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# THE DEGENERATIONS FOR MODULES AND DUAL MODULES 

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#### Abstract

Let $\Lambda$ be a finite-dimensional algebra over algebraically closed field $\mathbb{k}$ and $\Lambda^{o}$ be an algebra opposite to $\Lambda$. Denote by $\bmod _{d}^{\Lambda}$ and $\bmod _{d}^{\Lambda^{O}}$ the category of all $d$-dimensional right modules over $\mathbb{k}$-algebra $\Lambda$ and $d$-dimensional left modules over $\mathbb{k}$-algebra $\Lambda^{o}$. In this paper, we show that a module $M$ in $\bmod _{d}^{\Lambda}$ degenerates to another module $N$ in $\bmod _{d}^{\Lambda}$ if and only if dual module $M^{*}$ in $\bmod _{d}^{\Lambda^{0}}$ degenerates to dual module $N^{*}$ in $\bmod _{d}^{\Lambda^{\circ}}$.


## 1. Introduction

[^0]polynomials in $\mathbb{k}\left[X_{1}, \ldots, X_{S}\right]$ and subset of affine space $\mathbb{A}_{\mathbb{k}}^{s}$ which contains all s-tuple of elements of algebraically closed field $\mathbb{k}$ [3]. For this condition, the set $\mathcal{M}_{d}(\mathbb{k})$ of all $d \times d$-matrices over $\mathbb{k}$ can be considered as affine space $\mathbb{A}_{\frac{d_{k}}{}}$. This fact may be used to give a geometrical structure of affine variety for $\bmod _{d}^{\Lambda}$, the category of $d$-dimensional right module over finitedimensional $\mathbb{k}$-algebra $\Lambda$, by parameterizing $\bmod _{d}^{\Lambda}$ into $n$-tuple of matrices in $\mathcal{M}_{d}(\mathbb{k})$, where $n$ is number of $\mathbb{k}$-basis of $\mathbb{k}$-algebra $\Lambda$ [1]. Moreover, the general linear group $G L_{d}(\mathbb{k})$ acts on $\bmod _{d}^{\Lambda}$ and the $G L_{d}(\mathbb{k})$-orbits correspond to the isomorphism classes of $d$-dimensional $\Lambda$-modules [4]. We denote by $\mathcal{O}(M)$ the $G L_{d}(\mathbb{k})$-orbit of module $M$ in $\bmod _{d}^{\Lambda}$. A module $M$ in $\bmod _{d}^{\Lambda}$ degenerates to another module $N$ in $\bmod _{d}^{\Lambda}$ if $N$ belongs to Zariski closure $\overline{\mathcal{O}(M)}$ of $\mathcal{O}(M)$ in $\bmod _{d}^{\Lambda}$ and we denote this by $M \leq_{\operatorname{deg}} N$ [4]. By duality, we can easily show that $\bmod _{d}^{\Lambda^{0}}$ the category of $d$-dimensional left module over finite-dimensional $\mathbb{k}$-algebra $\Lambda^{0}$, where $\Lambda^{0}$ is the algebra opposite to $\Lambda$, is also affine variety. Moreover, the result $n$-tuple matrices which identify module $M^{*}$ in $\bmod _{d}^{\Lambda^{0}}$ is the transpose of $n$-tuple matrices which identify module $M$ in $\bmod _{d}^{\Lambda}$. In this paper, we will show that a module $M$ in $\bmod _{d}^{\Lambda}$ degenerates to another module $N$ in $\bmod _{d}^{\Lambda}$ if and only if dual module $M^{*}$ in $\bmod _{d}^{\Lambda^{0}}$ degenerates to dual module $N^{*}$ in $\bmod _{d}^{\Lambda^{0}}$.

## 2. The Modules Variety

Throughout this paper, $\mathbb{k}$ denotes the algebraically closed field. An algebra over $\mathbb{k}$, or $\mathbb{k}$-algebra, is a $\mathbb{k}$-vector space $A$ together with a bilinear associative multiplication. An algebra $A$ is said to be finite-dimensional if the $\mathbb{k}$-vector space $A$ is finite-dimensional. By $\Lambda$, we denote the $n$-dimensional
$\mathbb{k}$-algebra with identity. Let $\left\{a_{1}=1, a_{2}, \ldots, a_{n}\right\}$ be $\mathbb{k}$-basis of $\Lambda$. Note that the multiplication of two elements in $A$ is uniquely determined by the products of the basis vectors. Indeed, if $a=\sum_{i=1}^{n} \alpha_{i} a_{i}$ and $a=\sum_{j=1}^{n} \beta_{j} a_{j}$, then

$$
a b=\left(\sum_{i=1}^{n} \alpha_{i} a_{i}\right)\left(\sum_{i=1}^{n} \alpha_{i} a_{i}\right)=\sum_{i, j=1}^{n} \alpha_{i} \beta_{j} a_{i} a_{j}
$$

Now, we can decompose $a_{i} a_{j}$ with respect to the basis : $a_{i} a_{j}=\sum_{k=1}^{n} \gamma_{i j}^{k} a_{k}$.
This means that the structure of the algebra over the space $\Lambda$ with a fixed basis is uniquely given by a choice of $n^{3}$ elements $\gamma_{i j}^{k}(i, j, k=1, \ldots, n)$ of the field $\mathbb{k}$. These elements are called the structure constants of $\mathbb{k}$-algebra $\Lambda$. By $\Lambda^{0}$, we denote the opposite algebra of $\Lambda$, that is, the $\mathbb{k}$-algebra whose underlying set and vector space structure are just those $\Lambda$, but the multiplication $\cdot$ in $\Lambda^{o}$ is defined by formula $a \cdot b=b a$ for all $a, b \in \Lambda^{o}$ [2].

It is convenient to view the elements of $\Lambda$ as operator on $\mathbb{k}$-vector space $M$. This leads to the concept of a module. A right module over $\mathbb{k}$-algebra $\Lambda$, or a right $\Lambda$-module, is a $\mathbb{k}$-vector space $M$ together a $\mathbb{k}$-linear action of $\Lambda$ on $M, \Lambda \times M \rightarrow M,(a, m) \mapsto a m$ which satisfy the following conditions: for all $a, b \in \Lambda, m, n \in M, \beta \in \mathbb{k}$,
(a) $(m+n) a=m a+n a$,
(b) $m(a+b)=m a+n b$,
(c) $(\beta m) a=m(\beta a)=\beta(m a)$,
(d) $m(a b)=(m a) b$,
(e) $m 1_{\Lambda}=m$.

The definition of a left $\Lambda$-module is analogous. A module $M$ is said to be finite-dimensional if the dimension $\operatorname{dim}_{\mathbb{k}} M$ of underlying $\mathbb{k}$-vector space of $M$ is finite [2]. We denote by $\bmod _{d}^{\Lambda}$ the category of all $d$-dimensional right $\Lambda$-modules. By parameterizing each element in $\bmod _{d}^{\Lambda}$ into $n$-tuple of matrices in $\mathcal{M}_{d}(\mathbb{k})$, where $n$ is number of $\mathbb{k}$-basis of $\mathbb{k}$-algebra $\Lambda$, we have the following result [1].

Lemma 2.1. $\bmod _{d}^{\Lambda}$ is naturally an affine variety.
Proof. Let $a_{1}=1, a_{2}, \ldots, a_{n}$ be $\mathbb{k}$-basis of $\Lambda$. We have $a_{i} a_{j}=$ $\sum_{k=1}^{n} \gamma_{i j}^{k} a_{k}$, with the structure constant $\gamma_{i j}^{k} \in \mathbb{k}(i, j, k=1, \ldots, n)$. Let

$$
\begin{gathered}
\mathcal{M}_{\gamma}=\left\{\left(\mathbb{M}_{1}=\mathbb{I}_{d}, \ldots, \mathbb{M}_{n}\right) \in \prod_{i=1}^{n} \mathcal{M}_{d}(\mathbb{k}) \mid \mathbb{M}_{i} \mathbb{M}_{j}\right. \\
\left.=\sum_{k=1}^{n} \gamma_{i j}^{k} \mathbb{M}_{k},(i, j=1, \ldots, n)\right\}
\end{gathered}
$$

where $\mathbb{I}_{d}$ is $d \times d$-identity matrix over $\mathbb{k}$. We will show that there is oneone correspondence between $\bmod _{d}^{\Lambda}$ and $\mathcal{M}_{\gamma}$. Let $M$ be arbitrary elements in $\bmod _{d}^{\Lambda}$. Conditions (a) and (c) in the definition of module ensure that $\forall i \in\{1, \ldots, n\}$, the mapping $\mathcal{A}_{i}: m \mapsto m a_{i}, \forall m \in M$ is an endomorphism of $M$. By fixing an ordered $\mathbb{k}$-basis $\mathfrak{B}_{M}$ of $M$, we can regard $\mathcal{A}_{i}$ as $\mathbb{M}_{i} \in \mathcal{M}_{d}(\mathbb{k})$. Since $a_{1}=1$ and $a_{i} a_{j}=\sum_{k=1}^{n} \gamma_{i j}^{k} a_{k}, \mathbb{M}_{1}=\mathbb{I}_{d}$ and $\mathbb{M}_{i} \mathbb{M}_{j}$ $=\sum_{k=1}^{n} \gamma_{i j}^{k} \mathbb{M}_{k}, \quad(i, j=1, \ldots, n)$. So, $M$ can be identified with $\left(\mathbb{M}_{1}, \ldots, \mathbb{M}_{n}\right)$ $\in \mathcal{M}_{\gamma}$. Conversely, $\left(\mathbb{X}_{1}, \ldots, \mathbb{X}_{n}\right) \in \prod_{i=1}^{n} \mathcal{M}_{d}(\mathbb{k})$ if $\mathbb{X}_{1}, \ldots, \mathbb{X}_{n}$ satisfy $\mathbb{X}_{1}=\mathbb{I}_{d}$ and $\mathbb{X}_{i} \mathbb{X}_{j}=\sum_{k=1}^{n} \gamma_{i j}^{k} \mathbb{X}_{k}, \quad(i, j=1, \ldots, n)$. Therefore, there is a bijection between $\bmod _{d}^{\Lambda}$ and $\mathcal{M}_{\gamma}$.

To complete the proof, it is sufficient to show that $\mathcal{M}_{\gamma}$ is an affine variety. For each $\xi \in\{1, \ldots, n\}$, let $\mathbb{X}_{\xi}=\left(x_{r t}^{\xi}\right) \in \mathcal{M}_{d}(\mathbb{k})$ and $\mathbb{k}\left[x_{r t}^{j}\right]$, $(r, t=$ $1, \ldots, d ; j=1, \ldots, n$ ) be $\mathbb{k}$-algebra of all polynomials with $n d^{2}$ variables over $\mathbb{k}$. Let $\mathcal{U} \subset\left[x_{r t}^{j}\right]$ be two sides ideals which generated by all entries in the matrices $\mathbb{X}_{\mu} \mathbb{X}_{v}-\sum_{k=1}^{n} \gamma_{\mu \nu}^{k} \mathbb{X}_{k},(\mu, v=1, \ldots, n)$. Consequently, if $\left(\mathbb{M}_{1}=\right.$ $\left.\left(m_{r t}^{1}\right), \ldots, \mathbb{M}_{n}=\left(m_{r t}^{n}\right)\right) \in \mathcal{M}_{\gamma}$, then $\forall p\left(x_{r t}^{j}\right) \in \mathcal{U}, \quad p\left(m_{r t}^{j}\right)=0$. This means that $\bmod _{d}^{\Lambda} \cong \mathcal{M}_{\gamma}$ is the zero set of the ideal $\mathcal{U}$. Therefore, $\bmod _{d}^{\Lambda}$ is an affine variety.

Because of Lemma 2.1, we can say that $\bmod _{d}^{\Lambda}$ is a module variety. Clearly, the general linear group $G L_{d}(\mathbb{k})=\left\{\mathbb{M} \in \mathcal{M}_{d}(\mathbb{k}) \mid \operatorname{det} \mathbb{M} \neq 0\right\}$ acts on $\bmod _{d}^{\Lambda}$ via

$$
\mathbb{G} \cdot M=\mathbb{G} \cdot\left(\mathbb{M}_{1}, \ldots, \mathbb{M}_{n}\right)=\left(\mathbb{G M}_{1} \mathbb{G}^{-1}, \ldots, \mathbb{G M}_{n} \mathbb{G}^{-1}\right)
$$

$\forall \mathbb{G} \in G L_{d}(\mathbb{k})$ and for all $M \in \bmod _{d}^{\Lambda}$ which identified by $\left(\mathbb{M}_{1}, \ldots, \mathbb{M}_{n}\right) \in$ $\mathcal{M}_{\gamma}$. We denote by $\mathcal{O}(M)$ the $G L_{d}(\mathbb{k})$-orbit of module $M$ in $\bmod _{d}^{\Lambda}$.

Lemma 2.2. The orbits of $G L_{d}(\mathbb{k})$-action on $\bmod _{d}^{\Lambda}$ correspond to the isomorphism classes of d-dimensional $\Lambda$-modules.

Proof. Let $U$ and $V$ be $d$-dimensional $\Lambda$-modules which are isomorphism under $\rho$. By Lemma 2.1, we may assume that $U$ and $V$ in $\bmod _{d}^{\Lambda}$ are represented by $\left(\mathbb{U}_{1}, \ldots, \mathbb{U}_{n}\right)$ and $\left(\mathbb{V}_{1}, \ldots, \mathbb{V}_{n}\right)$ in $\mathcal{M}_{\gamma}$, respectively. Fix $\mathbb{k}$-bases in $U$ and $V$. Let $\mathbb{G}$ represent $\rho$ in the fixed $\mathbb{k}$-bases. We know that $\mathbb{G}$ is an isomorphism if only if $\mathbb{G} \in G L_{d}(\mathbb{k})$. The fact that $\rho$ is a $\Lambda$-module isomorphism gives the equation $\mathbb{V}_{i} \mathbb{G}=\mathbb{G} \mathbb{U}_{i}$ whence $\mathbb{V}_{i}=\mathbb{G}_{i} \mathbb{G}^{-1}$. Thus, the orbits of $G L_{d}(\mathbb{k})$-action on $\bmod _{d}^{\Lambda}$ correspond to the isomorphism classes of $d$-dimensional $\Lambda$-modules.

## 3. The Degeneration for Dual Modules

First, we shall establish the duality between the category of all $d$-dimensional right $\Lambda$-modules and category of all $d$-dimensional left $\Lambda^{o}$-modules. We denote by $\bmod _{d}^{\Lambda^{o}}$ the category of all $d$-dimensional right $\Lambda^{o}$-modules. To every module $M \in \bmod _{d}^{\Lambda}$ we assign a module $M^{*} \in \bmod _{d}^{\Lambda^{0}}$ constructed as follows. As $\mathbb{k}$-vector space, $M^{*}$ is the space $\operatorname{Hom}_{\mathbb{k}}(M, \mathbb{k})$ of linear functional on $M$. Operators from $\Lambda=\Lambda^{o}$ act on $M^{*}$ according the formula $(a f)(m)=f(m a)$ for all $a \in \Lambda, \quad f \in M^{*}$ and $m \in M$. From this action, $M^{*}$ becomes a left $\Lambda^{o}$-module [2].

Every linear map $\varphi: M \rightarrow N$ induces a conjugate map $\varphi^{*}: N^{*} \rightarrow M^{*}$ defined by $\left(\varphi^{*} g\right)(m)=g(\varphi m)$ for all $g \in N, m \in M$. We can check readily that if $\varphi$ is a $\Lambda$-module homomorphism, so is $\varphi$ [2].

Let $M \in \bmod _{d}^{\Lambda}$ and $\mathfrak{B}_{M}=\left\{v_{1}, \ldots, v_{d}\right\}$ be $\mathbb{k}$-basis for $M$. We define dual basis $\mathfrak{B}_{M^{*}}^{*}$ for $M^{*}$ by setting $\mathfrak{B}_{M^{*}}^{*}=\left\{f_{1}, \ldots, f_{d}\right\}$, where $f_{i}: M \rightarrow \mathbb{k}$ is the unique linear functional which satisfies $f_{i}\left(v_{i}\right)=1$ and $f_{i}\left(v_{j}\right)=0$ whenever $j \neq i$.

Lemma 3.1. $\bmod _{d}^{\Lambda^{0}}$ is an affine variety where $n$-tuple matrices which parameterize an element in $\bmod _{d}^{\Lambda^{\circ}}$ are the transpose of $n$-tuple matrices which parameterize an element in $\bmod _{d}^{\Lambda}$.

Proof. Let $a_{1}=1, a_{2}, \ldots, a_{n}$ be $\mathbb{k}$-basis of $\Lambda$ and $M^{*}$ be an arbitrary element in $\bmod _{d}^{\Lambda^{o}}$ which is dual module of $M \in \bmod _{d}^{\Lambda}$. Suppose that $\mathfrak{B}_{M}$ $=\left\{v_{1}, \ldots, v_{d}\right\}$ and $\mathfrak{B}_{M^{*}}^{*}=\left\{f_{1}, \ldots, f_{d}\right\}$ are $\mathbb{k}$-bases for $M$ and for $M^{*}$, respectively. Fix $s \in\{1, \ldots, n\}$. By Lemma 2.1, the mapping $\mathcal{A}_{s}: v \mapsto v a_{s}$, $\forall v \in M$ is an endomorphism of $M$ and let $\left[\mathcal{A}_{s}\right]_{\mathfrak{B}}=\mathbb{M}_{s}=\left(\eta_{i j}^{S}\right) \in \mathcal{M}_{d}(\mathbb{k})$,
that is, $M$ is identified with $\left(\mathbb{M}_{1}, \ldots, \mathbb{M}_{n}\right) \in \mathcal{M}_{\gamma}$. Then $\mathcal{A}_{s}\left(v_{i}\right)=\sum_{k=1}^{d} \eta_{k i} v_{k}$.
On the other hand, $\mathcal{A}_{s}^{*}: f \mapsto a_{s} f, \forall f \in M^{*}$ define an endomorphism of $M^{*}$. We compute $\mathcal{A}_{s}^{*}\left(f_{j}\right)$. By definition, $\forall \sum_{i=1}^{d} \beta_{i} v_{i}=v \in M$,

$$
\begin{aligned}
\left(\mathcal{A}_{s}^{*}\left(f_{j}\right)\right)(v) & =\sum_{i=1}^{d} \beta_{i}\left(f_{j}\left(\mathcal{A}_{s}\left(v_{i}\right)\right)\right) \\
& =\sum_{i=1}^{d} \beta_{i}\left(f_{j}\left(\sum_{k=1}^{d} \eta_{k i} v_{k}\right)\right)=\left(\sum_{k=1}^{d} \eta_{j k} f_{k}\right)(v) .
\end{aligned}
$$

It follows that $\mathcal{A}_{s}^{*}\left(f_{j}\right)$ is the linear functional $\sum_{k=1}^{d} \eta_{j k} f_{k}$. In particular, $\left[\mathcal{A}_{s}^{*}\right]_{\mathfrak{B}_{M^{*}}^{*}}=\left(\eta_{j i}\right)=\left(\mathbb{M}_{s}\right)^{T}$. Since this fact holds for all $s \in\{1, \ldots, n\}, M^{*}$ corresponds to the element $\left(\mathbb{M}_{1}^{T}, \ldots, \mathbb{M}_{n}^{T}\right) \in \mathcal{M}_{\gamma}$, i.e., the transpose of $n$-tuple matrices which parameterize $M$. The fact that $\mathcal{M}_{\gamma}$ is an affine variety follows from the proof of Lemma 2.1.

Because of Lemma 3.1, we can say that $\bmod _{d}^{\Lambda}$ is a dual module variety. By the same argument for $\bmod _{d}^{\Lambda}$, the general linear group $G L_{d}(\mathbb{k})$ acts on $\bmod _{d}^{\Lambda^{\circ}}$ via conjugation and the $G L_{d}(\mathbb{k})$-orbits correspond to the isomorphism classes of $d$-dimensional $\Lambda^{o}$-modules. Now, we give the definition of Zariski closure [4].

Definition 3.1. Let $M \in \bmod _{d}^{\Lambda}$ and $\mathcal{O}(M)$ be its $G L_{d}(\mathbb{k})$-orbits. Then the Zariski closure of $\mathcal{O}(M)$ is $\overline{\mathcal{O}(M)}=\left\{N \in \bmod _{d}^{\Lambda} \mid p(N)=0, \forall\right.$ polynomials $p$ such that $\left.p(\mathcal{O}(M))=0\right\}$.

Definition 3.2. Let $M, N \in \bmod _{d}^{\Lambda}$. Module $M$ degenerates to $N$, written $M \leq_{\operatorname{deg}} N$, if $\mathcal{O}(N) \subseteq \overline{\mathcal{O}(M)}$.

The following is the main theorem of this work.
Theorem 3.1. Given $M, N \in \bmod _{d}^{\Lambda}$. Let $M^{*}, N^{*} \in \bmod _{d}^{\Lambda}$ be dual modules of $M$ and $N$, respectively. Module $M^{*} \leq_{\operatorname{deg}} N^{*}$ if and only if $M \leq \leq_{\operatorname{deg}} N$.

Proof. Let $\left(\mathbb{M}_{1}, \ldots, \mathbb{M}_{n}\right),\left(\mathbb{N}_{1}, \ldots, \mathbb{N}_{n}\right) \in \mathcal{M}_{\gamma}$ be presentation of $M$ and $N$, respectively. By Lemma 3.1, $\left(\mathbb{M}_{1}^{T}, \ldots, \mathbb{M}_{n}^{T}\right),\left(\mathbb{N}_{1}^{T}, \ldots, \mathbb{N}_{n}^{T}\right) \in \mathcal{M}_{\gamma}$ are presentation of $M^{*}$ and $N^{*}$, respectively.

Assume that $M \leq_{\text {deg }} N$. Then $\mathcal{O}(N) \subseteq \overline{\mathcal{O}(M)}$. Let $\left(\mathbb{U}_{1}^{T}, \ldots, \mathbb{U}_{n}^{T}\right) \in$ $\mathcal{O}\left(N^{*}\right) \subset \mathcal{M}_{\gamma}$. This means that there exists $\mathbb{G}^{T} \in G L_{d}(\mathbb{k})$ such that

$$
\left(\mathbb{U}_{1}^{T}, \ldots, \mathbb{U}_{n}^{T}\right)=\left(\mathbb{G}^{T} \mathbb{N}_{1}^{T}\left(\mathbb{G}^{T}\right)^{-1}, \ldots, \mathbb{G}^{T} \mathbb{N}_{n}^{T}\left(\mathbb{G}^{T}\right)^{-1}\right)
$$

Hence,

$$
\begin{aligned}
\left(\mathbb{U}_{1}, \ldots, \mathbb{U}_{n}\right) & =\left(\left(\mathbb{U}_{1}^{T}\right)^{T}, \ldots,\left(\mathbb{U}_{n}^{T}\right)^{T}\right) \\
& =\left(\left(\mathbb{G}^{T} \mathbb{N}_{1}^{T}\left(\mathbb{G}^{T}\right)^{-1}\right)^{T}, \ldots,\left(\mathbb{G}^{T} \mathbb{N}_{n}^{T}\left(\mathbb{G}^{T}\right)^{-1}\right)^{T}\right) \\
& =\left(\mathbb{G}^{-1} \mathbb{N}_{1} \mathbb{G}, \ldots, \mathbb{G}^{-1} \mathbb{N}_{n} \mathbb{G}\right) \in \mathcal{O}(N)
\end{aligned}
$$

Since $\mathcal{O}(N) \subseteq \overline{\mathcal{O}(M)},\left(\mathbb{U}_{1}, \ldots, \mathbb{U}_{n}\right) \in \overline{\mathcal{O}(M)}$, that is,

$$
p\left(\left(\mathbb{U}_{1}, \ldots, \mathbb{U}_{n}\right)\right)=0, \quad \forall \text { polynomials } p \text { such that } p(\mathcal{O}(M))=0
$$

Now, for each polynomial $p$ with variables in entries of matrices, that is, $p\left(\left(x_{r t}^{1}\right), \ldots,\left(x_{r t}^{n}\right)\right) \in \mathbb{k}\left[x_{r t}^{j}\right](r, t=1, \ldots, d ; j=1, \ldots, n)$, we define polynomial $p^{*}$ by changing the order of variables through transposing $n$-tuple matrices, that is,

$$
p^{*}\left(\left(x_{r t}^{1}\right)^{T}, \ldots,\left(x_{r t}^{n}\right)^{T}\right)=p\left(\left(x_{r t}^{1}\right), \ldots,\left(x_{r t}^{n}\right)\right)
$$

This means that, $p\left(\left(x_{r t}^{1}\right), \ldots,\left(x_{r t}^{n}\right)\right)=0$ if and only if $p^{*}\left(\left(x_{r t}^{1}\right)^{T}, \ldots,\left(x_{r t}^{n}\right)^{T}\right)$ $=0$. Hence, $p(\mathcal{O}(M))=0$ if and only if $p^{*}\left(\mathcal{O}\left(M^{*}\right)\right)=0$. Consequently,

$$
p^{*}\left(\left(\mathbb{U}_{1}^{T}, \ldots, \mathbb{U}_{n}^{T}\right)\right)=0, \forall \text { polynomials } p^{*} \text { such that } p^{*}\left(\mathcal{O}\left(M^{*}\right)\right)=0 .
$$

Then $\left(\mathbb{U}_{1}^{T}, \ldots, \mathbb{U}_{n}^{T}\right) \in \mathcal{O}\left(M^{*}\right)$. Since $\left(\mathbb{U}_{1}^{T}, \ldots, \mathbb{U}_{n}^{T}\right)$ is arbitrary element in $\mathcal{O}\left(N^{*}\right)$, we conclude that

$$
\mathcal{O}\left(N^{*}\right) \subseteq \overline{\mathcal{O}\left(M^{*}\right)}
$$

Therefore, $M^{*} \leq_{\operatorname{deg}} N$. The proof for the converse is similar.

## 4. Concluding Remarks

We have shown that the $n$-tuple matrices which identify module $M^{*}$ in $\bmod _{d}^{\Lambda^{0}}$ is the transpose of $n$-tuple matrices which identify module $M$ in $\bmod _{d}^{\Lambda}$ and that a module $M$ in $\bmod _{d}^{\Lambda}$ degenerates to another module $N$ in $\bmod _{d}^{\Lambda}$ if and only if dual module $M^{*}$ in $\bmod _{d}^{\Lambda^{0}}$ degenerates to dual module $N^{*}$ in $\bmod _{d}^{\Lambda^{0}}$.

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[^0]:    An affine variety is the set of simultaneous zeroes of collection of © 2012 Pushpa Publishing House
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