Far East Journal of Mathematical Sciences (FJMS)
Volume 70, Number 1, 2012, Pages 87-120
Published Online: September 2012
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

THE BINARY GOLDBACH CONJECTURE WITH RESTRICTIONS ON THE PRIMES

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Abstract

It is proved that for any positive number X, any $P > 0$ and for all but $(\log X)^D$ prime numbers $k \leq X^{\frac{5}{48}-\varepsilon}$, the following is true: For any positive integers b_i, $i \in \{1, 2\}$, $(b_i, k) = 1$, all but $O(Xk^{-1}L^{-P})$ sufficiently large integers $N \leq X$ satisfying $N \equiv b_1 + b_2 (\operatorname{mod} k)$ can be written as $N = p_1 + p_2$, where p_i, $i \in \{1, 2\}$ are prime numbers that satisfy $p_i \equiv b_i (\operatorname{mod} k)$.

1. Introduction

The binary Goldbach conjecture states that every even integer larger than 2 can be written as the sum of two prime numbers. In 1975, Montgomery and Vaughan considered the corresponding exceptional set $E(X)$ defined as

$$E(X) := \#\{N \leq X : 2 | N,\ N \neq p_1 + p_2 \text{ for any primes } p_1,\ p_2\}.$$

2010 Mathematics Subject Classification: 11P32, 11P05, 11P55, 11N36.
Keywords and phrases: additive theory of prime numbers, circle method.
Submitted by Wei Dong Gao
Received December 21, 2011; Revised January 11, 2012

They could show that

$$E(X) < X^{1-\delta}$$

for a small positive number $\delta > 0$. It was later shown in [11] that δ can be chosen as large as $\delta = 0.086$. Lavrik [13] investigated a special case of the binary Goldbach conjecture requiring that the two prime summands belong to a given arithmetic progression. In particular, he considered the following exceptional set:

$$E_{k,b_1,b_2}(X) := \#\{N \le X : N \equiv b_1 + b_2 (\mathrm{mod}\, k),\ N \ne p_1 + p_2$$
$$\text{for any primes } p_i \equiv b_i (\mathrm{mod}\, k),\ i = 1, 2\},$$

$$E_k(X) = \max_{\substack{1 \le b_1, b_2 \le k \\ (b_1 b_2, k) = 1}} E_{k,b_1,b_2}(X).$$

He show that for $k \le (\log X)^c$ and any $A > 0$,

$$E_k(X) \ll X(\log X)^{-A} k^{-1}. \tag{1.1}$$

Using a different approach, Liu and Zhan [18] shown that the following estimate holds for all $k \le X^{\delta}$ for a small $\delta > 0$:

$$E_k(X) \ll X^{1-\delta_1} \phi^{-1}(k), \tag{1.2}$$

for a small, positive constant δ_1. In this paper, we show the following result:

Theorem 1. *There exists a positive constant D such that for all but* $O((\log X)^D)$ *prime numbers* $k \le X^{\frac{5}{48}-\varepsilon}$, *we have*

$$E_k(X) \ll Xk^{-1}(\log X)^{-P} \tag{1.3}$$

for any $P > 0$.

Compared to (1.2), we increase the permissible size of k, for all but $O((\log X)^D) k$, to $X^{\frac{5}{48}-\varepsilon}$ at the cost of a slightly smaller exceptional set $E_k(X)$.

The proof of Theorem 1 uses the concept of X-exceptional zeros introduced in [4]. We set

$$L = \log X, \quad L_2 = \log\log X, \quad L(s, \chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s},$$

where χ is a Dirichlet character. For a prime number k, $k \leq N$ and a fixed positive integer V, we define

$$I_k = [k, kL^V] \cup [k^2, k^2L^V], \quad P_k = \{m \in \mathbb{N} : m \equiv 0 (\operatorname{mod} k), m \in I_k\}.$$

For a fixed, very small δ, we call a Dirichlet character χ to a module q satisfying

$$X^{\delta} \leq q \leq X \tag{1.4}$$

an *X-exceptional* character if there exists at least one complex number $s = \sigma + it$ such that

$$\sigma > 1 - \frac{EL_2}{L}, \quad |t| \leq X, \quad L(s, \chi) = 0, \tag{1.5}$$

where E is a fixed, positive number to be defined later. We call s an *X-exceptional* zero and we call an integer q an *X-exceptional* integer if there exists an *X-exceptional* character χ modulo q.

Using the concept of *X-exceptional zeros*, Theorem 1 is a direct consequence of Theorems 2 and 3:

Theorem 2. *For a given prime number* $k \leq X^{\frac{5}{48}-\varepsilon}$, *if none of the integers* $q \in P_k$ *is X-exceptional, then* (1.3) *is true for this k.*

Theorem 3. *There exists a positive constant D such that for all but* $O((\log X)^D)$ *prime numbers* k, $1 \leq k \leq X$, *none of the integers* $q \in P_k$ *is X-exceptional.*

Theorem 3 follows directly from [4, Theorem 3]. The reminder of this paper is dedicated to the proof of Theorem 2.

The ternary Goldbach theorem with primes in arithmetic progressions has been extensively studied in [1, 3-5, 7-9, 14, 17, 18, 21-23, 25-27]. The case of large k was considered in [4, 5, 9, 27]. The methods applied in the publications cannot be simply applied to prove Theorem 1.

In particular, in [4, 5, 9, 27], estimates for Dirichlet polynomials were used to estimate the error term induced by the integral over the major arcs. The application of these estimates for Dirichlet polynomials requires a 'good' upper bound when estimating partial singular series associated with triplets of primitive characters. In particular, if r is the greatest common denominator of the modules of three primitive characters, a factor $r^{-1/2}$ is obtained when estimating the associated partial singular series. For the binary case, such a factor cannot be obtained for all major arcs. In particular, it can only be used for major arcs where the denominator of the center of the major arc is not divisible by the prime number k. In order to deal with some of the remaining major arcs, we use an estimate for exponential sums over primes in progressions from [27].

2. Outline of the Proof of Theorem 2

For the proof of Theorem 2, we only consider integers N lying in the range $X(\log X)^{-P} < N \leq X$. The contribution of the exceptional N satisfying $N \leq X(\log X)^{-P}$ can be estimated trivially. We denote by $[a_1, a_2]$ and (a_1, a_2) the least common multiple and the greatest common divisor of two integers a_1 and a_2, respectively. c is a positive constant that can take different values at different occasions. $d(N)$ is the divisor function. We know from [18] that Theorem 2 holds true for $k \leq X^{\delta}$, where δ is a small positive constant. Therefore, we assume throughout the document that

$$k > X^{\delta}, \tag{2.1}$$

where δ is chosen as in (1.4). In this paper, we do not distinguish between the quantities k and $\phi(k)$ used in (1.2) and (1.3), respectively, as k is a prime number. We use the familiar notation

$$r \sim R \Leftrightarrow R/2 < r \le R, \quad \sum_{1\le a\le q}{}^{*} := \sum_{\substack{1\le a\le q\\(a,q)=1}}.$$

We write $e(\alpha) = e^{2\pi i\alpha}$ and the variables p and p_i always denote prime numbers. For a given integer k, we set $k_q = (k, q)$. For a given integer m, we write $k^m \| q$ if $k^m \mid q$, $k^{m+1} \nmid q$. Further, if $k^m \| q$, then we define

$$s_q = qk^{-m}. \tag{2.2}$$

Throughout this paper, we keep the numbers b_1 and b_2 used in (1.2) fixed and omit them from all of the following definitions:

$$I(N, k) = \sum_{\substack{N/4<n_i\le N\\ n_1+n_2=N\\ p_i\equiv b_i (\mathrm{mod}\, k),\, i\in\{1,2\}}} \Lambda(n_1)\Lambda(n_2),$$

$$C(\chi, q, h, b, a) = \sum_{\substack{m=1\\ m\equiv b (\mathrm{mod}\, h)}}^{q} \chi(m)e\left(\frac{ma}{q}\right).$$

We set

$$Z(N, q, h, \chi_1, \chi_2)$$

$$= \frac{1}{\phi^2(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} C(\chi_1, q, h, b_1, a)C(\chi_2, q, h, b_2, a)e\left(\frac{-aN}{q}\right),$$

$$A(N, q, h) = Z(N, q, h, \chi_{0,q}, \chi_{0,q}), \quad T(\lambda) = \sum_{N/4<n\le N} e(\lambda n),$$

$$S(\alpha, k, b) = \sum_{\substack{N/4<n\le N\\ n\equiv b (\mathrm{mod}\, k)}} \Lambda(n)e(n\alpha), \quad S(\alpha) = \sum_{N/4<n\le N} \Lambda(n)e(n\alpha).$$

For a given k, we define

$$P = L^B, \quad P_1 = kL^B, \quad P_2 = k^2L^B, \quad Q = Nk^{-2}L^{-3B}, \tag{2.3}$$

where the constant $B > 0$ will be specified later. For a fixed prime integer k and N being an integer satisfying $N \equiv b_1 + b_2 (\operatorname{mod} k)$, we define the singular series

$$\sigma(N, k) = \frac{kN}{\phi(k)\phi(kN)} \prod_{\substack{p \geq 2 \\ (p, kN)=1}} \left(1 - \frac{1}{(p-1)^2}\right). \tag{2.4}$$

Using the circle method, we define the major arcs M as follows: We set

$$M(a, q) = \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right] \tag{2.5}$$

and

$$M = M_1 \cup M_2 \cup M_3,$$

$$M_1 = \bigcup_{q \leq P} \bigcup_{(a, q)=1} M(a, q),$$

$$M_2 = \bigcup_{\substack{q \leq P_1 \\ k || q}} \bigcup_{(a, q)=1} M(a, q),$$

$$M_3 = \bigcup_{\substack{q \leq P_2 \\ k^2 || q}} \bigcup_{(a, q)=1} M(a, q). \tag{2.6}$$

We note that in the summation defining of M_1, $k \nmid q$, as $k > P$. Finally, the minor arcs m are defined as:

$$m = \left[-\frac{1}{Q}, 1 - \frac{1}{Q}\right] \backslash M.$$

Thus, we can write

$$I(N,\ k) = \int_{-\frac{1}{Q}}^{1-\frac{1}{Q}} e(-N\alpha)\prod_{i=1}^{2} S(\alpha,\ k,\ b_i)d\alpha$$

$$= \int_{M} e(-N\alpha)\prod_{i=1}^{2} S(\alpha,\ k,\ b_i)d\alpha + \int_{m} e(-N\alpha)\prod_{i=1}^{2} S(\alpha,\ k,\ b_i)d\alpha$$

$$=: R_{N,M}(k) + R_{N,m}(k). \tag{2.7}$$

We can prove Theorem 2 by proving the following two statements:

1. If for a given prime number $k \le X^{\frac{5}{48}-\varepsilon}$ none of the integers $q \in P_k$ is X-exceptional, then for all $A > 0$, and all but $O(Xk^{-1}L^{-P})$ integers N satisfying $XL^{-P} \le N \le X$, and $N \equiv b_1 + b_2 (\bmod\, k)$,

$$R_{N,M}(k) = \sigma(N,\ k)\frac{N}{2} + O(Nk^{-1}L^{-A}). \tag{2.8}$$

2. For any given prime number $k \le X^{\frac{5}{48}-\varepsilon}$, any $A > 0$ and all but $O(Xk^{-1}L^{-P})$ integers N satisfying $N \le X$, and $N \equiv b_1 + b_2 (\bmod\, k)$, there is

$$|\ R_{N,m}(k)\ | \ll Nk^{-1}L^{-A}. \tag{2.9}$$

We see from (2.4) that the main term on the RHS of (2.8) is larger than $N\sigma(N,\ k) \gg Nk^{-1} \ge XL^{-P}k^{-1}$. Noting that (2.8) and (2.9) both hold for any $A > P$, Theorem 2 follows.

In the next section, we will use Dirichlet's theorem on rational approximation according to which any rational number α can be written as

$$\alpha = \frac{a}{q} + \lambda, \quad |\,\lambda\,| \le 1/qQ, \tag{2.10}$$

where $(a,\ q) = 1$, $q \le Q$, and Q is as defined above.

3. Minor Arcs

For the estimation of the contribution of the integral over the minor arcs, we need the following lemma:

Lemma 3.1. *For an integer r and any real number* α, *we write*

$$r\alpha = \frac{a_1}{q_1} + \lambda_1, \quad |\lambda_1| \le \frac{1}{q_1^2},$$

$$r^2\alpha = \frac{a_2}{q_2} + \lambda_2, \quad |\lambda_2| \le \frac{1}{q_2^2}.$$

Then for $(r, b) = 1$,

$$S\left(\frac{a}{q} + \lambda, r, b\right) \ll \left(q_1 + \frac{N}{rq_1} + \frac{N}{rq_2^{1/2}} + \frac{N^{5/6}}{r^{1/2}} + N^{1/2}q_2^{1/2}\right)L^3.$$

Proof. See [27].

Lemma 3.2. *For integers a, q, r, b satisfying* $(a, q) = 1$, $(r, b) = 1$, *and setting* $h = (r, q)$,

$$S\left(\frac{a}{q}, r, b\right) \ll \left(\frac{hN}{rq^{1/2}} + \frac{q^{1/2}N^{1/2}}{h^{1/2}} + \frac{N^{4/5}}{r^{2/5}}\right)L^3.$$

Proof. See [2].

We now argue as in [26] and apply Lemmas 3.1 and 3.2 to estimate the integral over the minor arcs. If $\alpha \in m$, then we see from (2.10) that $\alpha = \frac{a}{q} + \lambda$, where $|\lambda| \le q^{-1}Q^{-1}$, $(a, q) = 1$, where q does not belong to the summation range defining M_1, M_2, and M_3 in (2.6).

1. For $k \nmid q$, $q > k^{\frac{3}{2}}L^H$ with sufficiently large H and $k \le N^{2/11}L^{-F}$, we obtain from Lemma 3.2 via partial summation for $i = 1, 2$,

$$|S(\alpha, k, b_i)| \ll Xk^{-1}L^{-\frac{P+G+1}{2}} \tag{3.1}$$

for $F > F(A, P)$. We apply Lemma 3.1 in the remaining cases as follows:

2. For $k \nmid q$, and $q \le k^{3/2}L^H$, we see from the definition of m that $q = q_1 = q_2 > L^B$.

3. For $k \| q$, we see that $q > q_1 = q_2 = q/k > L^B$.

4. For $k^2 \| q$, we see $q > q_1 = q/k > q_2 = q/k^2 > L^B$.

Noting that in cases 2, 3 and 4, we have $L^B < q_2 \le q_1 \le q \le Nk^{-2}L^{-3B}$, we derive (3.1) from Lemma 3.1 for $k \le N^{2/11}L^{-F}$ and $F > F(A, P)$, $B > B(A, P)$.

It follows from Parseval's identity and (3.1),

$$\sum_{\substack{X/2<N\le X \\ N\equiv b_1+b_2 (\operatorname{mod} k)}} |R_{N,m}(k)|^2 \le \sum_{X/2<N\le X} |R_{N,m}(k)|^2$$

$$= \int_m |S(\alpha, k, b_1)S(\alpha, k, b_2)|^2 d\alpha$$

$$\ll \max_{\alpha\in m} |S(\alpha, k, b_1)|^2 \left(\int_0^1 |S(\alpha, k, b_2)|^2 d\alpha\right)$$

$$\ll X^3k^{-3}L^{-A-P},$$

which implies (2.9).

4. Preliminary Lemmas for the Major Arcs

Lemma 4.1. *For any natural number* $q = q_1q_2$, $(q_1, q_2) = 1$, *and characters*

$$\chi_c(\operatorname{mod} q) = \chi_{c_1}(\operatorname{mod} q_1)\chi_{c_2}(\operatorname{mod} q_2),$$

$$\chi_d(\operatorname{mod} q) = \chi_{d_1}(\operatorname{mod} q_1)\chi_{d_2}(\operatorname{mod} q_2),$$

there is:

(a)

$$Z(N, q, k_q, \chi_c, \chi_d)$$
$$= Z(N, q_1, k_{q_1}, \chi_{c_1}, \chi_{d_1})Z(N, q_2, k_{q_2}, \chi_{c_2}, \chi_{d_2}).$$

(b) *If* χ *modulo* p^β *is a both non-primitive and non-principal character, i.e.,* χ *is induced by* χ^* *modulo* p^α, $1 \le \alpha < \beta$, *then for* $(ab, p) = 1$, *and* $0 \le \gamma < \beta$, *we have*

$$C(\chi, p^\beta, p^\gamma, b, a) = 0.$$

Proof. The proof follows the proof of Lemma 3.2 in [4].

Lemma 4.2. *Set* $(a, q) = 1$, *and* $(b, q) = 1$ *throughout the parts* (a) *and* (b).

(a) *Let* χ *be a character modulo* q. *Then*

$$C(\chi, q, 1, b, a) \ll q^{1/2}.$$

(b)

$$C(\chi_{0,q}, q, k_q, b, a)$$
$$= \begin{cases} \mu(q/k_q)e\left(\dfrac{tba}{k_q}\right), & \textit{if } (q/k_q, k_q) = 1, \ tq/k_q \equiv 1(\operatorname{mod} k_q), \\ 0, & \textit{otherwise.} \end{cases}$$

(c) *For any two characters* χ_1, χ_2 *modulo* k^2, *we have*:

$$Z(N, k^2, k, \chi_1, \chi_2) \neq 0 \Rightarrow \chi_1, \chi_2, \textit{ are primitive characters modulo } k^2.$$

(d) *For any two primitive characters* χ_i *modulo* r_i, $i = 1, 2$ *with* $k^2 \| r$, *where* $[r_1, r_2] = r$, $r \mid q$, *and the principal character* χ_0 *modulo q, we have*:

$$Z(N,\, q,\, k,\, \chi_1\chi_0,\, \chi_2\chi_0) \neq 0 \Rightarrow k^2 \,||\, r_i, \quad 1 \leq i \leq 2.$$

(e) *For any* $\chi_1,\ \chi_2$ *modulo* k^2, *where k is prime*, *and an integer N satisfying* $\equiv b_1 + b_2 (\operatorname{mod} k)$,

$$Z(N,\, k^2,\, k,\, \chi_1,\, \chi_2) \ll k^{-1}.$$

Proof. The parts (a) and (b) are proved in [4, Lemma 3.3]. Parts (c) and (d) are derived from Lemma 4.1 in the same way [4, Lemma 3.3] is derived from [4, Lemma 3.2]. For the proof of part (e) we know from Lemma 4.2(c) that we only have to consider characters χ_i, $i = 1, 2,$ that are primitive modulo k^2. We see

$$C(\chi_i,\, k^2,\, k,\, b_i,\, a) = \chi_i(b_i) \sum_{s=1}^{k} \chi_i(1 + \bar{b}_i sk)\, e\left(\frac{ab_i + aks}{k^2}\right). \tag{4.1}$$

Thus,

$$Z(N,\, k^2,\, k,\, \chi_1,\, \chi_2) = \frac{\prod_{i=1}^{2} \chi_i(b_i)}{\phi^2(k^2)} \sum_{s_1}^{k} \sum_{s_2}^{k} \chi_1(1 + \bar{b}_1 s_1 k)\chi_2(1 + \bar{b}_2 s_2 k)$$

$$\times \sum_{a=1}^{k^2 *} e\left(\frac{a(b_1 + b_2 - N + s_1 k + s_2 k)}{k^2}\right). \tag{4.2}$$

Using that $b_1 + b_2 - N = Mk$, $M \in \mathbb{Z}$, we can write the inner sum in (4.2) as:

$$\sum_{a=1}^{k^2 *} e\left(\frac{ak(M + s_1 + s_2)}{k^2}\right) = k \sum_{a=1}^{k-1} e\left(\frac{a(M + s_1 + s_2)}{k}\right)$$

$$= \begin{Bmatrix} k(k-1), & \text{if } M + s_1 + s_2 \equiv 0 (\operatorname{mod} k), \\ -k, & \text{else.} \end{Bmatrix}$$

Obviously,

$$\sharp\{s_1,\ s_2 : 1 \le s_1,\ s_2 \le k,\ M + s_1 + s_2 \equiv 0 (\operatorname{mod} k)\} = k. \tag{4.3}$$

Since $k/\phi(k) \le 2$, we obtain from (4.2) and (4.3):

$$|\ Z(N,\ k^2,\ k,\ \chi_1,\ \chi_2)\ | \ll k^{-4}k^3 = k^{-1}.$$

Lemma 4.3. *For an integer k and integers* $b_1,\ b_2,\ N$ *satisfying* $N \equiv b_1 + b_2 (\operatorname{mod} k)$, *there is*

$$A(N,\ p^\beta,\ k) = \frac{1}{\phi^2(p^\beta)} \begin{Bmatrix} -1, & \beta = 1,\ (p,\ kN) = 1, \\ p - 1, & \beta = 1,\ (p,\ k) = 1,\ p \mid N, \\ p^\beta - p^{\beta-1}, & p \mid k,\ (p^\beta / k_{p^\beta},\ k_{p^\beta}) = 1, \\ 0, & \textit{else.} \end{Bmatrix}$$

Proof. This follows from Lemma 4.2(b).

Lemma 4.4. *Consider two primitive characters* $\chi_i \operatorname{mod} r_i (i = 1,\ 2)$, *the principal character* $\chi_0 \operatorname{mod} q,\ r = [r_1,\ r_2]$, *a number k which is either equal to* 1 *or a prime number, and a positive number* $M \le N$.

(a) *If* $k^m \,||\, r,\ m \in \{1,\ 2\}$, *then*

$$\sum_{\substack{q \le M \\ r \mid q}} |\ Z(N,\ q,\ k,\ \chi_1\chi_0,\ \chi_2\chi_0)\ | \ll k^{-1}L^2. \tag{4.4}$$

(b) *If* $(r,\ k) = 1$, *then*

$$\sum_{\substack{q \le M \\ r \mid q}} |\ Z(N,\ q,\ k,\ \chi_1\chi_0,\ \chi_2\chi_0)\ | \ll L^2. \tag{4.5}$$

(c) *If* $(r,\ k) = 1$, *then*

$$\sum_{\substack{q \le M \\ kr \mid q}} |\ Z(N,\ q,\ k,\ \chi_1\chi_0,\ \chi_2\chi_0)\ | \ll k^{-1}L^2. \tag{4.6}$$

Proof. (a) Applying Lemma 4.1(a), we can write $Z(N, q, k, ...) = Z(N, r', k, ...)A(N, l, 1)$, where $(r', l) = 1$, $r \mid r'$, and every prime factor that divides r' also divides r. From Lemma 4.1(b), we see that $Z(N, r', k, ...) = 0$ if $r' \neq r$. Using the notation introduced in (2.2) and again Lemma 4.1(a), we find $Z(N, r, k, ...) = Z(N, s_r, 1, ...)Z(N, k^m, k, ...)$. Thus, the proof can focus on terms $Z(N, q, ...)$ that can be written as $Z(N, q, k, ...) = Z(N, s_r, 1, ...)Z(N, k^m, k, ...)A(N, l, 1)$, where $(r, l) = 1$ and $(s_r, k) = 1$. In consequence, the right-hand side of (4.4) can be estimated as

$$\ll Z(N, s_r, 1, ...)Z(N, k^m, k, ...) \sum_{\substack{1 \le M/r \\ (l, k)=1}} A(N, l, 1), \tag{4.7}$$

where $(s_r, k) = 1$. We use Lemma 4.2(a) to estimate

$$Z(N, s_r, 1, ...) \ll L_2^2. \tag{4.8}$$

In order to estimate $Z(N, k^m, k, ...)$, for $m = 1$, we use the fact that by definition $|C(\chi, k, k, b, a)| \le 1$ whereas for $m = 2$, we use Lemma 4.2(e). Thus,

$$Z(N, k^m, k, ...) \ll k^{-1}. \tag{4.9}$$

Lemmas 4.1(a) and 4.3 imply

$$\sum_{\substack{l \le M/r \\ (l, k)=1}} |A(N, l, 1)| \ll \prod_{\substack{p \mid N \\ p \le M}} \left(1 + \frac{1}{p-1}\right) \sum_{l \le M/r} \phi^{-2}(l)$$

$$\ll N\phi^{-1}(N) \ll L. \tag{4.10}$$

The lemma follows from (4.7)-(4.10). For the proof of (b), we argue similarly and find that we need to estimate the expression

$$|Z(N, r, 1)| \sum_{l \le M/r} |A(N, l, 1)|. \tag{4.11}$$

Arguing similarly to the proof of part (a) and using Lemma 4.1, we see that

$$\sum_{l \le M/r} |A(N, l, 1)| \le \left(1 + \sum_{m \ge 1,\, k^m \le M/r} |A(N, k^m, k^m)|\right) \sum_{\substack{l \le M/r \\ (l,k)=1}} |A(N, l, 1)|. \tag{4.12}$$

For k prime, a trivial estimate shows

$$\sum_{m \ge 1,\, k^m \le M/r} |A(N, k^m, k^m)| \ll \sum_{m \ge 1,\, k^m \le M/r} k^{-m} \ll k^{-1}. \tag{4.13}$$

For $k = 1$, in (4.12), the sum $\sum_{m \ge 1,\, k^m \le M/r} |A(N, k^m, k^m)|$ is not needed in the estimate (4.12). Similar to (4.8),

$$|Z(N, r, 1)| \ll L_2^2. \tag{4.14}$$

Part (b) of the lemma follows from (4.10)-(4.14). For the proof of part (c), we follow the argument in (4.11) and estimate

$$|Z(N, r, 1)| \sum_{\substack{l \le M/r \\ k \mid l}} |A(N, l, k)|. \tag{4.15}$$

Using Lemma 4.2(b), we see that $A(N, l, k) = 0$ if $k^2 \mid l$. Applying again Lemma 4.3 and using (4.10), we obtain

$$\sum_{\substack{l \le M/r \\ k \mid l}} |A(N, l, k)| \ll |A(N, k, k)| \sum_{\substack{l \le M/rk \\ (l,k)=1}} |A(N, l, 1)| \ll k^{-1}L. \tag{4.16}$$

Part (c) of the lemma follows from (4.14)-(4.16).

Lemma 4.5. (a) *For any prime k,*

$$\sum_{\substack{q \ge U \\ (q,k)=1}} |A(N, q, 1)| \ll Ld(N)U^{-1}.$$

(b) *For any prime k and* $U \geq k$,

$$\sum_{\substack{q \geq U \\ k \mid q}} |A(N, q, k)| \ll L d(N) U^{-1}.$$

(c) *For any prime k*,

$$\sum_{q \geq 1} \frac{A(N, q, k_q)}{\phi^2(k/k_q)} = \sigma(N, k),$$

where $\sigma(N, k)$ *is defined in* (2.4).

Proof. For the proof of (a), we use Lemma 4.3 and obtain

$$\begin{aligned}\sum_{\substack{q \geq U \\ (q,k)=1}} |A(N, q, k)| &\leq \sum_{q \geq U} \frac{1}{\phi^2(q)} \phi((q, N)) \\ &\leq \sum_{d \mid N} \phi^{-1}(d) \sum_{q \geq U/d} \phi^{-2}(q) \\ &\ll L_2^3 U^{-1} d(N). \qquad (4.17)\end{aligned}$$

For the proof of (b), we similarly derive from Lemmas 4.1(a) and 4.3,

$$\begin{aligned}\sum_{\substack{q \geq U \\ k \mid q}} |A(N, q, k)| &\leq \phi^{-2}(k) \sum_{d \mid k} \phi(d) \sum_{\substack{q \geq U/d \\ (q,k)=1}} |A(N, q, k)| \\ &\ll \phi^{-2}(k) \sum_{d \mid k} \phi(d) L_2^3 d U^{-1} d(N) \\ &\ll d(N) L U^{-1}. \qquad (4.18)\end{aligned}$$

(c) We see from parts (a) and (b) of this lemma that the left-hand side of (4.17) is absolutely convergent. Thus, it is equal to its Euler product. Applying Lemma 4.3, we see

$$\frac{1}{\phi^2(k)}\sum_{q\geq 1}\frac{A(N,\ q,\ k_q)}{\phi^2(k/k_q)}$$

$$=\frac{1}{\phi^2(k)}\sum_{q\geq 1}A(N,\ q,\ k_q)\phi^2(k_q)$$

$$=\frac{1}{\phi^2(k)}\prod_{\substack{p\geq 2\\ (p,kN)=1}}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{(p,k)=1\\ p\mid N}}\left(1+\frac{1}{(p-1)}\right)$$

$$\cdot\prod_{p\mid k}\left(1+\sum_{\substack{b\geq 1\\ p^b\mid k}}(p^b-p^{b-1})\right)$$

$$=\frac{kN}{\phi(k)\phi(kN)}\prod_{\substack{p\geq 2\\ (p,kN)=1}}\left(1-\frac{1}{(p-1)^2}\right).$$

5. The Major Arcs

According to (2.6), we split the integral over the major arcs as follows:

$$R_{M,N}(k)=\int_{M_1+M_2+M_3}\prod_{i=1}^{2}S(\alpha,\ k,\ b_i)e(-\alpha N)d\alpha$$

$$=: G_1(k)+G_2(k)+G_3(k). \tag{5.1}$$

We first consider $G_1(k)$. As $k\nmid q$, we find

$$S\left(\frac{a}{q}+\lambda,\ b_i\right)=\sum_{g=1}^{q}{}^{*}e\left(\frac{ga}{q}\right)\sum_{\substack{N/4<n\leq N\\ n\equiv b_i\ (\mathrm{mod}\ k)\\ n\equiv g\ (\mathrm{mod}\ q)}}\Lambda(n)e(n\lambda)+O(L^2).$$

We introduce the Dirichlet characters ξ mod k and χ mod q and obtain

$$S\left(\frac{a}{q}+\lambda,\ b_i\right) = \frac{1}{\phi(k)\phi(q)} C(\chi_0,\ q,\ 1,\ b_i,\ a) T(\lambda)$$

$$+ \frac{1}{\phi(k)\phi(q)} \sum_{\xi \bmod k} \overline{\xi}(b_i) \sum_{\chi \bmod q} C(\overline{\chi},\ q,\ 1,\ b_i,\ a) W(\lambda,\ \xi\chi)$$

$$+ O(L^2), \tag{5.2}$$

where

$$W(\lambda,\ \chi) = \sum_{N/4<n\le N} \Lambda(n) e(n\lambda)\chi(n) - E_0(\chi) T(\lambda),$$

$$E_0(\chi) = \begin{cases} 1, & \text{if } \chi = \chi_0, \\ 0, & \text{otherwise.} \end{cases}$$

In the sequel, we will neglect the error term $O(L^2)$. We will see that its contribution will be dominated by other, larger error terms. Inserting (5.2) into (5.1), we obtain

$$G_1(k) = G_{1,M}(k) + G_{1,e}(k), \tag{5.3}$$

where

$$G_{1,M}(k) = \sum_{\substack{q\le P \\ k\nmid q}} \frac{1}{\phi^2(k)\phi^2(q)} \sum_{a=1}^{q^*} \prod_{i=1}^{2} C(\chi_0,\ q,\ 1,\ b_i,\ a) e\left(-\frac{a}{q}N\right)$$

$$\times \int_{-1/qQ}^{1/qQ} T^2(\lambda) e(-N\lambda) d\lambda,$$

$$G_{1,e}(k) = \sum_{\substack{q\le P \\ k\nmid q}} \frac{1}{\phi^2(k)\phi^2(q)} \sum_{a=1}^{q^*} e\left(-\frac{a}{q}N\right)$$

$$\times \int_{-1/qQ}^{1/qQ} \prod_{i=1}^{2} \left(\sum_{\xi \bmod k} \overline{\xi}(b_i) \sum_{\chi \bmod q} C(\overline{\chi},\ q,\ 1,\ b_i,\ a) W(\lambda,\ \xi\chi) \right)$$

$$\times e(-\lambda N)d\lambda$$

$$+\sum_{i=1}^{2}\sum_{\substack{q\le P\\ k\nmid q}}\frac{1}{\phi^2(k)\phi^2(q)}\sum_{a=1}^{q^*}e\left(-\frac{a}{q}N\right)$$

$$\times\int_{-1/qQ}^{1/qQ}\prod_{\substack{j=1\\ j\ne i}}^{2}\left(\sum_{\xi \bmod k}\overline{\xi}(b_j)\sum_{\chi \bmod q}C(\overline{\chi},\, q,\, 1,\, b_j,\, a)\right.$$

$$\left.\times W(\lambda,\, \xi\chi)\right)C(\chi_0,\, q,\, 1,\, b_i,\, a)T(\lambda)e(-\lambda N)d\lambda$$

$$=: \sum_1 + \sum_2. \tag{5.4}$$

We first evaluate the main term $G_{1,M}(k)$ using Lemma 4.4(b):

$$G_{1,M}(k) = \frac{1}{\phi^2(k)}\sum_{\substack{q\le P\\ k\nmid q}}A(N,\, q,\, 1)\int_{-1/2}^{1/2}T(\lambda)^2e(-N\lambda)d\lambda$$

$$+\, O\left(\frac{1}{\phi^2(k)}\sum_{\substack{q\le P\\ k\nmid q}}|\, A(N,\, q,\, 1)\,|\int_{1/qQ}^{1/2}\frac{1}{|\,\lambda\,|^2}\,d\lambda\right)$$

$$= \frac{1}{\phi^2(k)}\sum_{\substack{q\le P\\ k\nmid q}}A(N,\, q,\, 1)\frac{N}{2} + O\left(\frac{L^2QP}{\phi^2(k)}\right)$$

$$= \frac{1}{\phi^2(k)}\sum_{\substack{q\le P\\ k\nmid q}}A(N,\, q,\, 1)\frac{N}{2} + O(Xk^{-1}L^{-A}), \tag{5.5}$$

where we have used (2.1). We have also used $T(\lambda) \ll \dfrac{1}{|\,\lambda\,|}$ and

$$\int_{-1/2}^{1/2}T(\lambda)^2e(-N\lambda)d\lambda = \frac{N}{2} + O(1). \tag{5.6}$$

In the sequel, we will without further mentioning use the fact that for any character χ induced by a primitive character χ^*, we have $W(\lambda, \chi) = W(\lambda, \chi^*) + O(L^2)$. We also note for further usage that for a prime number k, each non-principal character χ mod k is a primitive character mod k and the principal character mod k is induced by the primitive character mod 1. Using Lemma 4.4(b), we estimate $\sum_1$:

$$\left|\sum_1\right|$$

$$\leq \frac{1}{\phi^2(k)} \sum_{\substack{q \leq P \\ k \nmid q}} \sum_{\chi_1 \bmod q} \sum_{\chi_2 \bmod q} \sum_{\xi_1 \bmod k} \sum_{\xi_2 \bmod k}$$

$$\times |Z(N, q, 1, \chi_1, \chi_2)| \int_{-1/qQ}^{1/qQ} \prod_{j=1}^{2} |W(\lambda, \chi_j \xi_j)| d\lambda$$

$$\leq \frac{1}{\phi^2(k)} \sum_{\substack{r_1 \leq P \\ k \nmid r_1}} \sum_{\substack{r_2 \leq P \\ k \nmid r_2}} \sum_{\chi_1 \bmod r_1}{}^{*} \sum_{\chi_2 \bmod r_2}{}^{*} \sum_{\xi_1 \bmod k} \sum_{\xi_2 \bmod k}$$

$$\times \int_{-1/[r_1, r_2]Q}^{1/[r_1, r_2]Q} \prod_{j=1}^{2} (|W(\lambda, \chi_j \xi_j)| + L^2) d\lambda \sum_{\substack{q \leq P_1 \\ [r_1, r_2] | q}} |Z(N, q, 1, \chi_1 \chi_0, \chi_2 \chi_0)|$$

$$\ll \frac{L^2}{\phi^2(k)} \sum_{\substack{r_1 \leq P \\ k \nmid r_1}} \sum_{\substack{r_2 \leq P \\ k \nmid r_2}} \sum_{\chi_1 \bmod r_1}{}^{*} \sum_{\chi_2 \bmod r_2}{}^{*} \left(\sideset{}{^*}\sum_{\xi_1 \bmod k} + \sum_{\xi_1 = \chi_0 (\bmod 1)} \right)$$

$$\times \left(\sideset{}{^*}\sum_{\xi_2 \bmod k} + \sum_{\xi_2 = \chi_0 (\bmod 1)} \right) \int_{-1/[r_1, r_2]Q}^{1/[r_1, r_2]Q} \prod_{j=1}^{2} (|W(\lambda, \chi_j \xi_j)| + L^2) d\lambda. \quad (5.7)$$

In the following, we will neglect the error terms L^2 in the last integral in (5.7) as their contribution will be dominated by other terms. We see from (2.2) and (5.7),

$$\sum_{1} \ll k^{-2}L^2\left(\sum_{\substack{r_1\le P_k\\ k\|r_1}}\sum_{\substack{r_2\le P_k\\ k\|r_2}}+\sum_{\substack{r_1\le P_k\\ k\|r_1}}\sum_{\substack{r_2\le P\\ k\nmid r_2}}+\sum_{\substack{r_1\le P\\ k\nmid r_1}}\sum_{\substack{r_2\le P\\ k\nmid r_2}}\right)$$

$$\times \sum_{\chi_1 \bmod r_1}{}^{*} \sum_{\chi_2 \bmod r_2}{}^{*} \int_{-1/[s_{r_1}, s_{r_2}]Q}^{1/[s_{r_1}, s_{r_2}]Q} \prod_{j=1}^{2} |W(\lambda, \chi_j)|\, d\lambda$$

$$=: \sum_{1,1} + \sum_{1,2} + \sum_{1,3}, \tag{5.8}$$

where each $\sum_{1,i}$ stands for one of the multiple sums in (5.8). We see

$$\sum_{1,1} \ll k^{-1}L^2W_A^2, \tag{5.9}$$

where

$$W_A = k^{-1/2}\sum_{\substack{r\le P_k\\ k\mid r}}\sum_{\chi(\bmod r)}{}^{*}\left(\int_{-k/rQ}^{k/rQ} |W(\lambda, \chi)|^2 d\lambda\right)^{1/2}.$$

Arguing similarly, we obtain

$$\sum_{1,2} + \sum_{1,3} \ll k^{-1}L^2(W_AW_B + W_B^2), \tag{5.10}$$

where

$$W_B = k^{-1/2}\sum_{\substack{r\le P\\ k\nmid r}}\sum_{\chi(\bmod r)}{}^{*}\left(\int_{-1/rQ}^{1/rQ} |W(\lambda, \chi)|^2 d\lambda\right)^{1/2}.$$

In the same way, we find

$$\sum_2 \ll k^{-1}L^2 \max_{|\lambda|\le 1/Q}\left(\int_{-1/Q}^{1/Q} |T(\lambda)|^2 dl\right)^{1/2}(W_A + W_B)$$

$$\ll k^{-1}L^2 X^{1/2}(W_A + W_B). \tag{5.11}$$

We see from (5.4) and (5.8)-(5.11):

$$G_{1,e}(k) \ll k^{-1}L^2(W_A^2 + W_A W_B + W_B^2 + X^{1/2}W_A + X^{1/2}W^B). \tag{5.12}$$

For $q \in M_2$, we see

$$S\left(\frac{a}{q}+\lambda,\, b_i\right) = \sum_{\substack{g=1 \\ g\equiv b_i (\mathrm{mod}\, k)}}^{q} {}^{*} e\left(\frac{ga}{q}\right) \sum_{\substack{N/4<n\le N \\ n\equiv b_i (\mathrm{mod}\, k) \\ n\equiv g (\mathrm{mod}\, q)}} \Lambda(n)e(n\lambda)$$

$$= \sum_{\substack{g=1 \\ g\equiv b_i (\mathrm{mod}\, k)}}^{q} {}^{*} e\left(\frac{ga}{q}\right) \sum_{\substack{N/4<n\le N \\ n\equiv g (\mathrm{mod}\, q)}} \Lambda(n)e(n\lambda)$$

$$= \frac{1}{\phi(q)} C(\chi_0,\, q,\, k,\, b_i,\, a)T(\lambda)$$

$$+ \frac{1}{\phi(q)} \sum_{\chi \bmod q} C(\overline{\chi},\, q,\, k,\, b_i,\, a)W(\lambda,\, \chi). \tag{5.13}$$

Inserting (5.13) into (5.1), we obtain

$$G_2(k) = G_{2,M}(k) + G_{2,e}(k), \tag{5.14}$$

where

$$G_{2,M}(k) = \sum_{\substack{q\le P_1 \\ k\,||\,q}} A(N,\, q,\, k)\int_{-1/2}^{1/2} T(\lambda)^2 e(-N\lambda)d\lambda$$

$$+ O\left(\sum_{\substack{q\le P_1 \\ k\,||\,q}} |\, A(N,\, q,\, k)\,| \int_{1/qQ}^{1/2} \frac{1}{|\lambda|^2} d\lambda\right)$$

$$= \sum_{\substack{q \le P_1 \\ k \| q}} A(N, q, k) \frac{N}{2} + O\left(\frac{qQL^2}{k} \right)$$

$$= \sum_{\substack{q \le P_1 \\ k \| q}} A(N, q, k) \frac{N}{2} + O(Xk^{-1}L^{-A}), \tag{5.15}$$

where we have used Lemma 4.4(a) and (5.6). Further, we see

$$G_{2,e}(k) = \sum_{\substack{q \le P_1 \\ k \| q}} \frac{1}{\phi^2(q)} \sum_{a=1}^{q}{}^* \int_{-1/qQ}^{1/qQ} \prod_{i=1}^{2} \left(\sum_{\chi \bmod q} C(\overline{\chi}, q, k, b_i, a) W(\lambda, \chi) \right)$$

$$\times e\left(-\frac{a}{q} N - \lambda N \right) d\lambda$$

$$+ \sum_{i=1}^{2} \sum_{\substack{q \le P_1 \\ k \| q}} \frac{1}{\phi^2(q)} \sum_{a=1}^{q}{}^* \int_{-1/qQ}^{1/qQ} \prod_{\substack{j=1 \\ j \ne i}}^{2} \left(\sum_{\chi \bmod q} C(\overline{\chi}, q, k, b_j, a) W(\lambda, \chi) \right)$$

$$\times C(\chi_0, q, k, b_i, a) T(\lambda) e\left(-\frac{a}{q} N - \lambda N \right) d\lambda$$

$$=: \sum_3 + \sum_4 . \tag{5.16}$$

We estimate $\sum_3$ as

$$\left| \sum_3 \right|$$

$$\le \sum_{\substack{q \le P_1 \\ k | q}} \sum_{\chi_1 \bmod q} \sum_{\chi_2 \bmod q} | Z(N, q, k, \chi_1, \chi_2) |$$

$$\times \int_{-1/qQ}^{1/qQ} \prod_{j=1}^{2} | W(\lambda, \chi_j) | d\lambda$$

$$\ll \left(\sum_{\substack{r_1 \le P_1 \\ k | r_1}} \sum_{\substack{r_2 \le P_1 \\ k | r_2}} + \sum_{\substack{r_1 \le P_1 \\ k | r_1}} \sum_{\substack{r_2 \le P_1/k \\ k \nmid r_2}} + \sum_{\substack{r_1 \le P_1/k \\ k \nmid r_1}} \sum_{\substack{r_2 \le P_1/k \\ k \nmid r_2}} \right) \sum_{\chi_1 \bmod r_1}{}^{*} \sum_{\chi_2 \bmod r_2}{}^{*}$$

$$\times \int_{-1/[r_1, r_2]Q}^{1/[r_1, r_2]Q} \prod_{j=1}^{2} (|W(\lambda, \chi_j)| + L^2) d\lambda \sum_{\substack{q \le P_1 \\ [r_1, r_2] | q \\ k | q}} |Z(N, q, k, \chi_1\chi_0, \chi_2\chi_0)|$$

$$=: \sum_{i=1}^{3} \sum_{3, i}. \tag{5.17}$$

The condition $k | q$ in the sum $\sum_{\substack{q \le P_1 \\ [r_1, r_2] | q \\ k | q}}$ is only necessary for the sum $\sum_{3, 3}$. In the other cases, $k | [r_1, r_2]$ implies $k | q$. We will make use of this condition when estimating $\sum_{3, 3}$. In the following, we again neglect the error term L^2 as it is dominated by other terms. We use Lemma 4.4(a) to estimate $\sum_{3, 1}$:

$$\sum_{3, 1} \ll k^{-1} L^2 \sum_{\substack{r_1 \le P_1 \\ k | r_1}} \sum_{\substack{r_2 \le P_2 \\ k | r_2}} \sum_{\chi_1 \bmod r_1}{}^{*} \sum_{\chi_2 \bmod r_2}{}^{*}$$

$$\times \int_{-1/[r_1, r_2]Q}^{1/[r_1, r_2]Q} \prod_{j=1}^{2} (|W(\lambda, \chi_j)| + L^2) d\lambda$$

$$:= k^{-1} L^2 W_C^2, \tag{5.18}$$

where

$$W_C := \sum_{\substack{r \le P_1 \\ k | r}} \sum_{\chi \bmod r}{}^{*} \left(\int_{-1/rQ}^{1/rQ} |W(\lambda, \chi)|^2 d\lambda \right)^{1/2}.$$

Similarly, we see using Lemmas 4.4(a) and (c),

$$\sum_{3,2} + \sum_{3,3} \ll k^{-1}L^2(W_C W_D + W_D^2), \tag{5.19}$$

where

$$W_D := \sum_{\substack{r \le P_1/k \\ k \nmid r}} \sum_{\chi \bmod r}{}^{*} \left(\int_{-1/rQ}^{1/rQ} |W(\lambda, \chi)|^2 d\lambda \right)^{1/2}.$$

For $\sum_4$, we obtain in the same way

$$\left| \sum_4 \right| \ll k^{-1}L^2(W_C + W_D)\left(\int_M |T^2(\lambda)| d\lambda \right)^{1/2}$$

$$\le k^{-1}L^2 X^{1/2}(W_C + W_D). \tag{5.20}$$

We see from (5.16)-(5.20),

$$G_{2,e}(k) \ll k^{-1}L^2(W_C^2 + W_C W_D + W_D^2 + X^{1/2}W_C + X^{1/2}W_D). \tag{5.21}$$

Using Lemma 4.2(d) and arguing similar to the estimation of $G_2(k)$, we see

$$G_3(k) = G_{3,M}(k) + G_{3,e}(k), \tag{5.22}$$

where

$$G_{3,M}(k) \ll \sum_{\substack{q \le P_2 \\ k^2 || q}} |A(N, q, k)| \int_{-1/qQ}^{1/qQ} T(\lambda)^2 e(-N\lambda) d\lambda, \tag{5.23}$$

$$G_{3,e}(k) \ll k^{-1}L^2(W_E^2 + X^{1/2}W_E), \tag{5.24}$$

where

$$W_E := \sum_{\substack{r \le P_2 \\ k^2 || r}} \sum_{\chi \bmod r}{}^{*} \left(\int_{-1/rQ}^{1/rQ} |W(\lambda, \chi)|^2 d\lambda \right)^{1/2}.$$

Further, we see from Lemma 4.2(b) that

$$k^2 \mid q \Rightarrow A(N, q, k) = 0. \tag{5.25}$$

Thus, we see from (5.5), (5.15), (5.23), and (5.25) for sufficiently large $B = B(A)$,

$$\begin{aligned}
&G_{1,M}(k) + G_{2,M}(k) + G_{3,M}(k) \\
&= \frac{1}{\phi^2(k)} \sum_{\substack{q \le P \\ k \nmid q}} A(N, q, 1)\frac{N}{2} + \sum_{\substack{q \le P_1 \\ k \| q}} A(N, q, k)\frac{N}{2} + O(Xk^{-1}L^{-A}) \\
&= \sigma(N, k)\frac{N}{2} \\
&\quad + O\left(X\left(\frac{1}{\phi^2(k)} \sum_{\substack{q > P \\ k \nmid q}} |A(N, q, 1)| + \sum_{\substack{q > P_1 \\ k \| q}} |A(N, q, k)| \right) + Xk^{-1}L^{-A} \right) \\
&= \sigma(N, k)\frac{N}{2} + O(Xd(N)k^{-1}L^{-A}),
\end{aligned} \tag{5.26}$$

for $B > B(A)$. Here, we have used Lemma 4.5 and (5.25). We note that for any prime k and integer b with $(k, b) = 1$,

$$\begin{aligned}
\sum_{\substack{N \le X \\ n \equiv b (\mathrm{mod}\, k)}} d(N) &= \sum_{a=1}^{k-1} \sum_{\substack{n_1 \le X \\ n_1 \equiv a (\mathrm{mod}\, k)}} \sum_{\substack{n_2 \le X/n_1 \\ n_2 \equiv \bar{b} a (\mathrm{mod}\, k)}} 1 \\
&\ll Xk^{-1} \sum_{a=1}^{k-1} \sum_{\substack{n_1 \le X \\ n_1 \equiv \bar{b} a (\mathrm{mod}\, k)}} n_1^{-1} \\
&\ll Xk^{-2} \sum_{a=1}^{k-1} \sum_{l \le (X - \bar{b} a)/k} l^{-1} \ll Xk^{-1}L.
\end{aligned}$$

Thus, $d(N) \ll L^{P+1}$ for all but $O(Xk^{-1}L^{-P})$ integers $XL^{-A} \leq N \leq X$ satisfying $N \equiv b_1 + b_2 (\operatorname{mod} k)$. We note that in (5.26) the constant A can take any positive value by adjusting the constant implied in the $O(..)$ term accordingly. Setting $A = A + P + 1$, we can thus derive from (5.26) that

$$G_{1, M}(k) + G_{2, M}(k) + G_{3, M}(k) = \sigma(N, k)\frac{N}{2} + O(Xk^{-1}L^{-A}), \qquad (5.27)$$

for all but $O(Xk^{-1}L^{-P})$ integers $N \leq x$ satisfying $N \equiv b_1 + b_2 (\operatorname{mod} k)$.

Further, using the relation $ab \ll a^2 + b^2$, we see from (5.12), (5.21), and (5.24):

$$G_{1, e}(k) + G_{2, e}(k) + G_{3, e}(k) \ll k^{-1}L^2 \sum_{F \in \{A, B, C, D, E\}} (W_F^2 + X^{1/2}W_F). \quad (5.28)$$

In summary, we see from (5.1), (5.3), (5.14), (5.22), (5.27), and (5.28) that the proof of (2.8) reduces to the proof of the following lemma:

Lemma 5.1. *For* $k \leq X^{\frac{5}{48}-\varepsilon}$, *then for* $F \in \{A, B, D\}$,

$$W_F \ll X^{\frac{1}{2}}L^{-A} \qquad (5.29)$$

for any $A > 0$. *For* $k \leq X^{\frac{5}{48}-\varepsilon}$ *and if none of the integers* $q \in P_k$ *is X-exceptional, then* (5.29) *holds for* $F \in \{C, E\}$.

6. Proof of Lemma 5.1

In order to prove the lemma for $F = A$, it is enough to show that

$$W_{A, R} \ll X^{\frac{1}{2}}k^{1/2}L^{-A}, \qquad (6.1)$$

where

$$W_{A, R} = \sum_{\substack{r \sim R \\ k \mid r}} \sum_{\chi (\operatorname{mod} r)}{}^{*} \left(\int_{-k/rQ}^{k/rQ} |W(\lambda, \chi)|^2 d\lambda \right)^{1/2}$$

for $R \leq Pk$. Applying Lemma 1 [6], we see

$$\int_{-k/rQ}^{k/rQ} |W(\lambda, \chi)|^2 d\lambda$$

$$\ll (RQ/k)^{-2} \int_{N/8}^{N} \left| \sum_{\substack{t<m\leq t+Qr/k \\ N/4<m\leq N}} \Lambda(m)\chi(m) - E_0(\chi) \sum_{\substack{t<m\leq t+Qr \\ N/4<m\leq N}} 1 \right|^2 dt. \quad (6.2)$$

We note that $E_0(\chi) = 0$ because of $R \geq k$ and the primitivity of the characters. We set $X = \max(N/4, t)$ and $X + Y = \min(N, t + rQ/k)$. We apply a slight modification of Heath-Brown's identity ([10])

$$-\frac{\zeta'}{\zeta}(s) = \sum_{j=1}^{K} \binom{K}{j} (-1)^{j-1} \zeta'(s)\zeta^{j-1}(s) M^j(s) - \frac{\zeta'}{\zeta}(s)(1 - \zeta(s)M(s))^K,$$

with $K = 5$ and

$$M(s) = \sum_{n \leq N^{1/5}} \mu(n) n^{-s}$$

to the sum

$$\sum_{X < m \leq X+Y} \Lambda(m)\chi(m).$$

Arguing exactly as in part III, [24] we find by applying Heath-Brown's identity and Perron's summation formula that the inner sum of (6.2) is a linear combination of $O(L^c)$ terms of the form

$$S_{I_{a_1}, \ldots, I_{a_{10}}}$$

$$= \frac{1}{2\pi i} \int_{-T}^{T} F\left(\frac{1}{2} + iu, \chi\right) \frac{(X + Y)^{(1/2+iu)} - X^{(1/2+iu)}}{\frac{1}{2} + iu} du + O(T^{-1}NL^2),$$

where $2 \le T \le N$,

$$F(s,\chi) = \prod_{j=1}^{10} f_j(s,\chi),\ f_j(s,\chi) = \sum_{n\in I_j} a_j(n)\chi(n)n^{-s},$$

$$a_j(n) = \begin{cases} \log n \text{ or } 1, & j=1, \\ 1, & 1 < j \le 5, \\ \mu(n), & 6 \le j \le 10, \end{cases} \qquad I_j = (N_j, 2N_j], \quad 1 \le j \le 10,$$

$$N \ll \prod_{j=1}^{10} N_j \ll N, \quad N_j \le N^{1/5}, \quad 6 \le j \le 10.$$

Since

$$\frac{(X+Y)^{(1/2+iu)} - X^{(1/2+iu)}}{\frac{1}{2}+iu} \ll \min(QRk^{-1}N^{-\frac{1}{2}},\ N^{\frac{1}{2}}(|u|+1)^{-1})$$

by taking $T = N$ and $T_0 = N(QR/k)^{-1}$, we conclude that for a sufficiently large $G = G(M)$, $S_{I_{a_1},\ldots,I_{a_{10}}}$ is bounded by

$$\ll QRk^{-1}N^{-\frac{1}{2}}\int_{-T_0}^{T_0}\left|F\left(\frac{1}{2}+iu,\chi\right)\right|du$$

$$+ N^{\frac{1}{2}}\int_{T_0\le|u|\le T}\left|F\left(\frac{1}{2}+iu,\chi\right)\right|\frac{du}{|u|} + L^2.$$

Thus, we see from (6.2) that in order to prove (6.1), it is enough to show that for $R \le Pk$:

$$\sum_{\substack{r\sim R \\ k|r}}\sum_{\chi}{}^{*}\int_0^{T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt \ll X^{1/2}k^{1/2}L^{-A}, \tag{6.3}$$

$$\sum_{\substack{r\sim R\\ k|r}}\sum_{\chi}{}^{*}\int_{T_1}^{2T_1}\left|F\left(\frac{1}{2}+it,\,\chi\right)\right|dt \ll X^{-1/2}QRk^{-1/2}T_1L^{-A},\; T_0<|T_1|\le T. \quad (6.4)$$

Inequalities (6.3) and (6.4) are both derived from the following lemma which is shown for $m = 1$ in [15, Lemma 5.2] and for the general case $m \geq 1$ in [16, Lemma 2.1].

Lemma 6.1. *Let* $F(s,\,\chi)$ *be defined as above. Then for any* $R \geq 1$ *and* $T_2 > 0$,

$$\sum_{\substack{r\sim R\\ m|r}}\sum_{\chi}{}^{*}\int_{T_2}^{2T_2}\left|F\left(\frac{1}{2}+it,\,\chi\right)\right|dt \ll \left(\frac{R^2}{m}T_2+\frac{R}{m^{1/2}}T_2^{1/2}N^{3/10}+N^{1/2}\right)L^{c}. \quad (6.5)$$

Using (2.1) and (2.3), the estimates (6.3) and (6.4) follow from Lemma 6.1 by setting $T_2 = T_0$ and $T_2 = T_1$, respectively, provided that $k \leq N^{1/5-\varepsilon}$.

In order to prove the lemma for $F = B$, it is sufficient to show that

$$W_{B,\,R} \ll X^{\frac{1}{2}}L^{-A}, \quad (6.6)$$

where $W_{B,\,R}$ is defined as

$$W_{B,\,R} = \sum_{r\sim R}\ \sum_{\chi(\bmod r)}{}^{*}\left(\int_{-1/rQ}^{1/rQ}|W(\lambda,\,\chi)|^2 d\lambda\right)^{1/2} \quad (6.7)$$

for $R \leq P$. We note that in (6.7), we omit the factor $k^{-1/2}$ included in the definition of W_B as we can derive the desired estimate without taking it into account. Arguing as in the case $F = A$, we can estimate the sum on the right-hand side of (6.7) by using the zero expansion of the von Mangoldt-function:

$$\sum_{X<m\le X+Y} \Lambda(m)\chi(m) - E_0(\chi) \sum_{X<m\le X+Y} 1$$

$$\ll \sum_{|\operatorname{Im}\rho|\le T} \left| \frac{(X+Y)^\rho}{\rho} - \frac{X^\rho}{\rho} \right| + O\left(\frac{X}{T} L^2\right)$$

$$\ll QR \sum_{|\operatorname{Im}\rho|\le T} N^{\beta-1} + O\left(\frac{X}{T} L^2\right), \tag{6.8}$$

where ρ runs over the non-trivial zeros of the L-function corresponding to χ mod r with $|\operatorname{Im}\rho| \le T$ and $\beta = \operatorname{Re}\rho$. We now use the fact that $L(\sigma + it, \chi)$ with χ mod r and $r \le L^D$ has no zeros in the region (see [20, VIII Satz 6.2])

$$\sigma \ge 1 - \delta(T) := 1 - \frac{c_0}{\log r + (\log(T+2))^{4/5}}, \quad |t| \le T,$$

where c_0 is an absolute constant. We now make using of the following lemma from [12]:

Lemma 6.2. *Let* $N^*(\alpha, T, q)$ *denote the number of zeros* $\sigma + it$ *of all* L*-functions to primitive characters modulo* q *within the region* $\sigma \ge \alpha$, $|t| \le T$. *Then for any positive integer m and* $1/2 \le \alpha \le 1$:

$$\sum_{\substack{q\le P \\ m|q}} N^*(\alpha, T, q) \ll \left(\frac{P^2 T}{m}\right)^{\left(\frac{12}{5}+\varepsilon\right)(1-\alpha)}.$$

Taking $T = N^{1/3}$, we apply Lemma 6.2 and derive from (6.2) and (6.8),

$$W_{B,R} \ll N^{1/2} \sum_{r\sim R} \sideset{}{^*}\sum_{\chi(\operatorname{mod} r)} \sum_{|\operatorname{Im}\rho|\le N^{1/3}} N^{\beta-1} + N^{1/6} k^3 L^{5B+2}$$

$$\ll N^{1/2}L^{c}\left(\max_{\frac{1}{2}\le\beta\le 1-\delta(T)} N^{\left(\frac{4}{5}+\varepsilon\right)(1-\beta)}N^{(\beta-1)}\right)^{2} + N^{\frac{1}{2}-\varepsilon}$$

$$\ll X^{1/2}\exp(-cL^{1/5}) \tag{6.9}$$

as $k \le X^{\frac{5}{48}-\varepsilon}$. For the proof of the case $F = C,$ we define $W_{C,R}$ similarly to $W_{A,R}$ in (6.1) as

$$W_{C,R} = \sum_{\substack{r\sim R\\ k|r}}\ \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, r)}\left(\int_{-1/rQ}^{1/rQ}|W(\lambda,\chi)|^{2}d\lambda\right)^{1/2}.$$

Taking $T = k^{2}L^{2V}$ and arguing as in (6.9), we obtain by using (1.5) and the assumptions of Theorem 2,

$$W_{C,R} \ll N^{1/2}\sum_{\substack{r\le P_1\\ k|r}}\ \sideset{}{^*}\sum_{X\sim r}\ \sum_{|\mathrm{Im}\,\rho|\le k^{2}L^{2V}} N^{\beta-1} + N^{1/2}L^{-A}$$

$$\ll N^{1/2}L^{C}\max_{\frac{1}{2}\le\beta\le 1-\frac{EL_2}{L}}\left(N^{\left(\frac{5}{12}-2\varepsilon\right)\left(\frac{12}{5}+\varepsilon\right)(1-\beta)}N^{\beta-1}\right) + N^{1/2}L^{-A}$$

$$\ll X^{1/2}L^{-A}, \tag{6.10}$$

for sufficiently large $E = E(A,\varepsilon), V = V(A),$ and $k \le N^{\frac{5}{36}-\varepsilon}$. In order to prove the lemma for $F = D,$ we need to show

$$W_{D,R} \ll X^{\frac{1}{2}}L^{-A},$$

where $W_{D,R}$ is defined as

$$W_{D,R} = \sum_{r\sim R} \sum_{\chi(\bmod r)}{}^{*} \left(\int_{-1/rQ}^{1/rQ} |W(\lambda, \chi)|^2 d\lambda \right)^{1/2} \tag{6.11}$$

for $R \le P_1/k = L^B$. We see from (6.7) and (6.11) that

$$W_{D,R} = W_{B,R}. \tag{6.12}$$

We also see that the respective maximum size of R is P for $F = B$ and P_1/k for $F = D$ and that by (2.3) $P = P_1/k$. Thus, for $F = D$ the lemma follows from (6.6) and (6.12).

For the case $F = E$, we argue as in the case $F = C$. Here, the upper bound $k \ll N^{\frac{5}{48}-\varepsilon}$ is required.

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