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BEST PROXIMITY POINT FOR MAPPINGS SATISFYING GENERALIZED CONTRACTIVE CONDITION OF RATIONAL TYPE ON A METRIC SPACE

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Abstract

In this paper, we introduce the notion of generalized contractive condition of a rational type and prove the existence of best proximity

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point in the setting of metric space which generalizes the result of Eldred and Veeramani [1] and Jaggi [2].

1. Introduction

Let A and B be nonempty subsets of a metric space (X, d) and a map $T : A \cup B \to A \cup B$ be called a *cyclic mapping* if $T(A) \subseteq B$ and $T(B) \subseteq A$.

If the fixed point equation $Tx = x$ does not possess a solution, then it is natural to find an $x \in A$ satisfying $d(x, Tx) = d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. A point $x \in A$ is called a *best proximity point* for T if $d(x, Tx) = d(A, B)$.

Eldred and Veeramani [1] introduced cyclic contraction maps.

Definition 1.1 [1]**.** Let A and B be nonempty subsets of a metric space (X, d). A map $T : A \cup B \to A \cup B$ is called a *cyclic contraction* if it satisfies

(i) $T(A) \subseteq B$ and $T(B) \subseteq A$;

(ii) for some $k \in (0, 1)$, we have $d(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B)$, for all $x \in A$, $y \in B$.

Using the concept of cyclic contraction Eldred and Veeramani [1] proved the existence of best proximity point.

Theorem 1.2 [1]**.** *Let A and B be nonempty closed subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be cyclic contraction. If either A or B is boundedly compact, then there exists $x \in A \cup B$ such that $d(x, Tx) = d(A, B)$.*

Jaggi [2] proved the following fixed point theorem.

Theorem 1.3 [2]**.** *Let T be a continuous self map defined on a complete metric space (X, d). Suppose that T satisfies the following contractive condition*:

$$d(Tx, Ty) \leq \frac{\alpha d(x, Tx) d(y, Ty)}{d(x, y)} + \beta d(x, y)$$

for all $x, y \in X$, $x \neq y$ *and for some* $\alpha, \beta \in [0, 1)$ *with* $\alpha + \beta < 1$. *Then T has a unique fixed point in X.*

In this paper, we introduce the notion of generalized contractive condition of a rational type and prove the existence of best proximity point in the setting of metric space which generalizes Theorem 1.2 by Eldred and Veeramani, and Jaggi.

2. Preliminaries

In this section, we give some basic definitions and concepts which are useful and related to the context of our results.

Let A and B be nonempty subsets of a metric space (X, d). Define

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

Definition 2.1. A subset K of a metric space (X, d) is said to be *boundedly compact* if each bounded sequence in K has a subsequence converging to a point in K.

Definition 2.2. Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to satisfy *generalized contractive condition of a rational type* if

(i) $T(A) \subseteq B$ and $T(B) \subseteq A$;

(ii)

$$d(Tx, Ty) \leq \frac{\alpha d(x, Tx) d(y, Ty)}{d(x, y)} + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, y)$$
$$+ (1 - (\alpha + \beta + \gamma + \delta)) d(A, B)$$

for all $x \in A$, $y \in B$ with $\alpha + \beta + \gamma + \delta < 1$, where $0 \leq \alpha, \beta, \gamma, \delta \leq 1$.

Note that if $\alpha = \beta = \gamma = 0$, then T is a cyclic contraction.

3. Main Results

First, we give simple but very useful approximation result.

Proposition 3.1. *Let A and B be nonempty subsets of a metric space X. Suppose that* $T : A \cup B \to A \cup B$ *is cyclic and satisfies*

$$d(Tx, Ty) \leq \frac{\alpha d(x, Tx) d(y, Ty)}{d(x, y)} + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, y)$$

$$+ (1 - (\alpha + \beta + \gamma + \delta)) d(A, B)$$

for all $x \in A$, $y \in B$ *with* $a + \beta + \gamma + \delta < 1$, *where* $0 \leq \alpha, \beta, \gamma, \delta \leq 1$. *Then for any* $x_0 \in A \cup B$, *we have* $d(x_n, Tx_n) \to d(A, B)$, *where* $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$.

Proof.

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq \frac{\alpha d(x_{n-1}, Tx_{n-1}) d(x_n, Tx_n)}{d(x_{n-1}, x_n)} + \beta d(x_{n-1}, Tx_{n-1})$$

$$+ \gamma d(x_n, Tx_n) + \delta d(x_{n-1}, x_n) + (1 - (\alpha + \beta + \gamma + \delta)) d(A, B)$$

$$= \frac{\alpha d(x_{n-1}, x_n) d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1})$$

$$+ \delta d(x_{n-1}, x_n) + (1 - (\alpha + \beta + \gamma + \delta)) d(A, B)$$

$$= (\alpha + \gamma) d(x_n, x_{n+1}) + (\beta + \delta) d(x_{n-1}, x_n)$$

$$+ (1 - (\alpha + \beta + \gamma + \delta)) d(A, B).$$

Therefore,

$$d(x_n, x_{n+1}) \leq \frac{\beta + \delta}{1 - (\alpha + \gamma)} d(x_{n-1}, x_n) + \left(1 - \frac{\beta + \delta}{1 - (\alpha + \gamma)}\right) d(A, B).$$

Put $k = \dfrac{\beta + \delta}{1 - (\alpha + \gamma)}$, then $k < 1$. Therefore,

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leq kd(x_{n-1}, x_n) + (1 - k)d(A, B) \\
&= k[kd(x_{n-2}, x_{n-1}) + (1 - k)d(A, B)] + (1 - k)d(A, B) \\
&= k^2 d(x_{n-2}, x_{n-1}) + (1 - k^2)d(A, B).
\end{aligned}
$$

Inductively, we have

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1) + (1 - k^n)d(A, B).$$

As $n \to \infty$, we obtain $d(x_n, x_{n+1}) \to d(A, B)$. □

The following result of Eldred and Veeramani [1] is a special case of the above Proposition 3.1.

Corollary 3.1. *Let A and B be nonempty subsets of a metric space X. Suppose* $T : A \cup B \to A \cup B$ *is a cyclic contraction map. Then starting with any* $x_0 \in A \cup B$, *we have* $d(x_n, Tx_n) \to d(A, B)$, *where* $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$.

Proposition 3.2. *Let A and B be nonempty closed subsets of a complete metric space X. Let* $T : A \cup B \to A \cup B$ *be cyclic and satisfy*

$$
\begin{aligned}
d(Tx, Ty) \leq \frac{\alpha d(x, Tx) d(y, Ty)}{d(x, y)} &+ \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, y) \\
&+ (1 - (\alpha + \beta + \gamma + \delta))d(A, B)
\end{aligned}
$$

for all $x \in A$, $y \in B$ *with* $\alpha + \beta + \gamma + \delta < 1$, *where* $0 \leq \alpha, \beta, \gamma, \delta \leq 1$. *Let* $x_0 \in A$ *and define* $x_{n+1} = Tx_n$. *Suppose* $\{x_{2n}\}$ *has a convergent subsequence in A. Then there exists* $x \in A$ *such that* $d(x, Tx) = d(A, B)$.

Proof. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ converge to some $x \in A$. Then $d(A, B) \leq d(x, x_{2n_k - 1}) \leq d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k - 1})$.

Thus $d(x, x_{2n_k-1}) \to d(A, B)$.

Now,

$$d(A, B) \leq d(x_{2n_k}, Tx)$$

$$\leq \frac{\alpha d(x_{2n_k-1}, Tx_{2n_k-1})d(x, Tx)}{d(x_{2n_k-1}, x)} + \beta d(x_{2n_k-1}, Tx_{2n_k-1})$$

$$+ \gamma d(x, Tx) + \delta d(x_{2n_k-1}, x) + (1-(\alpha+\beta+\gamma+\delta))d(A, B).$$

Taking limit as $n \to \infty$, we get

$$d(A, B) \leq d(x, Tx) \leq \frac{\alpha d(A, B)d(x, Tx)}{d(A, B)} + \beta d(A, B)$$

$$+ \gamma d(x, Tx) + \delta d(A, B) + (1-(\alpha+\beta+\gamma+\delta))d(A, B),$$

that is,

$$d(A, B) \leq d(x, Tx) \leq (\alpha+\gamma)d(x, Tx) + (1-(\alpha+\gamma))d(A, B). \tag{2}$$

From (2),

$$d(x, Tx) \leq (\alpha+\gamma)d(x, Tx) + (1-(\alpha+\gamma))d(A, B),$$

we have

$$(1-(\alpha+\gamma))d(x, Tx) \leq (1-(\alpha+\gamma))d(A, B),$$

$$d(x, Tx) \leq d(A, B). \tag{3}$$

From (2) and (3), we get $d(A, B) \leq d(x, Tx) \leq d(A, B)$. Thus $d(x, Tx) = d(A, B)$. □

The following result of Jaggi [2] is a special case of the above Proposition 3.2.

Corollary 3.2. *Let T be a continuous self map defined on a complete metric space (X, d). Suppose that T satisfies the following contractive condition*:

$$d(Tx, Ty) \leq \frac{\alpha d(x, Tx) d(y, Ty)}{d(x, y)} + \beta d(x, y)$$

for all $x, y \in X$, $x \neq y$ *and for some* $\alpha, \beta \in [0, 1)$ *with* $\alpha + \beta < 1$. *Then T has a unique fixed point in X.*

Proof. Let $A = B = X \Rightarrow A \bigcup B = X$. Then T is cyclic map. Define $x_{n+1} = Tx_n$. Then $\{x_n\}$ is a convergent sequence in A and hence $\{x_{2n}\}$ is a convergent sequence in A. Then by Proposition 3.2 there exists $x \in A$ such that $d(x, Tx) = d(A, B) = 0$. Therefore, $Tx = x$. □

The following result of Eldred and Veeramani [1] is a special case of the above Proposition 3.2.

Corollary 3.3. *Let A and B be nonempty closed subsets of a complete metric space X. Let* $T : A \bigcup B \to A \bigcup B$ *be cyclic contraction map. Let* $x_0 \in A$ *and define* $x_{n+1} = Tx_n$. *Suppose* $\{x_{2n}\}$ *has a convergent subsequence in A. Then there exists* $x \in A$ *such that* $d(x, Tx) = d(A, B)$.

Proposition 3.3. *Let A and B be nonempty subsets of a metric space X. Let* $T : A \bigcup B \to A \bigcup B$ *be cyclic and satisfy*

$$d(Tx, Ty) \leq \frac{\alpha d(x, Tx) d(y, Ty)}{d(x, y)} + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, y)$$

$$+ (1 - (\alpha + \beta + \gamma + \delta)) d(A, B)$$

for all $x \in A$, $y \in B$ *with* $\alpha + \beta + \gamma + \delta < 1$, *where* $0 \leq \alpha, \beta, \gamma, \delta \leq 1$. *Then for any* $x_0 \in A \bigcup B$ *and* $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$, *the sequences* $\{x_{2n}\}$ *and* $\{x_{2n+1}\}$ *are bounded.*

Proof. Suppose $x_0 \in A$ (the proof when $x_0 \in B$ is similar). Since by Proposition 3.1, $d(x_{2n}, x_{2n+1})$ converges to $d(A, B)$. So it is enough to prove that $\{x_{2n+1}\}$ is bounded.

Suppose $\{x_{2n+1}\}$ is not bounded. Then there exists N_0 such that

$$d(T^2x_0, T^{2N_0+1}x_0) > M \text{ and } d(T^2x_0, T^{2N_0-1}x_0) \leq M,$$

where

$$M > \max\left\{\frac{2d(x_0, Tx_0)}{\frac{1}{k^2}-1} + d(A, B), d(T^2x_0, Tx_0)\right\}$$

and $k = \dfrac{\beta + \delta}{1-(\alpha+\gamma)}$.

$$\begin{aligned} M &< d(T^2x_0, T^{2N_0+1}x_0) \\ &\leq kd(Tx_0, T^{2N_0}x_0) + (1-k)d(A, B) \\ &\leq k[kd(x_0, T^{2N_0-1}x_0) + (1-k)d(A, B)] + (1-k)d(A, B) \\ &= k^2d(x_0, T^{2N_0-1}x_0) + (1-k^2)d(A, B). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{M - d(A, B)}{k^2} + d(A, B) &< d(x_0, T^{2N_0-1}x_0) \\ &\leq d(x_0, T^2x_0) + d(T^2x_0, T^{2N_0-1}x_0) \\ &\leq d(x_0, T^2x_0) + M \\ &\leq d(x_0, Tx_0) + d(Tx_0, T^2x_0) + M \\ &\leq 2d(x_0, Tx_0) + M. \end{aligned}$$

Thus, $M < \dfrac{2d(x_0, Tx_0)}{\frac{1}{k^2}-1} + d(A, B)$ which is a contradiction. □

The following result of Eldred and Veeramani [1] is a special case of the above Proposition 3.3.

Corollary 3.4. *Let A and B be nonempty subsets of a metric space X. Let* $T : A \cup B \to A \cup B$ *be cyclic contraction map. Then for* $x_0 \in A \cup B$ *and define* $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots,$ *the sequences* $\{x_{2n}\}$ *and* $\{x_{2n+1}\}$ *are bounded.*

Theorem 3.4. *Let A and B be nonempty closed subsets of a metric space X. Let* $T : A \cup B \to A \cup B$ *be a cyclic map and satisfy*

$$d(Tx, Ty) \leq \frac{\alpha d(x, Tx) d(y, Ty)}{d(x, y)} + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, y)$$

$$+ (1 - (\alpha + \beta + \gamma + \delta)) d(A, B)$$

for all $x \in A$, $y \in B$ *with* $\alpha + \beta + \gamma + \delta < 1$, *where* $0 \leq \alpha, \beta, \gamma, \delta \leq 1$. *If either A or B is boundedly compact, then there exists* $x \in A \cup B$ *such that* $d(x, Tx) = d(A, B)$.

Proof. Suppose A is boundedly compact. Let $x_0 \in A$ and $x_{n+1} = Tx_n$. By Proposition 3.3, $\{x_{2n}\}$ is bounded. Since A is boundedly compact, we have $\{x_{2n}\}$ has a subsequence converges to a point in A. By Proposition 3.2, there exists $x \in A$ such that $d(x, Tx) = d(A, B)$. Similarly, we can prove when B is boundedly compact. This completes the proof. □

The following result of Eldred and Veeramani [1] is a special case of the above Theorem 3.4.

Corollary 3.5. *Let A and B be nonempty closed subsets of a metric space X. Let* $T : A \cup B \to A \cup B$ *be a cyclic contraction map. If either A or B is boundedly compact, then there exists* $x \in A \cup B$ *such that* $d(x, Tx) = d(A, B)$.

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