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SUM AND QUOTIENT OF PREFERENTIAL FUZZY SUBGROUPS

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Abstract

A preferential fuzzy subgroup is an equivalence class of fuzzy subgroups under preferential equality. In this paper, we determine conditions under which preferential equality of fuzzy subgroups is preserved by fuzzy intersection, product, sum and quotient. Examples are given to illustrate the necessity of such conditions ensuring preferential equality.

1. Introduction

An equivalence relation on the set of all fuzzy subgroups of a group which naturally generalizes equality of crisp subgroups has been proposed in

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the literature [8], [3] and [14]. We call such an equivalence relation *preferential equality* and an equivalence class of fuzzy subgroups, a *preferential fuzzy subgroup*. It was used in the classification of fuzzy subgroups of finite abelian groups. The very first paper [10] and the subsequent papers ([12], [7], [1], [4], [5] and others), dealing with properties of fuzzy subgroups relied heavily on the lattice properties of membership values in a simplistic way, inter alia, exploiting the "*sup-min*" operator. For instance, the intersection of two fuzzy subgroups μ and ν is given by $(\mu \wedge \nu)(g) = \mu(g) \wedge \nu(g)$; the sum is given by $(\mu + \nu)(g) = \bigvee_{g_1 + g_2 = g} (\mu(g_1) \wedge \nu(g_2))$, for $g \in G$, etc. This lattice theoretic approach does not throw light on group operation of fuzzy subgroups.

Also, the alternative route of the study of fuzzy subgroups through their α-cuts, as found in the literature, for instance [2], [7] and [1], suffers from at least two weak points. (i) When one considers an α-cut of a fuzzy subgroup, that α may or may not belong to the image set of the membership function. For several values of α in the unit interval, the α-cuts may be the same. (ii) Even when α_1 and α_2 are two consecutive membership values of a fuzzy subgroup, with $\alpha_1 > \alpha_2$ say, the analysis of α-cuts fails to reveal the number and nature of subgroups that contain α_1-cut and are contained in α_2-cut. That is the α-cuts of a fuzzy subgroup may not form a maximal chain in the lattice of subgroups. The motivation for studying fuzzy subgroups through pinned-flags, see Section 2 for a definition, is to address the above weak points and take a new stand-point. We recall that a pinned-flag is a pair (flag, keychain) where a flag is a maximal chain of subgroups and a keychain is precisely the membership values.

It was observed in [9] that the operations of intersection and direct sum of fuzzy subgroups do not preserve the preferential equality and some examples were provided to illustrate this fact in that same paper. As a consequence, the complete characterization of these operations in terms of preferential fuzzy subgroups was left open. We take up such a study in this paper.

In Section 2, we gather all the preliminaries such as equivalence of fuzzy subgroups, equivalence of pinned-flags and fix notation. We prove that there is a one-to-one correspondence between preferential fuzzy subgroups and equivalence classes of pinned-flags. In Section 3, conditions are developed for preservation of preferential equality by intersection and product of two fuzzy subgroups. Similarly, Section 4 deals with the operations of sum of two preferential fuzzy subgroups and quotient of a preferential fuzzy subgroup.

2. Preliminaries

We use $\mathbf{I} = [\mathbf{0}, 1],$ the real unit interval as a chain with the usual ordering in which $\wedge$ stands for infimum (inf) (or intersection) and $\vee$ stands for supremum (sup) (or union). A *fuzzy subset* of a set G is a mapping $\mu : G \to \mathbf{I}$. The union, intersection of two fuzzy sets, and complementation of a fuzzy set are defined using sup and inf pointwise, and $1 - \mu$ operator pointwise, respectively. We denote the set of all fuzzy subsets of G by $\mathbf{I}^G$, [13]. Throughout this paper, we take G to be a *finite abelian group* and G_0 to be the trivial subgroup $\{0\}$. Almost all results of this paper are applicable to any finite group, abelian or not, with normality condition on the components of any maximal chain of subgroups. By an α-cut of μ for a real number α in $\mathbf{I}$, we mean a subset $\mu^{\alpha} = \{x \in G : \mu(x) \geq \alpha\}$ of G. A fuzzy set μ is said to be a *fuzzy subgroup* if $\mu(x + y) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$ and $\mu(x) = \mu(-x)$ (see [1], [7], [4]). In this paper, we assume $\mu(0) = 1$. From this assumption, we notice that the only admissible fuzzy subgroup of the trivial group is $\mu(0) = 1$. By *core* and *support* of μ, we mean the crisp subsets of G given by $core(\mu) = \{x \in G : \mu(x) = 1\}$ and $\mathrm{supp}(\mu) = \{x \in G : \mu(x) \neq 0\}$, respectively.

We now recall from [8] that an *equivalence relation of two fuzzy subgroups* ~ on $\mathbf{I}^G$ is defined as $\mu \sim \nu$ if and only if the following two conditions are satisfied:

(i) $\forall x,\ y \in G,\ \mu(x) > \mu(y)$ if and only if $\nu(x) > \nu(y)$,

(ii) $\mu(x) = 0$ if and only if $\nu(x) = 0$. (2.1)

We call the above equivalence relation *preferential equality* and denote it by $\mu \sim \nu$ and the equivalence class containing μ is called a *preferential fuzzy subgroup* and is denoted by $[\mu]$. Throughout this paper, we refer to preferential equality as equivalence. Thus two equivalent fuzzy subgroups have the same set of α-cuts. We refer the reader to [9] for results on flag, keychain and pinned-flag, and state their definitions here. By a *flag* $\mathcal{C}$ on G, we mean a maximal chain of subgroups of the form

$$\{0\} = G_0 \subset G_1 \subset G_2 \cdots \subset G_n = G. \tag{2.2}$$

We call various G_i's the *components* of the flag $\mathcal{C}$. By a *keychain* ℓ, we mean an $(n+1)$-tuple $(\lambda_0, \lambda_1, ..., \lambda_n)$ of real numbers in **I** of the form

$$1 = \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0, \tag{2.3}$$

where λ_i's are not all necessarily distinct. The λ_i's are called *pins*. The length of keychain ℓ is $n+1$. The *minimum content* of the keychain ℓ, denoted by $\mathrm{minco}(\ell)$, is defined as $\inf\{\lambda_i : 0 \leq i \leq n,\ \lambda_i \neq 0\}$. Similarly the *maximum content*, denoted by $\mathrm{maxco}(\ell)$, is defined as $\sup\{\lambda_i : 0 \leq i \leq n,\ \lambda_i \neq 1\}$. In general, $\mathrm{maxco}(\ell) \geq \mathrm{minco}(\ell)$ (the only exception being when the keychain has only 0's and 1's). $\mathrm{maxco}(\ell) = \mathrm{minco}(\ell)$ only if there are at most three pins. By a *pinned-flag* on G, we mean a pair $(\mathcal{C},\ \ell)$ of a flag $\mathcal{C}$ on G and a keychain ℓ from **I**, written as follows:

$$G_0^1 \subset G_1^{\lambda_1} \subset G_2^{\lambda_2} \cdots \subset G_n^{\lambda_n}. \tag{2.4}$$

We call $G_i^{\lambda_i}$ for $i = 0, 1, ..., n,$ the *ith component of the pinned-flag*.

With any pinned-flag $(\mathcal{C},\ \ell)$, we can associate a fuzzy subgroup $\mu = \vee\{\lambda_i \chi_{G_i} : 0 \leq i \leq n\}$. It is easily checked that μ is a fuzzy subgroup of

G and further μ is unique with respect to the pinned-flag $(\mathcal{C}, \ell)$. Conversely, suppose μ is a fuzzy subgroup of G with α-cuts $G_1 \subset G_2 \subset \cdots \subset G_k = G$ and the corresponding membership values $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ in **I**. By refining the chain $G_1 \subset G_2 \cdots \subset G_k = G$ to a maximal chain $\mathcal{C}$ of subgroups and in the process repeating certain λ_i 's if necessary, we get a pinned-flag $(\mathcal{C}, \ell)$ associated with μ, not necessarily unique, where ℓ is the keychain whose pins are λ_i 's with repetitions as the case may be.

The next discussion describes the relationship between fuzzy subgroups and pinned-flags under the equivalence defined above.

Proposition 2.1. *Let* μ *and* ν *be two fuzzy subgroups that are preferential equal and let* μ *have the pinned-flag* $G_0^{\lambda_0} \subset G_1^{\lambda_1} \subset \cdots \subset G_n^{\lambda_n}$ *and* ν *have the pinned-flag* $H_0^{\beta_0} \subset H_1^{\beta_1} \subset \cdots \subset H_m^{\beta_m}$. *Then* $n = m$ *and for each* i, λ_i *repeats* m_i *times if and only if* β_i *repeats* m_i *times*.

Proof. It is easily shown that $n = m$ using the Jordan-Holder Theorem. Now assume λ_i repeats m_i times. Consider the pinned-flag of $\mu : G_0^{\lambda_0} \subset G_1^{\lambda_1} \subset \cdots \subset G_{i-1}^{\lambda_{i-1}} \subset G_i^{\lambda_i} \subset \cdots \subset G_{i+m_i-1}^{\lambda_i} \subset G_{i+m_i}^{\lambda_{i+m_i}} \subset \cdots \subset G_n^{\lambda_n}$ with $\lambda_{i-1} > \lambda_i$ and $\lambda_i > \lambda_{i+m_i}$. We first claim that $G_{i-1} = H_{i-1}$. Suppose the claim is false. Then there exist $x \in G_{i-1} - H_{i-1}$ and $y \in H_{i-1} - G_{i-1}$, by maximality of the chains of subgroups, such that $\lambda_{i-1} = \mu(x) > \mu(y)$ and $\nu(y) = \beta_{i-1} > \nu(x)$. But by equivalence, $\nu(x) > \nu(y)$. Thus we have a contradiction. Hence $G_{i-1} = H_{i-1}$. Similarly, $G_{i+m_i-1} = H_{i+m_i-1}$.

Next, we show that if $\lambda_{i-1} > \lambda_i$, then $\beta_{i-1} > \beta_i$. The first claim implies that $G_{i-1} = H_{i-1}$. If $\beta_{i-1} = \beta_i$, let $a \in H_i - H_{i-1}$ and $b \in H_{i-1} - H_{i-2}$. Then $\beta_i = \nu(a) = \beta_{i-1} = \nu(b)$, implying $\mu(a) = \mu(b)$ with $b \in G_{i-1}$. So $\mu(b) \geq \lambda_{i-1} > \lambda_i$, implying $\mu(a) > \lambda_i$. Thus $a \in G_{i-1} = H_{i-1}$, a contradiction. Therefore, $\beta_{i-1} > \beta_i$.

Finally, we show that for any $z \in H_{i+m_i-1}$, we have $\nu(z) \geq \beta_i$ and for $z \notin H_{i+m_i-1}$, we have $\nu(z) < \beta_i$. Now suppose there exists $z \in H_{i+m_i-1}$ such that $\beta_i = \nu(y) > \nu(z) = \beta_{i+m_i-1}$ for $y \in H_i - H_{i-1}$. Then $\mu(y) > \mu(z) \geq \lambda_i$ by equivalence. This implies $y \in G_{i-1} = H_{i-1}$. Therefore, $\nu(y) \geq \beta_{i-1} > \beta_i = \nu(y)$, an absurdity. Thus, for all $z \in H_{i+m_i-1}$, we have $\nu(z) \geq \beta_i$. Suppose $z \in H_j - H_{j-1}$ for $i \leq j \leq i + m_i - 1$. If $\nu(z) > \beta_i$, then $\beta_j > \beta_i$, which is a contradiction. Hence $\nu(z) = \beta_i$. Since $\lambda_i > \lambda_{i+m_i}$, we have $\beta_i > \beta_{i+m_i}$ by an earlier part of the proof. This shows that β_i repeats m_i times. Similarly for the converse. This completes the proof.

The above proposition leads to

Definition 2.2. Consider two pinned-flags as given below:

$$(\mathcal{C}_1, \ell_1) : G_0^{\lambda_0} \subset G_1^{\lambda_1} \subset \cdots \subset G_n^{\lambda_n}, \; (\mathcal{C}_2, \ell_2) : H_0^{\beta_0} \subset H_1^{\beta_1} \subset \cdots \subset H_m^{\beta_m}, \tag{2.5}$$

where $\lambda_0 = \beta_0 = 1$. We say $(\mathcal{C}_1, \ell_1)$ is *equivalent* to $(\mathcal{C}_2, \ell_2)$, and write $(\mathcal{C}_1, \ell_1) \sim (\mathcal{C}_2, \ell_2)$, if and only if:

(i) $n = m$;

(ii) $\lambda_i > \lambda_j$ if and only if $\beta_i > \beta_j$ for $0 \leq i, j \leq n$;

(iii) if $\lambda_i > \lambda_{i+1}$ or $\beta_i > \beta_{i+1}$, then $G_i = H_i$ for $i = 0, 1, \ldots, n-1$;

(iv) $\lambda_i = 0$ if and only if $\beta_i = 0$ for $1 \leq i, j \leq n$.

With the definition above, we have

Theorem 2.3. *Let* μ *and* ν *be the fuzzy subgroups associated with the pinned-flags* $(\mathcal{C}_1, \ell_1)$ *and* $(\mathcal{C}_2, \ell_2)$ *given above, respectively. Then* $(\mathcal{C}_1, \ell_1) \sim (\mathcal{C}_2, \ell_2)$ *if and only if* $\mu \sim \nu$.

Corollary 2.4. *There is a one-to-one correspondence between the preferential fuzzy subgroups and the equivalence classes of pinned-flags on any finite abelian group G.*

The above result enables us to reduce the study of preferential fuzzy subgroups to that of pinned-flags.

3. Intersection and Product Under Equivalence

In [9], it was shown that the operation of fuzzy intersection does not preserve preferential equality. Similar examples showing that the fuzzy product does not preserve preferential equality can be exhibited. In this section, we study conditions under which these operations preserve preferential equality.

Note: If ℓ is a keychain of μ, then we find it convenient to write $\text{maxco}(\mu)$ (resp. $\text{minco}(\mu)$) for $\text{maxco}(\ell)$ (resp. $\text{minco}(\ell)$). We define the *product* $\mu \times \nu$ by $(\mu \times \nu)(x, y) = \mu(x) \wedge \nu(y)$, see for instance [12]. For the rest of this paper, we assume that any keychain used has at least two distinct pins.

Proposition 3.1. *Let* $\mu \sim \mu_1$ *and* $\nu \sim \nu_1$ *with* $\text{maxco}(\nu) < \text{minco}(\mu)$ *and* $\text{maxco}(\nu_1) < \text{minco}(\mu_1)$. *Then* $\mu \wedge \nu \sim \mu_1 \wedge \nu_1$ *and* $\mu \times \nu \sim \mu_1 \times \nu_1$.

Proof. Let $(\mu \wedge \nu)(x) > (\mu \wedge \nu)(y)$. First, suppose $(\mu \wedge \nu)(y) > 0$.

Case (i). Assume $\nu(y) \leq \mu(y)$. Since $\mu(y) > 0$, we have $\mu_1(y) > 0$. Also, $\nu(y) \neq 1$ implies $\nu_1(y) \neq 1$. Thus $\mu_1(x) > \nu_1(y)$ by the maxco-minco property of the hypothesis. By equivalence, $\nu_1(x) > \nu_1(y)$. Hence $(\mu_1 \wedge \nu_1)(x) > (\mu_1 \wedge \nu_1)(y)$.

Case (ii). Assume $\mu(y) < \nu(y)$. We must have $\nu(y) = 1$ otherwise the maxco-minco property is violated. Thus $\nu_1(y) = 1$. By equivalence, $\mu_1(x) > \mu_1(y)$. Also, $\nu(x) > \mu(y)$ implies $\nu(x) = 1 = \nu_1(x)$ by the maxco-minco property. Hence $(\mu_1 \wedge \nu_1)(x) > (\mu_1 \wedge \nu_1)(y)$.

Second, if $(\mu \wedge \nu)(y) = 0.$ Using equivalence, it follows easily that $(\mu_1 \wedge \nu_1)(x) > (\mu_1 \wedge \nu_1)(y).$ Clearly, $\mu \wedge \nu(x) = 0$ if and only if $\mu_1 \wedge \nu_1(x)$ $= 0$. Thus $\mu \wedge \nu \sim \mu_1 \wedge \nu_1.$

For the product, the proof follows mutatis mutandis.

Example 3.2. Let $G = \mathbb{Z}_{12}$ and define μ, μ_1, ν and ν_1 by their pinned-flags, respectively, as follows:

$$\mu : 0^1 \subset \mathbb{Z}_3^{1/2} \subset \mathbb{Z}_6^{1/3} \subset \mathbb{Z}_{12}^0, \text{ and } \mu_1 : 0^1 \subset \mathbb{Z}_3^{3/4} \subset \mathbb{Z}_6^{1/5} \subset \mathbb{Z}_{12}^0.$$

$$\nu : 0^1 \subset \mathbb{Z}_2^{3/10} \subset \mathbb{Z}_4^{1/4} \subset \mathbb{Z}_{12}^{1/20}, \text{ and } \nu_1 : 0^1 \subset \mathbb{Z}_2^{1/10} \subset \mathbb{Z}_4^{1/12} \subset \mathbb{Z}_{12}^{1/15}.$$

Clearly, $\mu \sim \mu_1$ and $\nu \sim \nu_1$ and $\text{maxco}(\nu) < \text{minco}(\mu)$ and $\text{maxco}(\nu_1) <$ $\text{minco}(\mu_1).$ It is easy to see that $\mu \wedge \nu$ is given by $0^1 \subset \mathbb{Z}_2^{3/10} \subset \mathbb{Z}_6^{1/20} \subset$ $\mathbb{Z}_{12}^0$ and $\mu_1 \wedge \nu_1$ is given by $0^1 \subset \mathbb{Z}_2^{1/10} \subset \mathbb{Z}_6^{1/15} \subset \mathbb{Z}_{12}^0.$ Thus $\mu \wedge \nu \sim$ $\mu_1 \wedge \nu_1.$ We also observe that the maxco-minco property does not necessarily lead to trivial cases as $\mu \wedge \nu$ is neither equal to nor equivalent to any of the given fuzzy subgroups.

Now suppose that $\mu \sim \mu_1$ and $\nu \sim \nu_1$ such that all 4 fuzzy subgroups have the same flag. It still does not follow that fuzzy intersection and fuzzy product preserve preferential equality. See the example below.

Example 3.3. Let $G = \mathbb{Z}_{12}$ and define μ, μ_1, ν and ν_1 by their pinned-flags, respectively, as follows:

$$\mu : 0^1 \subset \mathbb{Z}_2^{1/2} \subset \mathbb{Z}_6^{1/3} \subset \mathbb{Z}_{12}^{1/5} \text{ and } \mu_1 : 0^1 \subset \mathbb{Z}_2^{7/8} \subset \mathbb{Z}_6^{5/6} \subset \mathbb{Z}_{12}^{1/4},$$

$$\nu : 0^1 \subset \mathbb{Z}_2^{1/2} \subset \mathbb{Z}_6^{1/2} \subset \mathbb{Z}_{12}^0 \text{ and } \nu_1 : 0^1 \subset \mathbb{Z}_2^{3/4} \subset \mathbb{Z}_6^{3/4} \subset \mathbb{Z}_{12}^0.$$

Clearly, $\mu \sim \mu_1$ and $\nu \sim \nu_1.$ But $\mu \times \nu(2, 0) = 1/3 < 1/2 = \mu \times \nu(0, 6)$ while $\mu_1 \times \nu_1(2, 0) = 5/6 > 3/4 = \mu_1 \times \nu_1(0, 6).$ Similarly, for the fuzzy intersection.

However, we have

Proposition 3.4. *Let* $\mu \sim \mu_1$ *and* $\nu \sim \nu_1$ *such that all* 4 *fuzzy subgroups have the same flag. Suppose also that the keychain of each fuzzy subgroup has no repeating nonzero non-unit pins. Then* $\mu \wedge \nu \sim \mu_1 \wedge \nu_1$ *and* $\mu \times \nu \sim \mu_1 \times \nu_1$.

Proof. We prove the proposition for the intersection, then the product follows similarly. Let $\mu \wedge \nu(x) > \mu \wedge \nu(y)$. Suppose first that $\mu(y) \leq \nu(y)$. By equivalence, $\mu_1(x) > \mu_1(y)$. Let $H_0 \subset H_1 \subset \cdots \subset H_n$ be a flag for all 4 fuzzy subgroups. If $y \in H_i - H_{i-1}$, then $x \in H_j$ for $j < i$. Thus $\nu_1(x) \geq \nu_1(y)$. If $\mu(y) \neq 0$, then $\nu(y) \neq 0$ which implies $\nu_1(y) \neq 0$. So if $\nu_1(x) = \nu_1(y)$, then $\nu_1(x) = 1 = \nu_1(y)$ by hypothesis. Therefore, $\nu_1(x) \geq \mu_1(x) > \mu_1 \wedge \nu_1(y)$ which implies $\mu_1 \wedge \nu_1(x) > \mu_1 \wedge \nu_1(y)$. The same result is obtained if $\nu_1(x) > \nu_1(y)$. If $\mu(y) = 0$, then $\mu_1(y) = 0$ by equivalence, thus it is immediate that $\mu_1 \wedge \nu_1(x) > \mu_1 \wedge \nu_1(y)$. The same result is obtained if $\mu(y) \geq \nu(y)$. This completes the proof as the result on supports is obvious.

Analogously

Proposition 3.5. *Let* $\mu \sim \mu_1$ *and* $\nu \sim \nu_1$ *such that all* 4 *fuzzy subgroups have the same flag. Suppose also that* $\lambda_i = \beta_j$ *if and only if* $\lambda'_i = \beta'_j$, *where* $\lambda_i, \lambda'_i, \beta_j, \beta'_j$ *are pins in the keychains of* μ, μ_1, ν, ν_1, *respectively, for* $i = 0, 1, \ldots, n$. *Then* $\mu \wedge \nu \sim \mu_1 \wedge \nu_1$ *and* $\mu \times \nu \sim \mu_1 \times \nu_1$.

Proof. The proof is easy and is thus omitted.

If the keychains do contain repeating pins, then we have the following:

Proposition 3.6. *Let* $\mu \sim \mu_1$ *and* $\nu \sim \nu_1$ *such that* $\lambda_i > \beta_k$ *if and only if* $\lambda'_i > \beta'_k$ *and* $\lambda_i < \beta_k$ *if and only if* $\lambda'_i < \beta'_k$, *where* $\lambda_i, \lambda'_i, \beta_i, \beta'_i$ *are pins in the keychains of* μ, μ_1, ν, ν_1, *respectively. Then* $\mu \wedge \nu \sim \mu_1 \wedge \nu_1$ *and* $\mu \times \nu \sim \mu_1 \times \nu_1$.

Proof. We prove the proposition for the product, then the intersection follows mutatis mutandis. Let $\mu_1(a) \wedge \nu_1(b) > \mu_1(x) \wedge \nu_1(b)$. Suppose first that $\mu_1(x) \geq \nu_1(y)$. By equivalence, $\nu(b) > \nu(y)$. Let $B_0 \subset B_1 \subset \cdots \subset B_n$ be a flag for $\mu_1 \times \nu_1$. Let i be the least positive integer such that $(x, y) \in B_i$. Let $\lambda'_j \wedge \beta'_k$ be the pin corresponding to B_i. Then $\lambda'_j \wedge \beta'_k = \mu_1(x) \wedge \nu_1(y)$ $\geq \nu_1(y) = \beta'_k$. So $\lambda'_j \geq \beta'_k$ implies $\lambda_j \geq \beta_k$ by hypothesis. Also, there exist subscripts s and t such that $\lambda'_s \wedge \beta'_t = \mu_1(a) \wedge \nu_1(b) > \nu_1(y) = \beta'_k$, implying that $\lambda'_s \geq \beta'_k$ which implies $\mu(a) = \lambda_s > \beta_k = \nu(y)$ by hypothesis, and $\nu(b) > \nu(y)$ by equivalence. Thus $\mu(a) \wedge \nu(b) > \mu(x) \wedge \nu(y)$. Similarly if $\mu_1(x) \leq \nu_1(y)$. It is also clear that $\mu \times \nu(x, y) = 0$ if and only if $\mu_1 \times \nu_1(x, y)$ $= 0$. This completes the proof.

4. Sum and Quotients Under Equivalence

In this section, we show that the sum preserves equivalence under the conditions of the previous section. We also show that one of the quotients preserves equivalence unconditionally, and that a second quotient preserves equivalence under the conditions of the previous section.

1° Sums under equivalence:

Proposition 4.1. *Let* $\mu \sim \mu_1$ *and* $\nu \sim \nu_1$ *with* $\operatorname{maxco}(\nu) < \operatorname{minco}(\mu)$ *and* $\operatorname{maxco}(\nu_1) < \operatorname{minco}(\mu_1)$. *Then* $\mu + \nu \sim \mu_1 + \nu_1$.

Proof. Let $(\mu + \nu)(x) > (\mu + \nu)(y)$. Then there exist x_1, x_2, y_1, y_2 such that $LHS = \mu(x_1) \wedge \nu(x_2) > RHS = \mu(y_1) \wedge \nu(y_2)$, where $x = x_1 + x_2$ and $y = y_1 + y_2$. Suppose that $\mu(y_1) \wedge \nu(y_2) > 0$. Assume $\mu(y_1) \wedge \nu(y_2) =$ $\nu(y_2)$. Then $\nu_1(x_2) > \nu_1(y_2)$ by equivalence. If $\mu_1(x_1) \leq \nu_1(y_2)$, then $\nu_1(y_2) = 1$ by the maxco-minco property of the hypothesis. This is obviously impossible. Thus $\mu_1(x_1) > \nu_1(y_2)$. Hence $\mu_1(x_1) \wedge \nu_1(x_2) > \mu_1(y_1) \wedge \nu_1(y_1)$. So $(\mu_1 + \nu_1)(x) > (\mu_1 + \nu_1)(y)$.

Suppose $\mu(y_1) \wedge \nu(y_2) = \mu(y_1)$. An argument similar to the above shows that $(\mu_1 + \nu_1)(x) > (\mu_1 + \nu_1)(y)$. Next, assume that $\mu(y_1) \wedge \nu(y_2) = 0$. Clearly, $\mu_1(y_1) \wedge \nu_1(y_2) = 0$. Thus $(\mu_1 + \nu_1)(x) > 0 = (\mu_1 + \nu_1)(y)$. This completes the proof.

Proposition 4.2. *Let* $\mu \sim \mu_1$ *and* $\nu \sim \nu_1$ *such that* $\lambda_i > \beta_k$ *if and only if* $\lambda'_i > \beta'_k$ *and* $\lambda_i < \beta_k$ *if and only if* $\lambda'_i < \beta'_k$, *where* $\lambda_i, \lambda'_i, \beta_i, \beta'_i$ *are pins in the keychains of* μ, μ_1, ν, ν_1, *respectively*. *Then* $\mu + \nu \sim \mu_1 + \nu_1$.

Proof. Let $(\mu_1 + \nu_1)(a) > (\mu_1 + \nu_1)(b)$. Then there exist a_1, a_2 with $a = a_1 + a_2$ and b_1, b_2 with $b = b_1 + b_2$ such that $\mu_1(a_1) \wedge \nu_1(a_2) > \mu_1(b_1) \wedge \nu_1(b_2)$. First, suppose $\mu_1(b_1) \wedge \nu_1(b_2) = \nu_1(b_2)$. If $\nu_1(b_2) \neq 0$, then $\nu(a_2) > \nu(b_2)$ by equivalence, and $\lambda'_i = \mu_1(a_1) > \nu_1(b_2) = \beta'_k$ which implies $\lambda_i > \beta_k$ by hypothesis. Thus $\mu(a_1) > \nu(b_2)$, hence $\mu(a_1) \wedge \nu(a_2) > \mu(b_1) \wedge \nu(b_2)$. If $\nu_1(b_2) = 0$, then $\nu(b_2) = 0$ by equivalence; thus $\mu(a_1) \wedge \nu(a_2) > \mu(b_1) \wedge \nu(b_2)$.

Next, suppose $\mu_1(b_1) \wedge \nu_1(b_2) = \mu_1(b_1)$. An argument similar to the above shows that $\mu(a_1) \wedge \nu(a_2) > \mu(b_1) \wedge \nu(b_2)$. The fact that the two sums have the same support is clear. Thus $(\mu + \nu)(a) > (\mu + \nu)(b)$. This completes the proof.

Example 4.3. Let $G = \mathbb{Z}_{12}$ and define μ, μ_1, ν and ν_1 by their pinned-flags, respectively, as follows:

$$\mu : 0^1 \subset \mathbb{Z}_2^{7/10} \subset \mathbb{Z}_4^{1/2} \subset \mathbb{Z}_{12}^{1/5} \text{ and } \mu_1 : 0^1 \subset \mathbb{Z}_2^{3/4} \subset \mathbb{Z}_4^{1/3} \subset \mathbb{Z}_{12}^{1/6},$$

$$\nu : 0^1 \subset \mathbb{Z}_3^{5/6} \subset \mathbb{Z}_6^{3/5} \subset \mathbb{Z}_{12}^{1/5} \text{ and } \nu_1 : 0^1 \subset \mathbb{Z}_3^{2/3} \subset \mathbb{Z}_6^{1/2} \subset \mathbb{Z}_{12}^{1/6}.$$

It is easily checked that $\mu \sim \mu_1$ and $\nu \sim \nu_1$. Also, the conditions of the above proposition are satisfied. Thus we expect equivalence to be preserved by the sum. $\mu + \nu$ and $\mu_1 + \nu_1$ are, respectively, given by $\mu + \nu : 0^1 \subset \mathbb{Z}_2^{7/10} \subset \mathbb{Z}_6^{5/6} \subset \mathbb{Z}_{12}^{1/2}$, $\mu_1 + \nu_1 : 0^1 \subset \mathbb{Z}_2^{3/4} \subset \mathbb{Z}_6^{2/3} \subset \mathbb{Z}_{12}^{1/3}$. Thus indeed $\mu + \nu \sim \mu_1 + \nu_1$.

Proposition 4.4. *Let* $\mu \sim \mu_1$ *and* $\nu \sim \nu_1$ *such that all* 4 *fuzzy subgroups have the same flag*. *Suppose also that the keychain of each fuzzy subgroup has no repeating nonzero non-unit pins*. *Then* $\mu + \nu \sim \mu_1 + \nu_1$.

Proof. Let $(\mu + \nu)(x) > (\mu + \nu)(y)$. Then there exist x_1, x_2, y_1, y_2 such that $LHS = \mu(x_1) \wedge \nu(x_2) > RHS = \mu(y_1) \wedge \nu(y_2)$, where $x = x_1 + x_2$ and $y = y_1 + y_2$. Suppose that $\mu(y_1) \wedge \nu(y_2) > 0$. Assume $\mu(y_1) \wedge \nu(y_2) = \nu(y_2)$. Then $\nu_1(x_2) > \nu_1(y_2)$ by equivalence. Let $H_0 \subset H_1 \subset \cdots \subset H_n$ be a flag for all 4 fuzzy subgroups. If $y \in H_i - H_{i-1}$, then $x \in H_j$ for $j < i$. Thus $\mu_1 + \nu_1(x) \geq \mu_1 + \nu_1(y)$. If $\mu_1(x_1) \wedge \nu_1(x_2) = \mu_1(y_1) \wedge \nu_1(y_2)$, then from the hypothesis about pins and equivalence, we get a contradiction of the assumption about the sum. Thus $(\mu_1 + \nu_1)(x) > (\mu_1 + \nu_1)(y)$. The case $(\mu + \nu)(y) = 0$ is easy. It is also clear that $(\mu + \nu)(a) = 0$ if and only if $(\mu_1 + \nu_1)(a) = 0$. Hence $\mu + \nu \sim \mu_1 + \nu_1$. Analogously we state without proof

Proposition 4.5. *Let* $\mu \sim \mu_1$ *and* $\nu \sim \nu_1$ *such that all* 4 *fuzzy subgroups have the same flag*. *Suppose also that* $\lambda_i = \beta_j$ *if and only if* $\lambda'_i = \beta'_j$, *where* $\lambda_i, \lambda'_i, \beta_j, \beta'_j$ *are pins in the keychains of* μ, μ_1, ν, ν_1, *respectively*, *for* $i = 0, 1, \ldots, n$. *Then* $\mu + \nu \sim \mu_1 + \nu_1$.

2° Quotients under equivalence:

Definition 4.6. (1) Suppose μ and ν are two fuzzy subgroups of G. Then the *fuzzy quotient* μ/ν is defined as a fuzzy subgroup of the quotient group $G/core(\nu)$ given by $(\mu/\nu)(x\,core(\nu)) = \sup\{\mu(a) : a\,core(\nu) = x\,core(\nu), a \in G\}$, [5], [6], [15].

(2) Suppose μ and ν are two fuzzy subgroups of G. Then the *fuzzy p-quotient* μ/ν is defined as a fuzzy subset of the group G given by $(\mu/\nu)(x) = \sup\{a_\lambda \nu(x) : a_\lambda \in \mu\}$, where $a_\lambda \nu(x) = \lambda \wedge \nu(-a + x)$ [4].

It is easily checked that the fuzzy p-quotient is given by $(\mu/\nu)(x) = \sup\{\mu(a) \wedge \nu(-a + x) : a \in \operatorname{supp} \mu\}$.

Proposition 4.7. *Let* $\mu \sim \mu_1$ *and* $\nu \sim \nu_1$. *Then for the fuzzy quotient, we have* $\mu/\nu \sim \mu_1/\nu_1$.

Proof. Let $(\mu/\nu)(x\, core(\nu)) > (\mu/\nu)(y\, core(\nu))$. There exist x_1, y_1 such that $x_1\, core(\nu) = x\, core(\nu)$, $y_1\, core(\nu) = y\, core(\nu)$ and $\mu(x_1) > \mu(y_1)$. By equivalence, $\mu_1(x_1) > \mu_1(y_1)$ and $core(\nu) = core(\nu_1)$. Since $(\mu_1/\nu_1) \cdot (y\, core(\nu_1)) = \mu_1(y_1)$, it follows that $(\mu_1/\nu_1)(x\, core(\nu)) > (\mu_1/\nu_1) \cdot (y\, core(\nu))$. It is also immediate that $(\mu/\nu)(x\, core(\nu)) = 0$ if and only if $(\mu_1/\nu_1)(x\, core(\nu)) = 0$. Thus $\mu/\nu \sim \mu_1/\nu_1$.

Proposition 4.8. *The fuzzy p-quotient* $(\mu/\nu)(x) = \sup\{a_\lambda \nu(x) : a_\lambda \in \mu\}$ *is a fuzzy subgroup of G.*

Proof. See [4].

Proposition 4.9. *If* $\operatorname{supp} \nu \subset \operatorname{supp} \mu$, *then the fuzzy p-quotient*

$$(\mu/\nu)(x) = \begin{cases} (\mu + \nu)(x) & \textit{if } \mu(x) > 0, \\ 0 & \textit{if } \mu(x) = 0. \end{cases}$$

Since the sum of fuzzy subgroups does not in general preserve preferential equality, it follows that the fuzzy p-quotient does not in general preserve preferential equality.

Proposition 4.10. *Let* $\mu \sim \mu_1$ *and* $\nu \sim \nu_1$ *with* $\operatorname{maxco}(\nu) < \operatorname{minco}(\mu)$ *and* $\operatorname{maxco}(\nu_1) < \operatorname{minco}(\mu_1)$. *Then, for the fuzzy p-quotient,* $\mu/\nu \sim \mu_1/\nu_1$.

Proof. Let $(\mu/\nu)(x) > (\mu/\nu)(y)$. There exist $a, b \in \operatorname{supp} \mu$ such that $\mu(a) \wedge \nu(-a + x) > \mu(b) \wedge \nu(-b + y)$. Suppose $\mu(b) \wedge \nu(-b + y) = \mu(b)$. For $\mu(b) > 0$, we have $\mu_1(a) > \mu_1(b)$ by equivalence. Also, $\nu(-b + y) \geq \mu(b)$ and $\nu(-a + x) > \mu(b)$ imply that $\nu(-a + x) = 1 = \nu(-a + y)$ by the maxco-minco property. Thus, by equivalence, $\nu_1(-a + x) = 1 = \nu_1(-a + y)$. This implies that $\mu_1(a) \wedge \nu_1(-a + x) > \mu_1(b) \wedge \nu_1(-b + y)$. For $\mu(b) = 0$, we have $\mu_1(b) = 0$ and $\nu(-a + x) > 0$ imply $\nu_1(-a + x) > 0$. Therefore, $\mu_1(a) \wedge \nu_1(-a + x) > \mu_1(b) \wedge \nu_1(-b + y)$. Next, suppose $\mu(b) \wedge \nu(-b + y) = \nu(-b + y)$. Equivalence implies $\nu_1(-a + x) > \nu(-b + y)$. $\mu(a) > 0$ implies $\mu_1(a) > 0$ by equivalence. Thus $\mu_1(a) > \nu_1(-b + y)$ by the maxco-minco property. Hence $\mu_1(a) \wedge \nu_1(-a + x) > \mu_1(b) \wedge \nu_1(-b + y)$, implying $(\mu_1/\nu_1) \cdot (x) > (\mu_1/\nu_1)(y)$. It is clear that $(\mu/\nu)(x) = 0$ if and only if $(\mu_1/\nu_1)(x) = 0$. This completes the proof.

Proposition 4.11. (1) *Let* $\mu \sim \mu_1$ *and* $\nu \sim \nu_1$ *such that* $\lambda_i > \beta_k$ *if and only if* $\lambda'_i > \beta'_k$ *and* $\lambda_i < \beta_k$ *if and only if* $\lambda'_i < \beta'_k$, *where* $\lambda_i, \lambda'_i, \beta_i, \beta'_i$ *are pins in the keychains of* μ, μ_1, ν, ν_1, *respectively. Then for the fuzzy p-quotient, we have* $\mu/\nu \sim \mu_1/\nu_1$.

(2) *Let* $\mu \sim \mu_1$ *and* $\nu \sim \nu_1$ *such that all* 4 *fuzzy subgroups have the same flag. Suppose also that the keychain of each fuzzy subgroup has distinct nonzero non-unit pins. Then, for the fuzzy p-quotient,* $\mu/\nu \sim \mu_1/\nu_1$.

(3) *Let* $\mu \sim \mu_1$ *and* $\nu \sim \nu_1$ *such that all* 4 *fuzzy subgroups have the same flag. Suppose also that* $\lambda_i > \beta_j$ *if and only if* $\lambda'_i = \beta'_j$, *where* $\lambda_i, \lambda'_i, \beta_j, \beta'_j$ *are pins in the keychains of* μ, μ_1, ν, ν_1, *respectively, for* $i = 0, 1, \ldots, n$. *Then, for the fuzzy p-quotient,* $\mu/\nu \sim \mu_1/\nu_1$.

The proofs for the above three facts are straightforward and thus omitted.

Proposition 4.12. *Let* $\mu \sim \mu_1$ *and* $\nu \sim \nu_1$ *such that* $0^1 \subset G_1 \subset \cdots \subset G_n = G$ *and* $0^1 \subset H_1 \subset \cdots \subset H_n = G$ *are the flags for* μ *and* ν, *respectively. Let* i *be the least positive integer such that* $G_i \neq H_i$, *and suppose* $\operatorname{minco}(\mu) > \operatorname{maxco}(\nu)$ *and* $\operatorname{minco}(\mu_1) > \operatorname{maxco}(\nu_1)$ *from the ith component* *on. Assume that the nonzero non-unit pins for each keychain are distinct at all kth positions for* $k < i$. *Then* (i) $\mu \wedge \nu \sim \mu_1 \wedge \nu_1$; (ii) $\mu \times \nu \sim \mu_1 \times \nu_1$; (iii) $\mu + \nu \sim \mu_1 + \nu_1$; (iv) *for the fuzzy p-quotient, we have* $\mu/\nu \sim \mu_1/\nu_1$.

Proof. The proof combines earlier propositions.

Example 4.13. Let $G = \mathbb{Z}_{72}$ and define μ, μ_1, ν and ν_1 by their pinned-flags, respectively, as follows:

$$\mu : 0^1 \subset \mathbb{Z}_2^{1/2} \subset \mathbb{Z}_6^{1/3} \subset \mathbb{Z}_{12}^{1/4} \subset \mathbb{Z}_{24}^{1/5} \subset \mathbb{Z}_{72}^{0}$$

and

$$\mu_1 : 0^1 \subset \mathbb{Z}_2^{3/4} \subset \mathbb{Z}_6^{1/5} \subset \mathbb{Z}_{12}^{1/6} \subset \mathbb{Z}_{24}^{1/7} \subset \mathbb{Z}_{72}^{0},$$

$$\nu : 0^1 \subset \mathbb{Z}_2^{5/6} \subset \mathbb{Z}_6^{1/6} \subset \mathbb{Z}_{18}^{1/7} \subset \mathbb{Z}_{36}^{1/8} \subset \mathbb{Z}_{72}^{1/10}$$

and

$$\nu_1 : 0^1 \subset \mathbb{Z}_2^{7/8} \subset \mathbb{Z}_6^{1/4} \subset \mathbb{Z}_{18}^{1/10} \subset \mathbb{Z}_{36}^{1/11} \subset \mathbb{Z}_{72}^{1/12}.$$

It is easily checked that $\mu \sim \mu_1$ and $\nu \sim \nu_1$. Also, the conditions of the above proposition are satisfied with $i = 3$. Thus we expect equivalence to be preserved by the operations. We check only the fuzzy sum. $\mu + \nu$ and $\mu_1 + \nu_1$ are, respectively, given by $\mu + \nu : 0^1 \subset \mathbb{Z}_2^{5/6} \subset \mathbb{Z}_6^{1/3} \subset \mathbb{Z}_{12}^{1/4} \subset \mathbb{Z}_{24}^{1/5} \subset \mathbb{Z}_{72}^{1/7}$ and $\mu_1 + \nu_1 : 0^1 \subset \mathbb{Z}_2^{7/8} \subset \mathbb{Z}_6^{1/4} \subset \mathbb{Z}_{12}^{1/6} \subset \mathbb{Z}_{24}^{1/7} \subset \mathbb{Z}_{72}^{1/10}$ showing that the sum preserves equivalence.

In conclusion, we may also have other combinations of the propositions analogous to the above. For example, we may combine the maxco-minco property with the property "$\lambda_i > \beta_j$ if and only if $\lambda'_i > \beta'_j$" to have a more complex proposition. Further, we may also combine all conditions discussed in this paper to give an even better proposition involving preservation of preferential equality by operations. Finally, it is also possible to change the order of combinations to arrive at useful propositions and examples. For example, we may start with the minco-maxco property, follow it with the condition of the same components and then other conditions may be added towards the end. However, to keep the paper brief and interesting, we omit such elaborate combinations.

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