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SOME ALGEBRAIC PROPERTIES OF DUAL GENERALIZED QUATERNIONS ALGEBRA

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Abstract

Some algebraic properties of dual generalized quaternions are presented, and De-Moivre's and Euler's formulas for these quaternions are investigated. The solutions of equation $Q^n = 1$ are discussed and it is shown that it has not solutions for a general unit dual generalized quaternion.

1. Introduction

Dual quaternions are powerful mathematical tools for the spatial analysis of rigid body motions. Dual numbers and dual quaternions were introduced

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in the 19th century by Clifford [2], as a tool for his geometrical investigation. Study [8] and Kotel'nikov [6] systematically applied the dual number and dual vector in their studies of line geometry and kinematics and independently discovered the transfer principle. The use of dual numbers, matrices and dual quaternions in instantaneous spatial kinematics are investigated in [9, 11]. Also, dual quaternion algebra is used to express point-line displacement operation [12].

The Euler's and De-Moivre's formulas for the complex numbers are generalized for quaternions in [1]. These formulas are also investigated for the cases of split and dual quaternions in [5, 7]. In [10], by the help of Euler's formula, circles in the plane are obtained and the sphere in 3-space is found by means of the exponential expansions. In this paper, after a review of some properties of dual generalized quaternions, De-Moivre's and Euler's formulas for these quaternions are studied. Solutions of the equation $Q^n = 1$ are discussed and it is shown that there are no solutions for a general unit dual generalized quaternion. The relations between the powers of these quaternions are given.

2. Preliminaries

In this section, we give a brief summary of the generalized quaternions and dual generalized quaternions. For detailed information about these quaternions, we refer the reader to [3, 4].

Definition 2.1. A *generalized quaternion* *q* is defined as

$$q = a_{.} + a_1 i + a_2 j + a_3 k,$$

where $a_{.}, a_1, a_2$ and a_3 are real numbers and $1, i, j, k$ of q may be interpreted as the four basic vectors of Cartesian set of coordinates; and they satisfy the non-commutative multiplication rules

$$i^2 = -\alpha, \quad j^2 = -\beta, \quad k^2 = -\alpha\beta,$$

$$ij = k = -ji, \quad jk = \beta i = -kj$$

and

$$ki = \alpha j = -ik, \quad \alpha, \beta \in \mathbb{R}.$$

The set of all generalized quaternions are denoted by $H_{\alpha\beta}$.

Definition 2.2. A dual generalized quaternion Q is written as

$$Q = A.1 + A_1 i + A_2 j + A_3 k,$$

where $A.$, A_1, A_2 and A_3 are dual numbers and i, j, k are quaternionic units which satisfy in the above equalities. As a consequence of this definition, a generalized dual quaternion Q can also be written as:

$$Q = q + \varepsilon q^*, \quad q, q^* \in H_{\alpha\beta},$$

where q and q^*, real and pure dual generalized quaternion components, respectively. A quaternion $Q = A.1 + A_1 i + A_2 j + A_3 k$ is pieced into two parts with scalar piece $S_Q = A.$ and vectorial piece $\vec{V}_Q = A_1 i + A_2 j + A_3 k.$ We also write $Q = S_Q + \vec{V}_Q$. The conjugate of $Q = S_Q + \vec{V}_Q$ is then defined as $\overline{Q} = S_Q - \vec{V}_Q$. If $S_Q = 0$, then Q is called *pure dual generalized quaternion*, we may be called its *dual generalized vector*.

Dual quaternionic multiplication of two dual quaternions $Q = S_Q + \vec{V}_Q$ and $P = S_P + \vec{V}_P$ is defined:

$$\begin{aligned} QP &= S_Q S_P - g(\vec{V}_Q, \vec{V}_P) + S_P \vec{V}_Q + S_Q \vec{V}_P + \vec{V}_Q \wedge \vec{V}_P \\ &= A.B. - (\alpha A_1 B_1 + \beta A_2 B_2 + \alpha\beta A_3 B_3) + A.(B_1, B_2, B_3) \\ &\quad + B.(A_1, A_2, A_3) \\ &\quad + (\beta(A_2 B_3 - A_3 B_2), \alpha(A_3 B_1 - A_1 B_3), (A_1 B_2 - A_2 B_1)). \end{aligned}$$

The norm of Q is defined as $N_Q = Q\overline{Q} = \overline{Q}Q = A^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2$. If $N_Q = 1$, then Q is called a *unit generalized dual quaternion*. The set of all dual generalized quaternions (DGQ) is denoted by $\widetilde{H}_{\alpha\beta}$.

3. De Moiver's Formula for DGQ

We investigate the properties of the dual generalized quaternions in two different cases:

Case 1. Let α, β be positive numbers.

Definition 3.1. Let $\hat{S}_D^3$ be the set of all unit dual generalized quaternions and $\hat{S}_D^2$ be the set of unit dual generalized vector, that is,

$$\hat{S}_D^3 = \{Q \in \tilde{H}_{\alpha\beta} : N_Q = 1\} \subset \tilde{H}_{\alpha\beta},$$

$$\hat{S}_D^2 = \{\vec{V}_Q = (A_1, A_2, A_3) : g(\vec{V}_Q, \vec{V}_Q) = \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 = 1\}.$$

Definition 3.2. Every nonzero unit dual generalized quaternion can be written in the polar form

$$Q = A. + A_1 i + A_2 j + A_3 k$$

$$= \cos\phi + \vec{W}\sin\phi,$$

where $\cos\phi = A$, $\sin\phi = \sqrt{\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}$. $\phi = \varphi + \varepsilon\varphi^*$ is a dual angle and the unit dual generalized vector $\vec{W}$ is given by

$$\vec{W} = \frac{A_1 i + A_2 j + A_3 j}{\sqrt{\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}} = \frac{A_1 i + A_2 j + A_3 j}{\sqrt{1 - A^2}},$$

with $\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 \neq 0$.

For any $\vec{W} \in \hat{S}_D^2$, since $g(\vec{W}, \vec{W}) = 1$ and $\vec{W} \wedge \vec{W} = 0$, $\vec{W}^2 = -1$. Therefore, any $\vec{W} \in \hat{S}_D^2$ is of order 4, i.e., $\vec{W}^4 = 1$. We have a natural generalization of Euler's formula for dual generalized quaternions

$$e^{\vec{W}\phi} = 1 + \vec{W}\phi - \frac{\phi^2}{2!} - \vec{W}\frac{\phi^3}{3!} + \frac{\phi^4}{4!} + \cdots$$

$$= \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4} - \cdots\right) + \vec{W}\left(\phi - \frac{\phi^3}{3} + \frac{\phi^5}{5!} - \cdots\right)$$

$$= \cos\phi + \vec{W}\sin\phi,$$

for any dual number ϕ. For detailed information about Euler's formula, see [10].

Note that $\vec{W}$ is a unit dual generalized vector to which a directed line in $E^3_{\alpha\beta}$ corresponds by means of the generalized Study map.

Lemma 3.1. *For any* $\vec{W} \in \hat{S}^2_D$, *we have*

$$(\cos\phi + \vec{W}\sin\phi)(\cos\psi + \vec{W}\sin\psi) = \cos(\phi + \psi) + \vec{W}\sin(\phi + \psi).$$

Proof. See [5] for a similar proof. □

Theorem 3.2 (De-Moivre's formula)**.** *Let* $Q = e^{\vec{W}\phi} = \cos\phi + \vec{W}\sin\phi \in \hat{S}^3_D$, *where* $\phi = \varphi + \varepsilon\varphi^*$ *is dual angle and* $\vec{W} \in \hat{S}^2_D$. *Then for every integer n*,

$$Q^n = \cos n\phi + \vec{W}\sin n\phi.$$

Proof. We use induction on positive integers n. Assume that $Q^n = \cos n\phi + \vec{W}\sin n\phi$ holds. Then

$$\begin{aligned} Q^{n+1} &= (\cos\phi + \vec{W}\sin\phi)^n(\cos\phi + \vec{W}\sin\phi) \\ &= (\cos n\phi + \vec{W}\sin n\phi)(\cos\phi + \vec{W}\sin\phi) \\ &= \cos(n\phi + \phi) + \vec{W}\sin(n\phi + \phi) \\ &= \cos(n+1)\phi + \vec{W}\sin(n+1)\phi. \end{aligned}$$

Hence, the formula is true. Moreover, since

$$Q^{-1} = \cos\phi - \vec{W}\sin\phi,$$

$$Q^{-n} = \cos(-n\phi) + \vec{W}\sin(-n\phi)$$

$$= \cos n\phi - \vec{W}\sin n\phi,$$

the formula holds for all integers. □

Special case. If $\alpha = \beta = 1,$ then Theorem 3.2 holds for dual quaternions (see [5]).

Every generalized dual quaternion is separated into two cases:

(1) Dual generalized quaternion with dual angles $(\phi = \varphi + \varepsilon\varphi^*);$ i.e.,

$$Q = \sqrt{N_Q}(\cos\phi + \vec{W}\sin\phi).$$

(2) Dual generalized quaternions with real angles $(\phi = \varphi,\ \varphi^* = 0);$ i.e.,

$$Q = \sqrt{N_Q}(\cos\varphi + \vec{W}\sin\varphi).$$

Theorem 3.3. *Let* $Q = \cos\varphi + \vec{W}\sin\varphi \in \hat{S}_D^3.$ *De-Moivre's formula implies that there are uncountably many unit dual generalized quaternions satisfying* $Q^n = 1$ *for* $n \geq 3.$

Proof. For every $\vec{W} \in \hat{S}_D^2,$ the unit dual generalized quaternion

$$Q = \cos\frac{2\pi}{n} + \vec{W}\sin\frac{2\pi}{n}$$

is of order n. For $n = 1$ or $n = 2,$ the dual generalized quaternion Q is independent of $\vec{W}.$ □

Example 3.1.

$$Q = \frac{1}{\sqrt{2}} + \frac{1}{2}\left(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\beta}}, \varepsilon\right) = \cos\frac{\pi}{4} + \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{\alpha}}i + \frac{1}{\sqrt{\beta}}j + \varepsilon k\right]\sin\frac{\pi}{4}$$

is of order 8 and

$$Q_2 = -\frac{1}{2} + \left(2\varepsilon, \frac{1}{\sqrt{2\beta}}, \frac{1}{2\sqrt{\alpha\beta}}\right) = \cos\frac{2\pi}{3} + \frac{1}{\sqrt{3}}\left[2\varepsilon i + \sqrt{\frac{2}{\beta}}\, j + \frac{1}{\sqrt{\alpha\beta}}\, k\right]\sin\frac{2\pi}{3}$$

is of order 3.

Also, we find the nth root of $Q = \cos\varphi + \vec{W}\sin\varphi \in \hat{S}_D^3$. The equation $X^n = Q$ has n roots. Thus,

$$X = \cos\left(\frac{\varphi + 2k\pi}{n}\right) + \vec{W}\sin\left(\frac{\varphi + 2k\pi}{n}\right), \quad k = 0, \ldots, n-1.$$

Corollary 3.4. *The equation* $Q^n = 1$ *does not have any solution for a general unit dual generalized quaternions.*

Example 3.2. Let $Q = \cos\left(\frac{\pi}{3} + \varepsilon\right) + \vec{W}\sin\left(\frac{\pi}{3} + \varepsilon\right)$ be a unit dual generalized quaternion. There is no n $(n > 0)$ such that $Q^n = 1$.

Theorem 3.5. *Let Q be a unit dual generalized quaternion with the polar form* $Q = \cos\varphi + \vec{W}\sin\varphi$. *If* $m = \frac{2\pi}{\varphi} \in \mathbb{Z}^+ - \{1\}$, *then* $n \equiv p(\operatorname{mod} m)$ *is possible if and only if* $Q^n = Q^p$.

Proof. Let $n \equiv p(\operatorname{mod} m)$. Then we have $n = a.m + p$, where $a \in \mathbb{Z}$,

$$\begin{aligned} Q^n &= \cos n\varphi + \vec{W}\sin n\varphi \\ &= \cos(am + p)\varphi + \vec{W}\sin(am + p)\varphi \\ &= \cos\left(a\frac{2\pi}{\varphi} + p\right)\varphi + \vec{W}\sin\left(a\frac{2\pi}{\varphi} + p\right)\varphi \\ &= \cos(p\varphi + a2\pi) + \vec{W}\sin(p\varphi + a2\pi) \\ &= \cos(p\varphi) + \vec{W}\sin(p\varphi) \\ &= Q^p. \end{aligned}$$

Now suppose $Q^n = \cos n\varphi + \vec{W} \sin n\varphi$ and $Q^p = \cos p\varphi + \vec{W} \sin p\varphi.$ Since $Q^n = Q^p,$ we have $\cos n\varphi = \cos p\varphi$ and $\sin n\varphi = \sin p\varphi,$ which means $n\varphi = p\varphi + 2\pi a,$ $a \in \mathbb{Z}.$ Thus, $n = a\dfrac{2\pi}{\varphi} + p,$ $n \equiv p(\operatorname{mod} m).$ □

Example 3.3. Letting $Q = \dfrac{1}{\sqrt{2}} + \dfrac{1}{2}\left(\dfrac{1}{\sqrt{\alpha}}, \dfrac{1}{\sqrt{\beta}}, \varepsilon\right) \in \hat{S}_D^3$ from Theorem 3.5, $m = \dfrac{2\pi}{\pi/4} = 8,$ we have

$$
\begin{aligned}
&Q = Q^9 = Q^{17} = \cdots \\
&Q^2 = Q^{10} = Q^{18} = \cdots \\
&Q^3 = Q^{11} = Q^{19} = \cdots \\
&Q^4 = Q^{12} = Q^{20} = \cdots = -1 \\
&\ldots \\
&Q^8 = Q^{16} = Q^{24} = \cdots = 1.
\end{aligned}
$$

Case 2. Let α be a positive number and β be a negative number.

In this case, for a dual generalized quaternion $Q = A_. + A_1 i + A_2 j + A_3 k,$ we can consider three different cases:

Case A. The norm of dual generalized quaternion is negative, i.e.,

$$N_Q = A_.^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 < 0,$$

since $0 < A_.^2 < -\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2$ thus $\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 < 0.$ In this case, the polar form of Q is defined as

$$Q = r(\sinh \Psi + \vec{W} \cosh \Psi),$$

where we assume

$$r = \sqrt{|N_Q|} = \sqrt{|A.^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2|},$$

$$\sinh \Psi = \frac{A.}{\sqrt{|N_Q}}, \cosh \Psi = \frac{\sqrt{-\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2}}{\sqrt{|N_Q}}.$$

The unit dual vector $\vec{W}$ (axis of quaternion) is defined as

$$\vec{W} = (w_1, w_2, w_3) = \frac{1}{\sqrt{-\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2}}(A_1, A_2, A_3).$$

Special case. If $\alpha = 1, \beta = -1$ and the norm be negative number, i.e.,

$$N_Q = A^2 + A_1^2 - A_2^2 - A_3^2 < 0,$$

then the quaternion Q is a dual split quaternion with negative norm and its polar forms

$$Q = \sqrt{|N_Q|}(\sinh \Psi + \vec{W} \cosh \Psi).$$

Theorem 3.6 (De-Moivre's formula)**.** *Let* $Q = r(\sinh \Psi + \vec{W} \cosh \Psi)$ *be a dual generalized quaternion with* $N_Q < 0$. *Then for every integer n,*

$$Q^n = r^n(\sinh n\Psi + \vec{W} \cosh n\Psi).$$

Proof. The proof follows immediately from the induction. □

Case B. The norm of dual generalized quaternion is positive and the norm of its vector part to be negative, i.e.,

$$N_Q > 0, \quad \vec{V}_Q = \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 < 0.$$

In this case, the polar form of Q is defined as

$$Q = r(\cosh \Phi + \vec{W} \sinh \Phi),$$

where we assume

$$r = \sqrt{N_Q} = \sqrt{A.^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2},$$

$$\cosh\Phi = \frac{A.}{\sqrt{N_Q}}, \ \sinh\Phi = \frac{\sqrt{-\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2}}{\sqrt{N_Q}}.$$

The unit dual vector $\vec{W}$ (axis of quaternion) is defined as

$$\vec{W} = (w_1, w_2, w_3) = \frac{1}{\sqrt{-\alpha A_1^2 - \beta A_2^2 - \alpha\beta A_3^2}}(A_1, A_2, A_3).$$

Theorem 3.7. *Let* $Q = r(\cosh\Phi + \vec{W}\sinh\Phi)$ *be a dual generalized quaternion with* $N_Q > 0$ *and* $\vec{V}_Q < 0$. *Then for every integer n*,

$$Q^n = r^n(\cosh n\Phi + \vec{W}\sinh n\Phi).$$

Proof. We use induction on positive integers n. Assume that $Q^n = r^n(\cosh n\Phi + \vec{W}\sinh n\Phi)$ holds. Then

$$\begin{aligned} Q^{n+1} &= r^n(\cosh n\Phi + \vec{W}\sinh n\Phi)\cdot r(\cosh\Phi + \vec{W}\sinh\Phi) \\ &= r^{n+1}(\cosh n\Phi + \vec{W}\sinh n\Phi)(\cosh\Phi + \vec{W}\sinh\Phi) \\ &= r^{n+1}(\cosh n\Phi\cosh\Phi + \sinh n\Phi\sinh\Phi \\ &\quad + \vec{W}(\cosh n\Phi\sinh\Phi + \sinh n\Phi\cosh\Phi) \\ &= r^{n+1}(\cosh(n+1)\Phi + \vec{W}\sinh(n+1)\Phi). \end{aligned}$$

Hence, the formula is true. □

Case C. The norm of dual generalized quaternion is positive and the norm of its vector part to be positive, i.e.

$$N_Q > 0, \quad \vec{V}_Q = \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2 > 0.$$

In this case, the polar form of Q is defined as

$$Q = r(\cos\Theta + \vec{W}\sin\Theta),$$

where we assume

$$r = \sqrt{N_Q} = \sqrt{A_\cdot^2 + \alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2},$$

$$\cos\Theta = \frac{A_\cdot}{\sqrt{N_Q}}, \quad \sin\Theta = \frac{\sqrt{\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}}{\sqrt{N_Q}}.$$

The unit dual vector $\vec{W}$ (axis of quaternion) is defined as

$$\vec{W} = (w_1, w_2, w_3) = \frac{1}{\sqrt{\alpha A_1^2 + \beta A_2^2 + \alpha\beta A_3^2}}(A_1, A_2, A_3).$$

Theorem 3.8. *Let* $Q = r(\cos\Theta + \vec{W}\sin\Phi)$ *be a dual generalized quaternion with* $N_Q > 0$ *and* $\vec{V}_Q > 0$. *Then for every integer n*,

$$Q^n = r^n(\cos n\Theta + \vec{W}\sin n\Theta).$$

Proof. The proof follows immediately from the induction. □

Conclusion

In this paper, we defined and gave some of algebraic properties of dual generalized quaternion and investigated the Euler's and De-Moivre's formulas for these quaternions in several cases. The relation between the powers of DGQ is given in Theorem 3.5. We also showed that the equation $Q^n = 1$ does not have solution for a general unit dual generalized quaternion (Corollary 3.4).

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