we have

$$f = f_z = -\frac{dW}{dl}. \tag{18}$$

Combining Equations (17) and (18), we arrive at

$$f = -q_2 \frac{dU}{dl}. \tag{19}$$

Substituting Equation (4) into Equation (19) then using similar calculation, we may obtain the same result for the force as above. That demonstrates two ways, making use of field intensity or the principle of virtual work, are consistent.

Actually, it is easy to reason this consistency out. Putting Equation (6) in Equation (19) yields

$$f = q_2 E_z \tag{20}$$

which is in the same form as Equation (8).

8. Conclusion

The electrical potential on coaxial circle produced by a charged ring can be expressed by complete elliptic integral. Then the force between the two charged coaxial rings is calculated. The force has a maximum magnitude, which solution can be obtained by numerical method. The force expression demonstrates both symbolic symmetry and geometric symmetry. Taking as a special case, the force between the ring and a point charge on the central axis is deduced from the general result. Two ways, making use of field intensity and the principle of virtual work, are pointed out. The consistency of these two methods in physics is displayed.

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