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POLYNOMIAL UPPER BOUNDS ON LARGE AND MODERATE DEVIATIONS FOR DIFFEOMORPHISMS WITH WEAK HYPERBOLIC PRODUCT STRUCTURE

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Abstract

We establish polynomial upper bounds on large and moderate deviations for diffeomorphisms with weak hyperbolic product structure studied in [12], which is the intersection of two transversal families of weak stable and weak unstable disks, with countably many branches and variable return times. Applications of our results are some almost Anosov diffeomorphisms with uniformly contracting direction of which restriction on one dimensional center unstable direction behaves as a Manneville-Pomeau map.

1. Introduction

The purpose of this paper is to study upper bounds on large deviations for diffeomorphisms of a manifold. Let M be a finite dimensional

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Riemannian manifold, $f : M \circlearrowleft$ a map of M, and μ an f-invariant ergodic probability measure. For any $L^1(\mu)$ observation $\varphi : M \to \mathbb{R}$, let $S_n\varphi := \sum_{i=0}^{n-1} \varphi \circ f^i$. By Birkhoffs ergodic theorem, $\frac{1}{n} S_n\varphi$ converges to the mean $\int \varphi d\mu$ for μ-almost everywhere. The theory of large deviations concerns bounds on the probability

$$Dev_n(\varphi, \varepsilon; \mu) := \mu\left(\left\{\left|\frac{1}{n}\left(S_n\varphi - n\int \varphi\, d\mu\right)\right| \geq \varepsilon\right\}\right).$$

In the context of uniform hyperbolic systems, Kifer [16] and Young [30] proved large deviation results for Anosov diffeomorphisms and Axiom A attractors (see also the results of Kifer [16], Orey and Pelikan [23], Lopes [18]).

Beyond the context of uniformly hyperbolic systems, Araújo and Pacifico proved large deviations for non-uniformly expanding maps with non-flat singularities or criticalities and some partially hyperbolic diffeomorphisms with mostly expanding central direction [1]. Powerful methods for studying large and moderate deviations are presented by Rey-Bellet and Young [3], Melbourne and Nicol [22], and Melbourne [21]. The key object in [3, 21, 22] is a generalized horseshoe in the sense of Young [31]. This object allows us to collapse stable disks to deduce an expanding map for which large and moderate deviations results recover the same results for the original system. Applications in [3, 21, 22] include several important classes of chaotic systems which are Hénon maps [2], piecewise hyperbolic maps [31], dispersing billiards [31], some partially hyperbolic diffeormophisms [7, 8], Manneville-Pomeau maps, planar periodic Lorentz gases and dispersing Lorentz flows with vanishing curvature [21, 22].

We should also mention the results on large deviations for a general class of unimodal interval maps by Keller and Nowicki [15], for systems with indifferent fixed points by Pollicott and Sharp [25], Pollicott et al. [26], and for expansive homeomorphisms with specification by Maes and Verbitskiy

[19]. We note several important references on large deviations by Dembo and Zeitouni [9], Eliss [10], and Hennion and Hervé [14].

In this paper, we propose some scheme to obtain polynomial upper bounds on large and moderate deviations for diffeomorphisms equipped with some weaker conditions on the generalized horseshoe in [31]. Our results are applied to some partially hyperbolic diffeomorphisms of which the restriction on one dimensional center unstable direction behaves as the Manneville-Pomeau type maps [11-13].

This paper is organized as follows: In the next two subsections, we state main results and introduce examples of some partially hyperbolic diffeomorphisms to which our results are applied. The proof of the main results will be done in Section 2.

1.1. Statements and results

Let $f : M \circlearrowleft$ be a diffeomorphism of a finite dimensional Riemannian manifold *M*. Let *d* denote the distance on *M* induced by the Riemannian metric. An embedded disk $\gamma \subset M$ is called a *weak unstable disk* if for any $x, y \in \gamma$, $d(f^{-n}(x), f^{-n}(y)) \to 0$ as $n \to \infty$. An embedded disk $\gamma \subset M$ is called a *weak stable disk* if for any $x, y \in \gamma$, $d(f^{n}(x), f^{n}(y)) \to 0$ as $n \to \infty$. We say that $\Gamma^u := \{\gamma^u\}$ is a *continuous family* of C^1-*weak unstable disks* if there exist a compact set K^s, $k \in \mathbb{N}$, a unit disk D^u of $\mathbb{R}^k$ and a map $\Phi^u : K^s \times D^u \to M$ such that

(i) Φ^u maps $K^s \times D^u$ homeomorphically onto its image,

(ii) $x \mapsto \Phi^u(x \times \cdot)$ is a continuous map from K^s into $Emb^1(D^u, M)$, the space of C^1-embeddings of D^u into *M*, and

(iii) each γ^u is a weak unstable disk and satisfies that $\gamma^u = \Phi^u(x \times D^u)$.

A *continuous family* of C^1-*weak stable disks* is defined similarly.

We say that a subset Ξ has a *weak hyperbolic product structure* if there exist a continuous family of C^1-weak unstable disks $\Gamma^u := \{\gamma^u\}$ and a continuous family of C^1-weak stable disks $\Gamma^s := \{\gamma^s\}$ such that

(i) $\dim \gamma^u + \dim \gamma^s = \dim M$,

(ii) the γ^u-disks are transversal to the γ^s-disks with the angles between them bounded away from 0,

(iii) each γ^u-disk meets each γ^s-disk exactly one point, and

(iv) $\Xi = (\bigcup \gamma^u) \bigcap (\bigcup \gamma^s)$.

Throughout this subsection, we will always assume that *f* admits a subset $\Lambda \subset M$ with conditions (C1) and (C2) below. For any submanifold $\gamma \subset M$, m_γ denotes the Lebesgue measure on γ.

(C1) Λ has a weak hyperbolic product structure with the defining families Γ^u and Γ^s. Furthermore, for any $\gamma \in \Gamma^u$, $m_\gamma(\gamma \bigcap \Lambda) > 0$.

A subset $\Lambda_0 \subset \Lambda$ is called an *s-subset* if there exists $\Gamma_0^s \subset \Gamma^s$ such that Λ_0 has a weak hyperbolic product structure with defining families Γ^u and Γ_0^s. A *u-subset* is defined similarly. For $x \in \Lambda$, let $\gamma^u(x)$(resp. $\gamma^s(x)$) denote the element of Γ^u (resp. Γ^s) which contains *x*.

(C2) There exist disjoint *s*-subsets $\Lambda_1, \Lambda_1, ..., \subset \Lambda$ such that

(a) for any $\gamma \in \Gamma^u$, $m_\gamma(\gamma \bigcap (\Lambda \backslash \bigcup_{i \geq 1} \Lambda_i)) = 0$,

(b) for any $i \in \mathbb{N}$, there exists $R_i \in \mathbb{N}$ such that $f^{R_i}(\Lambda_i)$ is a *u*-subset of Λ and

(c) for any $x \in \Lambda_i$, $f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x))$ and $f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x))$.

By (C2), we can define a *return time function* $R : \Lambda \to \mathbb{N}$ by $R|_{\Lambda_i} := R_i$. A *return map* $f^R : \Lambda \circlearrowleft$ is defined by

$$f^R(x) := f^{R(x)}(x)$$

for any $x \in \Lambda$ with $R(x) < \infty$. We give a notion of separation time (cf. [32]). Let $\mathbb{Z}^+ := \{0\} \cup \mathbb{N}$. For $x, y \in \Lambda$, the *separation time* $s(x, y)$ is defined by

$$s(x, y) := \inf\{n \in \mathbb{Z}^+ \,|\, (f^R)^n(x) \text{ and } (f^R)^n(y) \text{ belong to distinct } \Lambda_i\text{'s}\}$$

with the convention: $s(x, y) = \infty$ if the corresponding set is empty. For any $n \in \mathbb{N}$, let $(f^n)^u$ denote the restriction of f^n to $\gamma \in \Gamma^u$, and $\det(D_x(f^n)^u)$ denote the Jacobian of the derivative $D_x(f^n)^u$ of $(f^n)^u$ at $x \in \Lambda$.

There exist $C > 0$ and $0 < \beta < 1$ such that the following conditions (C3) and (C4) are satisfied on the set Λ:

(C3) For any $\gamma \in \Gamma^u$, $i \in \mathbb{N}$ and $x, y \in \gamma \cap \Lambda_i$,

$$\log \frac{|\det(D_x(f^{R_i})^u)|}{|\det(D_y(f^{R_i})^u)|} \le C\beta^{s(f^{R_i}(x),\, f^{R_i}(y))}.$$

Let (X_1, m_1) and (X_2, m_2) be finite measure spaces. We say that a measurable bijection $T : (X_1, m_1) \to (X_2, m_2)$ is *nonsingular* or *absolutely continuous* if it maps sets of m_1-measure 0 to sets of m_2-measure 0. If T is absolutely continuous, then there exists the Jacobian $J(T) = J_{m_1, m_2}(T) = d(T_*^{-1}m_2)/dm_1$ of T with respect to m_1 and m_2.

(C4) For any γ and $\gamma' \in \Gamma^u$, if $\Theta : \gamma \cap \Lambda \to \gamma' \cap \Lambda$ is defined by $\Theta(z) = \gamma^s(z) \cap \gamma'$ for any $z \in \gamma \cap \Lambda$, then Θ is absolutely continuous and satisfies

(a) $\dfrac{d(\Theta_*^{-1} m_{\gamma'})}{dm_\gamma}(x) = \displaystyle\prod_{i=0}^{\infty} \frac{|\det(D_{f^i(x)} f^u)|}{|\det(D_{f^i(\Theta(x))} f^u)|}$,

(b)

$$\left| \sum_{k=R_i}^{\infty} \log \frac{|\det(D_{f^k(x)} f^u)|}{|\det(D_{f^k(\Theta(x))} f^u)|} - \sum_{k=R_i}^{\infty} \log \frac{|\det(D_{f^k(y)} f^u)|}{|\det(D_{f^k(\Theta(y))} f^u)|} \right|$$

$\leq C\beta^{s(f^{R_i}(x),\ f^{R_i}(y))}$ $(\forall x,\ y \in \gamma \cap \Lambda_i)$.

Remark 1.1. Property (P4) in [31] implies (C3). For diffeomorphisms introduced in the next subsection (see [12, 13]), it is difficult to find a subset with (P4) in [31], however, it is easy to find a subset with (C3).

Remark 1.2. Property (P4) together with (P5) in [31] implies (C4). Indeed, it is shown in the proof of Lemma 1 in [31].

We give a definition of an SRB measure. An embedded disk $\gamma \subset M$ is called a *unstable disk* if for any $x, y \in \gamma$, $d(f^{-n}(x), f^{-n}(y)) \to 0$ exponentially fast as $n \to \infty$. An invariant probability measure μ is said to be a *Sinai-Ruelle-Bowen* (*SRB*) *measure* if (i) μ has positive Lyapunov exponents, and (ii) the conditional measures of μ on unstable disks are absolutely continuous with respect to the Lebesgue measures on these disks (see [17] for the precise meaning of (ii)).

Remark 1.3 [12]**.** Let $f : M \circlearrowleft$ be a diffeomorphism which admits a subset Λ with conditions (C1)-(C4). Suppose that there exists $\gamma \in \Gamma^u$ such that $\int_{\gamma \cap \Lambda} R dm_\gamma < \infty$. Then f has an invariant probability measure ν whose conditional measures on weak unstable disks $\bigcup_{i=0}^{\infty} f^i(\Gamma^u)$ are absolutely continuous with respect to the Lebesgue measures on these disks. The measure ν for diffeomorphisms introduced in the next subsection is an SRB measure.

We further require the following conditions (C5) and (C6) to establish upper bounds on large and moderate deviations of a measure ν as in Remark 1.3 for f:

(C5) For any $i, i' \in \mathbb{N}$, $\ell \in \{0, 1, \ldots, R_i - 1\}$, and $\ell' \in \{0, 1, \ldots, R_{i'} - 1\}$, there exists $N = N(i, i', \ell, \ell') \in \mathbb{N}$ such that $f^{-n}(f^{\ell}(\Lambda_i)) \cap f^{\ell'}(\Lambda_{i'}) \neq \varnothing$ for any $n \geq N$.

Let $\mathcal{H}_\eta$ denote the set of Hölder continuous functions on M with Hölder exponent η. To state condition (C6) below, for any $\varphi \in \mathcal{H}_\eta$, we define functions $\chi, \psi : \Lambda \times \Lambda \times \mathbb{Z}^+ \to \mathbb{R} \cup \{\pm\infty\}$ as follows: for any $\gamma^s \in \Gamma^s$, $x, x' \in \gamma^s \cap \Lambda$ and $\ell \in \{0, \ldots, R(x) - 1\}$,

$$\chi(x, x', \ell) := \sum_{k=\ell+1}^{\infty} (\varphi(f^k(x)) - \varphi(f^k(x'))),$$

$$\psi(x, x', \ell) := \chi(x, x', \ell) + \varphi(f^{\ell}(x)).$$

(C6) For any $\varphi \in \mathcal{H}_\eta$, there exists $C_\varphi > 0$ such that

(a) for any $\gamma^s \in \Gamma^s$, $x, x' \in \gamma^s \cap \Lambda$ and $\ell \in \{0, \ldots, R(x) - 1\}$,

$$|\chi(x, x', \ell)| + |\varphi(f^{\ell}(x))| \leq C_\varphi,$$

(b) for any $\gamma^u, \gamma^{u'} \in \Gamma^u$, $\gamma^s, \gamma^{s'} \in \Gamma^s$, $i \in \mathbb{N}$, $x, x', y, y' \in \Lambda_i$ with $x, y \in \gamma^u$, $x', y' \in \gamma^{u'}$, $x, x' \in \gamma^s$ and $y, y' \in \gamma^{s'}$ and $\ell \in \{0, \ldots, R_i - 1\}$,

$$|\psi(x, x', \ell) - \psi(y, y', \ell)| \leq C_\varphi \beta^{s(x, y)}.$$

Remark 1.4. It follows from the same arguments as in the proof of Lemma 1(3) in [31] that Properties (P3) and (P4) in [31] imply (C6). Diffeomorphisms in the next subsection satisfy (C6).

Let μ be an f-invariant probability measure. For any Borel function

$\varphi : M \to \mathbb{R}$, $n \in \mathbb{N}$, $\tau \in (0, 1]$ and $\varepsilon > 0$, let

$$Dev_n^\tau(\varphi, \varepsilon; \mu) := \mu\left(\left\{\left|\frac{1}{n^r}\left(S_n\varphi - n\int \varphi\, d\mu\right)\right| \geq \varepsilon\right\}\right),$$

where $S_n\varphi := \sum_{i=0}^{n-1} \varphi \circ f^i$.

For any $n \in \mathbb{Z}^+$, let $\{R > n\} := \{x \in \Lambda \,|\, R(x) > n\}$. Let $\mathbb{R}^+ := [0, \infty)$. For any two sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$, $a_n = O(b_n)$ means that there exists $K > 0$ such that for any $n \in \mathbb{N}$, $a_n \leq Kb_n$.

The main result of this paper is the following:

Theorem A. *Let* $f : M \circlearrowleft$ *be a diffeomorphism which admits a subset* Λ *with conditions* (C1)-(C6). *Suppose that there exists* $\gamma \in \Gamma^u$ *such that* $\int_{\gamma\cap\Lambda} R dm_\gamma < \infty$. *Further, suppose that there exists a positive increasing function* $v : \mathbb{R}^+ \to \mathbb{R}$ *such that* (i) $\sum_{\ell=1}^{\infty} v(\ell) m_\gamma(\{R > \ell\}) < \infty$ *and* (ii) *a sequence* $\left\{\frac{v(\ell)}{v(\ell+1)}\right\}_{\ell=1}^{\infty}$ *is also increasing, and that there exists* $p = p(v) \in [1, \infty)$ *such that* $\sum_{\ell=1}^{\infty} \ell^{-\frac{1}{2}} \cdot v\left(\frac{\ell}{2}\right)^{-\frac{1}{2p}} < \infty$. *Then there exists an invariant probability measure* ν *whose conditional measures on weak unstable disks* $\bigcup_{i=0}^{\infty} f^i(\Gamma^u)$ *are absolutely continuous with respect to the Lebesgue measures on these disks such that for any* $\varphi \in \mathcal{H}_\eta$, $\varepsilon > 0$, *and* $\tau \in \left(\frac{1}{2}, 1\right]$,

$$Dev_n^\tau(\varphi, \varepsilon; \nu) = O(n^{p(1-2\tau)}).$$

We notice that if *f* admits a subset Λ with (C1)-(C6) such that there exist $\gamma \in \Gamma^u$ and $\lambda \in (2, \infty)$ such that $m_\gamma(\{R > n\}) = O(n^{-\lambda})$, then Theorem A is applicable to such an *f*. In particular, for any $\lambda' \in (2, \lambda)$, if we define an increasing function $v : \mathbb{R}^+ \to \mathbb{R}$ by $v(t) = 1$ $(t \in [0, 1))$ and $v(t) = t^{\lambda' - 1}$ $(t \in [0, \infty)),$ then we can take p to be any $p \in [1, \lambda' - 1).$

In Theorem A, if we take $\tau = 1,$ then we have polynomial upper bounds on level 1 large deviations. Next results on level 2 polynomial upper large deviation bounds are reduced from level 1 polynomial upper large deviation bounds of Theorem A. This is done by the same arguments as in [25]. For any $x \in M,$ let δ_x denote the dirac measure at x, and for any $n \in \mathbb{N},$ set $S_n \delta_x := \sum_{i=0}^{n-1} \delta_{f^i(x)}.$

Theorem B. *Assume M to be compact in the hypothesis of Theorem* A. *Let* $\mathcal{M}(M)$ *denote the set of probability measures on M with the weak* $*$ *topology*. *Then for any compact set* $\mathcal{C} \subset \mathcal{M}(M)$ *with* $\nu \notin \mathcal{C},$

$$\nu\left(\left\{x \in M \,\middle|\, \frac{1}{n} S_n \delta_x \in \mathcal{C}\right\}\right) = O(n^{-p}).$$

For the case when $m_\gamma(\{R > n\}) = O(n^{-\lambda})$ for some $\gamma \in \Gamma^u$ and $\lambda \in (1, 2],$ we have the following results:

Theorem C. *Let* $f : M \circlearrowleft$ *be a diffeomorphism which admits a subset* Λ *with conditions* (C1)-(C6). *Suppose that there exist* $\gamma \in \Gamma^u$ *and* $\lambda \in (1, 2]$ *such that* $m_\gamma(\{R > n\}) = O(n^{-\lambda}).$ *Then there exists an invariant probability measure* ν *whose conditional measures on weak unstable disks* $\bigcup_{i=0}^{\infty} f^i(\Gamma^u)$ *are absolutely continuous with respect to the Lebesgue measures on these disks such that for any* $\varphi \in \mathcal{H}_\eta,$ $\varepsilon > 0,$ $\tau \in \left(\frac{1}{2}, 1\right]$ *and* $\lambda' \in (1, \lambda),$

$$Dev_n^\tau(\varphi, \varepsilon; \nu) = O(n^{-\lambda' + 3 - 2\tau}).$$

In this case, by the same arguments as in [25], polynomial upper bounds on level 1 large deviations of Theorem C reduce same upper bounds on level 2 large deviations:

Theorem D. *Assume M to be compact in the hypothesis of Theorem* C. *Then for any compact set* $\mathcal{C} \subset \mathcal{M}(M)$ *with* $\nu \notin \mathcal{C}$,

$$\nu\left(\left\{x \in M \,\middle|\, \frac{1}{n} S_n \delta_x \in \mathcal{C}\right\}\right) = O(n^{-\lambda'+1}).$$

1.2. Examples

Let g be a $C^{1+\alpha}$-diffeomorphism of the two dimensional torus $\mathbb{T}^2$. We impose on g the following assumptions A1-A3:

A1. g is an almost Anosov diffeomorphism with uniform contracting direction, i.e., there exist a non-hyperbolic fixed point P of g, a norm $\|\cdot\|$ on $\mathbb{T}^2$, $\kappa \in (0, 1)$ and a $D_x g$ invariant decomposition $T_x \mathbb{T}^2 = E^s(x) \oplus E^u(x)$ into subspaces $E^s(x)$ and $E^u(x)$ which satisfy

$$\| D_x g|_{E^s(x)} \| \le \kappa, \quad \| D_x g|_{E^u(x)} \| \begin{cases} = 1 & (x = P), \\ > 1 & (x \ne P). \end{cases}$$

For any $\varepsilon \in (0, 1]$, let $I_\varepsilon := [-\varepsilon, \varepsilon]$, and $Emb^r(I_1, \mathbb{T}^2)(r \in \mathbb{N})$ denote the set of C^r embeddings of I_1 into $\mathbb{T}^2$ with the C^r-topology. By Assumption A1, it follows from [28, Theorem IV.1] that there exist two continuous maps $\phi^s : \mathbb{T}^2 \to Emb^1(I_1, \mathbb{T}^2)$ and $\phi^u : \mathbb{T}^2 \to Emb^1(I_1, \mathbb{T}^2)$ with $\phi^\sigma(\{x\} \times 0) = x$ $(x \in \mathbb{T}^2, \sigma = s, u)$ such that for any $\varepsilon \in (0, 1]$, the *local stable* and *local center unstable manifolds* $V_\varepsilon^s(x) := \phi^s(\{x\} \times I_\varepsilon)$ and $V_\varepsilon^u(x) := \phi^u(\{x\} \times I_\varepsilon)$ satisfy $T_x V_\varepsilon^\sigma(x) = E^\sigma(x)$ for $\sigma = s, u$ (for more details, see [28]).

A2. ϕ^u is a continuous map from $\mathbb{T}^2$ to $Emb^2(I_1, \mathbb{T}^2)$ with respect to the C^2 topology.

By [12, Lemma 4.1], we have that $g^{-1}(V_\varepsilon^u(P)) \subset V_\varepsilon^u(P)$. Then g restricted to $V_\varepsilon^u(P)$, $g|_{V_\varepsilon^u(P)}$, is a map from $V_\varepsilon^u(P)$ to $g(V_\varepsilon^u(P))$. Using the $\phi^u(\{P\} \times \cdot)$, we can identify $V_\varepsilon^u(P)$ with I_ε. Then P corresponds to the origin 0 in I_ε, and thus 0 is a fixed point for $g|_{V_\varepsilon^u(P)}$.

A3. If we identify $V_\varepsilon^u(P)$ with I_ε, then the graph of $g|_{V_\varepsilon^u(P)}$ can be represented as

$$g|_{V_\varepsilon^u(P)}(x) = x + x|x|^\alpha + o(x^2).$$

The map g has a Markov partition $\mathcal{R}$, and admits subsets $\{\Lambda^{(j)}\}_{j=1}^J \subset \mathbb{T}^2$ such that each $\Lambda^{(j)}$ is some element of $\mathcal{R}$ and does not contain the fixed point P. Each $\Lambda^{(j)}$ satisfies conditions (C1)-(C5) with some modifications as in [12] and [13], and, using the same arguments as in [12] and [13], condition (C6) with the same modification holds on the set $\Lambda^{(j)}$. In this case, it is possible to make the proof progresses of Theorems A-D to have the following result. We note that the map g has a unique ergodic SRB measure [13].

Theorem E. *For any* $\alpha \in (0, 1)$, *the map* g *above has a unique ergodic SRB measure* ν *such that for any* $\alpha' \in (\alpha, 1)$, *the following hold*:

(1) *for any* $\varphi \in \mathcal{H}_n$, $\varepsilon > 0$, *and* $\tau \in \left(\frac{1}{2}, 1\right]$,

$$Dev_n^\tau(\varphi, \varepsilon; \nu) = \begin{cases} O(n^{\left(\frac{1}{\alpha'}-1\right)(1-2\tau)}) & \left(\alpha \in \left(0, \frac{1}{2}\right), \alpha' \in \left(\alpha, \frac{1}{2}\right)\right), \\ O(n^{\left(-\frac{1}{\alpha'}+3-2\tau\right)}) & \left(\alpha \in \left[\frac{1}{2}, 1\right), \alpha' \in (\alpha, 1)\right), \end{cases}$$

(2) *for any compact set* $\mathcal{C} \subset \mathcal{M}(M)$ *with* $\nu \notin \mathcal{C}$,

$$\nu\left(\left\{x \in M \,\middle|\, \frac{1}{n} S_n \delta_x \in \mathcal{C}\right\}\right) = O(n^{-\frac{1}{\alpha'}+1}).$$

2. Proof of Theorems A and C

We give below the arguments in [31] to define a tower map $F : \Delta \circlearrowleft$ induced from $f^R : \Lambda \circlearrowleft$ which reduces a quotient map $\overline{F} : \overline{\Delta} \circlearrowleft$ by identifying points on each $\gamma^s \in \Gamma^s$. Then, for the proof of Theorem A, we use results from [20] and [24] to get polynomial upper bounds on large and moderate deviations for $\overline{F}$.

Throughout this section, we assume that $f : M \circlearrowleft$ is a diffeomorphism which admits a subset Λ with conditions (C1)-(C6), and that there exists $\gamma \in \Gamma^u$ such that $\int_{\gamma \cap \Lambda} R dm_\gamma < \infty$.

2.1. A tower induced from $f^R : \Lambda \circlearrowleft$

A tower Δ is a union of the ℓth floors Δ_ℓ for $\ell \in \mathbb{Z}^+$. We define $\Delta_0 := \Lambda \times \{0\}$. Let Δ_ℓ be a copy of a part of Λ by

$$\Delta_\ell := \{(x, \ell) \mid x \in \Lambda,\ \ell < R(x)\}.$$

Let $\Delta_{\ell, i}$ be a copy of Λ_i by

$$\Delta_{\ell, i} := \{(x, \ell) \mid x \in \Lambda_i,\ \ell < R(x)\}.$$

Then a system F on the tower $\Delta := \bigcup_{\ell \in \mathbb{Z}^+} \Delta_\ell$ is defined by

$$F(x, \ell) := \begin{cases} (x, \ell + 1) & \text{if } \ell + 1 < R(x), \\ (f^R(x), 0) & \text{if } \ell + 1 = R(x). \end{cases}$$

Here $f^R : \Lambda \circlearrowleft$ is the return map as in Subsection 1.1. Let $\pi : \Delta \to \bigcup_{k=0}^{\infty} f^k(\Lambda)$ be a natural projection defined by $\pi(x, \ell) = f^\ell(x)$ for

$(x, \ell) \in \Delta.$ Then we have that

$$f \circ \pi = \pi \circ F. \tag{2.1}$$

We define the separation time $s_\Delta(\cdot, \cdot)$ on Δ as follows: First, for any $x, y \in \Delta_0$, $s_\Delta(x, y)$ is defined by $s_\Delta(x, y) := s(x_0, y_0)$, where $x = (x_0, 0)$ and $y = (y_0, 0)$. Second, for any $x, y \in \Delta_\ell$, $s_\Delta(x, y)$ is defined by $s_\Delta(x, y) := s(x_0, y_0)$, where $x = (x_0, \ell)$, $y = (y_0, \ell) \in \overline{\Delta}_\ell$ and $(x_0, 0)$ and $(y_0, 0)$ are the unique preimages of x, y by F^ℓ, i.e., $F^\ell(x_0, 0) = x$ and $F^\ell(y_0, 0) = y$. Otherwise, $s_\Delta(x, y) = 0$. We will denote the separation time $s_\Delta(\cdot, \cdot)$ on Δ by $s(\cdot, \cdot)$.

Lemma 2.1. *For any* $\varphi \in \mathcal{H}_\eta$, *there exist* $\tilde{\psi}, \tilde{\chi} : \Delta \to \mathbb{R}$ *such that the following hold*:

(1) $\varphi \circ \pi = \tilde{\psi} + \tilde{\chi} - \tilde{\chi} \circ F$,

(2) $\tilde{\chi}$ *is bounded*,

(3) *for any* $x = (x_0, \ell)$, $y = (y_0, \ell) \in \Delta_\ell$ *such that* x_0 *and* y_0 *are in the same weak stable disk*, $\tilde{\psi}(x) = \tilde{\psi}(y)$,

(4) *for any* $x, y \in \Delta$, $|\tilde{\psi}(x) - \tilde{\psi}(y)| \le 2C_\varphi \beta^{s(x, y)}$, *where* C_φ *is a constant for* φ *in* (C6).

Proof. We fix an arbitrary $\hat{\gamma} \in \Gamma^u$. For any $y \in \Lambda$, let $\hat{y} := \gamma^s(y) \cap \hat{\gamma}$. For any $x = (x_0, \ell) \in \Delta$, we define $\hat{x} = (\hat{x}_0, \ell)$. We define a function $\tilde{\chi}$ on Δ by

$$\tilde{\chi}(x) := \sum_{j=0}^{\infty} (\varphi(\pi \circ F^j(x)) - \varphi(\pi \circ F^j(\hat{x})))$$

for any $x \in \Delta$. Let $x = (x_0, \ell) \in \Delta$. By (2.1), we have that

$$\pi \circ F^j(x) = f^j \circ \pi(x) = f^{j+\ell}(x_0).$$

Similarly, $\pi \circ F^j(\hat{x}) = f^{j+\ell}(\hat{x}_0)$. Since x_0 and $\hat{x}_0$ are in the same weak stable disk of Λ, by (C6)(a), we have that $|\tilde{\chi}(x)| \leq C_\varphi$. This proves (2) of the lemma. We define

$$\tilde{\psi} := \varphi \circ \pi - \tilde{\chi} + \tilde{\chi} \circ F.$$

Then we estimate that

$$\tilde{\psi}(x) = \sum_{j=0}^{\infty} \left(\varphi\left(\pi \circ F^j\left(F(\hat{x})\right)\right) - \varphi\left(\pi \circ F^j\left(\widehat{F(x)}\right)\right)\right) + \varphi\left(\pi(\hat{x})\right). \tag{2.2}$$

Thus $\tilde{\psi}$ satisfies (3) of the lemma. We show (4) of the lemma holds for the function $\tilde{\psi}$ above. Let $x, y \in \Delta$ be s.t. x and y belong to the same $\Delta_{\ell, i}$. Then the first coordinates of pairs of points $F(\hat{x})$, $F(\hat{y})$, and $\widehat{F(x)}, \widehat{F(y)}$ are in the same weak unstable disk, and the first coordinates of the pairs of points $F(\hat{x}), \widehat{F(x)}$, and $F(\hat{y}), \widehat{F(y)}$ are in the same weak sable disk. Thus, by (C6)(b), (2.1) and (2.2), we have that $|\tilde{\psi}(x) - \tilde{\psi}(y)| \leq C_\varphi \beta^{s(x,y)}$. If $x, y \in \Delta$ belong to distinct $\Delta_{\ell, i}$, then the same conclusion of the previous case holds since the function $\tilde{\chi}$ and φ are bounded by (C6)(a). □

2.2. Reduction of $F : \Delta \circlearrowleft$ to the expanding map $\overline{F} : \overline{\Delta} \circlearrowleft$

Let $\overline{\Delta}_\ell := \Delta_\ell / \sim$, where $(x, \ell) \sim (y, \ell)$ if $y \in \gamma^s(x)$. $\overline{\Delta}_{\ell, i}$ is defined similarly. Then we define $\overline{\Delta} := \bigcup_{\ell \in \mathbb{Z}^+} \overline{\Delta}_\ell$. Since f^R sends weak stable disks to weak stable disks by (C2)(c), the quotient map $\overline{F} : \overline{\Delta} \circlearrowleft$ is well defined topologically.

We define a measure $\overline{m}$ on $\overline{\Delta}$ in a way that $\overline{F}$ is nonsingular, and the Jacobian of $\overline{F}$ with respect to $\overline{m}$ is well defined and satisfies the distortion inequality as in (C3). To do this, it suffices to define a measure $\overline{m}$ on $\overline{\Lambda} := \Lambda / \sim$ by the following way [4, 31]. We then let a measure $\overline{m}|_{\overline{\Delta}_0}$ on

$\overline{\Delta}_0$ to be the measure induced from the natural identification of $\overline{\Delta}_0$ with $\overline{\Lambda}$ and let a measure $\overline{m}|_{\overline{\Delta}_\ell}$ on $\overline{\Delta}_\ell$ to be the measure induced from the identification of $\overline{\Delta}_\ell$ with a subset of $\overline{\Delta}_0$.

We take an arbitrary $\hat{\gamma} \in \Gamma^u$. For any $x \in \Lambda$, let $\hat{x} := \gamma^s(x) \cap \hat{\gamma}$. We define a function Φ by

$$\Phi(x) := \sum_{k=0}^{\infty} (\varphi^u(f^k(x)) - \varphi^u(f^k(\hat{x}))),$$

where $\varphi^u(x) := \log|\det(D_x f^u)|$. On each $\gamma \in \Gamma^u$, define $\overline{m}_\gamma := 1_{\gamma \cap \Lambda} e^{\Phi} m_\gamma$, where 1_A denotes the characteristic function of a set A. If for some $\gamma' \in \Gamma^u$, $f^{R_i}(\Lambda_i \cap \gamma) \subset \gamma'$, then for $x \in \Lambda_i \cap \gamma$, we write

$$J(f^R)(x) = J_{\overline{m}_\gamma, \overline{m}_{\gamma'}}(f^{R_i}|_{\Lambda_i \cap \gamma})(x).$$

In our setting, the following lemma is proved in [12] under conditions (C1)-(C4) using the same arguments as in [31].

Lemma 2.2 [12, Lemma 3.4]**.** (1) *For any* $\gamma, \gamma' \in \Gamma^u$, *let* $\Theta : \gamma \cap \Lambda \to \gamma' \cap \Lambda$ *be as in* (C4). *Then* $\Theta_* \overline{m}_\gamma = \overline{m}_{\gamma'}$,

(2) $J(f^R)(x) = J(f^R)(y)$ *for any* $y \in \gamma^s(x)$,

(3) *for any* $i \in \mathbb{N}$, $\gamma \in \Gamma^u$ *and* $x, y \in \gamma \cap \Lambda_i$,

$$\left| \log \frac{J(f^R)(x)}{J(f^R)(y)} \right| \le 5C\beta^{s(f^R(x), f^R(y))}.$$

We define $F^R : \Delta_0 \circlearrowleft$ by $F^R(x, 0) = (f^R(x), 0)$ for $x \in \Lambda$. Then the quotient map $\overline{F^R} : \Delta_0 \circlearrowleft$ is also well defined similarly. Let $\overline{\pi} : \Delta \to \overline{\Delta}$ be the projection. By Lemma 2.2(2), we can define the Jacobian $J(\overline{F^R})$ of

$\overline{F^R} : \Delta_0 \circlearrowleft$ with respect to $\overline{m}$ by $J(\overline{F^R})(\overline{x}) := J(f^R)(x)$ for any $\overline{x} \in \overline{\Delta}_0$ and $(x, 0) \in \overline{\pi}^{-1}(\overline{x})$. The Jacobian $J\overline{F}$ of $\overline{F}$ w.r.t. $\overline{m}$ is defined by $J\overline{F} \equiv 1$ on $\overline{\Delta} \setminus \overline{F}^{-1}\overline{\Delta}_0$, and $J\overline{F}(\overline{x}, \ell) = J(\overline{F^R})(\overline{x})$ if $(\overline{x}, \ell) \in \overline{F}^{-1}\overline{\Delta}_0$.

We define the separation time $\overline{s}(\cdot, \cdot)$ on $\overline{\Delta}$ as follows: First, for any $\overline{x}, \overline{y} \in \overline{\Delta}_0$, $\overline{s}(\overline{x}, \overline{y})$ is defined by $\overline{s}(\overline{x}, \overline{y}) := s(x, y)$, where $(x, 0) \in \overline{\pi}^{-1}(\overline{x})$ and $(y, 0) \in \overline{\pi}^{-1}(\overline{y})$. Second, for any $\overline{x}, \overline{y} \in \overline{\Delta}_\ell$, $\overline{s}(\overline{x}, \overline{y})$ is defined by $\overline{s}(\overline{x}, \overline{y}) := \overline{s}(\overline{x}_0, \overline{y}_0)$, where $\overline{x}_0, \overline{y}_0 \in \overline{\Delta}_0$ are the unique preimages of $\overline{x}$, $\overline{y}$ by $\overline{F}^\ell$, i.e., $\overline{F}^\ell(\overline{x}_0) = \overline{x}$ and $\overline{F}^\ell(\overline{y}_0) = \overline{y}$. Otherwise, $\overline{s}(\overline{x}, \overline{y}) = 0$.

The next lemma is proved in [12] using Lemma 2.2(3).

Lemma 2.3 [12, Lemma 3.5]. *There exists* $C_1 > 1$ *such that for any* $k \in \mathbb{N}$ *and* $\overline{x} \in \overline{D} \in \overline{\mathcal{D}}_k$,

$$\frac{1}{C_1} \le \sum_{\overline{x}' : \overline{F}^k(\overline{x}') = \overline{x}} \frac{1}{J(\overline{F}^k)(\overline{x}')} \le C_1.$$

We summarize the properties of $\overline{F} : \overline{\Delta} \circlearrowleft$ as follows: (a) $\overline{F^R} : \overline{\Delta}_{0,i} \to \overline{F^R}(\overline{\Delta}_{0,i})$ is bijective $(\mathrm{mod}\, \overline{m})$ and $\overline{F^R}(\overline{\Delta}_{0,i})$ is a union of some $\overline{\Delta}_{0,i}$'s $(\mathrm{mod}\, \overline{m})$, and furthermore, there exists $\eta_0 > 0$ such that $\inf_{i \in \mathbb{N}} \{\overline{m}(\overline{F^R}(\overline{\Delta}_{0,i}))\} \ge \eta_0$ (by (C2)), (b) $\overline{\mathcal{D}} = \{\overline{\Delta}_{\ell,i}\}$ is a partition such that $\vee_{j=0}^{\infty} \overline{F}^{-j}\overline{\mathcal{D}}$ is the partition into points, (c) $\overline{m}(\overline{\Delta}_0) < \infty$, (d) $\overline{m}(A) = \overline{m}(\overline{F}(A))$ for any $A \subset \overline{\Delta}_{\ell,i}$ with $\overline{F}(A) \subset \overline{\Delta}_{\ell+1,i}$, $\overline{m}(A) = \overline{m}(\overline{F}(A))$, (e) for any $i \in \mathbb{N}$, $\overline{F^R}|_{\overline{\Delta}_{0,i}}$ and its inverse are nonsingular with respect to $\overline{m}$, (f) there exists $C_1 > 1$ such that for any $i \in \mathbb{N}$ and $\overline{x}, \overline{y} \in \overline{\Delta}_{0,i}$,

$$\left| \frac{J(\overline{F^R})(\overline{x})}{J(\overline{F^R})(\overline{y})} - 1 \right| \le C_1 \beta^{\overline{s}(\overline{F^R}(\overline{x}),\, \overline{f^R}(\overline{y}))}$$

(by Lemma 2.2(3)), (g) $\int_{\overline{\Delta}_0} R \circ \overline{\pi}^{-1} d\overline{m} < \infty,$ and (h) for any $\ell, \ell' \in \mathbb{Z}^+$ and $i, i' \in \mathbb{N},$ there exists $N \in \mathbb{N}$ such that $\overline{F}^{-n}(\overline{\Delta}_{\ell, i}) \cap \overline{\Delta}_{\ell', i'} \neq \varnothing$ for any $n \geq N$ (by (C5)).

The next lemma follows from the same argument in [20, 31, 32].

Lemma 2.4. $\overline{F}$ *has an invariant probability measure* $\overline{\nu}$ *which is mixing such that* $d\overline{\nu} = \overline{\varrho} d\overline{m},$ *where* $\overline{\varrho}$ *satisfies* $C_2^{-1} \leq \overline{\varrho} \leq C_2$ *for some* $C_2 > 0$ *with*

$$|\overline{\varrho}(\overline{x}) - \overline{\varrho}(\overline{y})| \leq C_2 \beta^{\overline{s}(\overline{x}, \overline{y})} \quad (\overline{x}, \overline{y} \in \overline{\Delta}_{\ell, i}).$$

We define the *transfer operator* associated with $\overline{F}$ and the measure $\overline{m}$ by

$$\overline{\mathcal{L}}(\overline{\varphi})(\overline{x}) := \sum_{\overline{x}' : \overline{F}(\overline{x}') = \overline{x}} \frac{\overline{\varphi}(\overline{x}')}{J(\overline{F})(\overline{x}')} \tag{2.3}$$

for $\overline{\varphi} \in L^2(\overline{m})$ and $\overline{x} \in \overline{\Delta}.$ Let $L^{\infty}(\overline{m})$ be the set of functions which are essentially bounded with respect to $\overline{m}.$ We denote the essential sup norm with respect to $\overline{m}$ by $\|\cdot\|_{\infty}.$ Let

$$C_{\beta}(\overline{\Delta}) = \{\overline{\varphi} : \overline{\Delta} \to \mathbb{R} \mid \exists C_{\overline{\varphi}} > 0 \text{ s.t. } |\overline{\varphi}(\overline{x}) - \overline{\varphi}(\overline{y})|$$

$$\leq C_{\overline{\varphi}}(\beta)^{\overline{s}(\overline{x}, \overline{y})}, \forall \overline{x}, \overline{y} \in \overline{\Delta}_{\ell}\}.$$

For any $\overline{\psi} \in C_{\beta}(\overline{\Delta}),$ we define

$$\|\overline{\phi}\| := \max\{\|\overline{\phi}\|_{\infty}, C_{\overline{\phi}}\}, \tag{2.4}$$

where $C_{\overline{\phi}}$ is as in the definition of $C_{\beta}(\overline{\Delta}).$ We note that $\overline{\varrho} \in C_{\beta}(\overline{\Delta})$ and $\|\overline{\varrho}\| \leq C_2,$ where $\overline{\varrho}$ is as in Lemma 2.4. We denote the essential sup norm with respect to $\overline{m}$ restricted on $\overline{\Delta}_{\ell}$ by $\|\cdot\|_{\infty, \ell}.$ The following result is proved in [20].

Theorem 2.5 [20, Proposition 3.13, Corollary 3.15]**.** *Let* $w := \{w(\ell)\}_{\ell \in \mathbb{Z}^+}$ *be a positive increasing sequence such that* (i) $\sum_{\ell=1}^{\infty} w(\ell)\overline{m}(\overline{\Delta}_\ell) < \infty$ *and* (ii) *the sequence* $\left\{\frac{w(\ell)}{w(\ell+1)}\right\}_{\ell=1}^{\infty}$ *is also increasing. Then there exist* $k_1 = k_1(w) \in \mathbb{N}$ *and* $C_3 = C_3(w, k_1) > 0$ *such that for any* $\overline{\phi} \in C_\beta(\overline{\Delta})$ *with* $\int \overline{\phi} d\overline{m} = 1$, *any* $n \in \mathbb{N}$ *with* $n = k_1 j + r$ *for some* $j \in \mathbb{N}$ *and* $r \in \{0, ..., k_1 - 1\}$, *and any* $\ell \in \mathbb{Z}^+$,

$$\| \overline{\mathcal{L}}^n(\overline{\phi}) - \overline{\varrho} \|_{\infty, \ell} \leq C_3(\overline{\phi}) \frac{w(\ell)}{w(k_1 j)},$$

where $\overline{\varrho}$ *is as in Lemma* 2.4.

Throughout this section, we fix a positive increasing function $v : \mathbb{R}^+ \to \mathbb{R}$ such that (i) for some $\gamma \in \Gamma^u$, $\sum_{\ell=1}^{\infty} v(\ell) m_\gamma(\{R > \ell\}) < \infty$, and (ii) the sequence $\left\{\frac{v(\ell)}{v(\ell+1)}\right\}_{\ell=1}^{\infty}$ is also increasing. By Lemma 2.2(1), we have that $\sum_{\ell=0}^{\infty} v(\ell)\overline{m}(\overline{\Delta}_\ell) < \infty$. Then we let $k_1 = k_1(v) \in \mathbb{N}$ and $C_3 = C_3(v, k_1) > 0$ as in Theorem 2.5. The following result follows from Theorem 2.5 (see [12, Lemma 3.8]).

Theorem 2.6. *For any* $\overline{\phi} \in C_\beta(\overline{\Delta})$ *with* $\int \overline{\phi} d\overline{m} = 1$, *any* $n \in \mathbb{N}$ *and* $\ell \in \mathbb{Z}^+$,

$$\| \overline{\mathcal{L}}^n(\overline{\phi}) - \overline{\varrho} \|_{\infty, \ell} \leq C(\overline{\phi}) \frac{v(\ell)}{v\left(\frac{n}{2}\right)},$$

where

$$C(\overline{\phi}) := \max\left\{ \frac{v\left(\frac{k_1}{2}\right)}{v(0)} \left(\max_{r \in \{0, ..., k_1 - 1\}} \{\| \overline{\mathcal{L}}^r(\overline{\phi}) \|_\infty\} + \| \overline{\varrho} \|_\infty \right), C_3 \| \overline{\phi} \| \right\}.$$

Theorem 2.7 [20]. *Let* $(\overline{F}, \overline{\nu})$ *be as above. If there exists* $\lambda > 1$ *such that* $\overline{m}(\{R > n\}) = O(n^{-\lambda})$, *then for any* $\lambda' \in (1, \lambda)$, $\overline{\varphi}_1 \in L^\infty(\overline{m})$, *and* $\overline{\varphi}_2 \in C_\beta(\overline{\Delta})$, *there exists* $C_4 = C_4(\overline{\varphi}_1, \overline{\varphi}_2, \lambda') > 0$ *such that for any* $n \in \mathbb{N}$,

$$\left| \int (\overline{\varphi}_1 \circ \overline{F}^n)\overline{\varphi}_2 d\overline{\nu} - \int \overline{\varphi}_1 d\overline{\nu} \int \overline{\varphi}_2 d\overline{\nu} \right| \leq C_4 n^{-\lambda'+1}.$$

Let $\overline{\mathcal{B}}$ be the Borel σ-algebra on $\overline{\Delta}$. Let $E(\overline{\psi} \,|\, \overline{\mathcal{B}})$ denote the $\overline{\nu}$-conditional expectation of $\overline{\psi}$ with respect to $\mathcal{B}_0$. For any $p \geq 1$ and $\overline{\psi} : \overline{\Delta} \to \mathbb{R}$, $\| \overline{\psi} \|_p := \left(\int | \overline{\psi} |^p \, d\overline{\nu} \right)^{1/p}$. The following result follows from the same arguments as in [24].

Theorem 2.8 [24]. *Let* $(\overline{F}, \overline{\nu})$ *be as above. For any* $p \geq 1$ *and* $\overline{\xi} : \overline{\Delta} \to \mathbb{R}$ *with* $\| \overline{\xi} \|_p < \infty$, *there exist* $K_{2p} > 0$ *and* $C_5 > 0$ *such that for any* $n \in \mathbb{N}$,

$$\| S_n \overline{\xi} \|_{2p} \leq (K_{2p})^{\frac{1}{2p}} n^{\frac{1}{2}} \left(\| \overline{\xi} \circ \overline{F} \|_{2p} + C_5 \sum_{k=1}^{n} k^{-\frac{1}{2}} \| E(\overline{\xi} \circ \overline{F}^k \,|\, \overline{\mathcal{B}}) \|_{2p} \right).$$

It follows from [5] and [6] that the constant K_{2p} of Theorem 2.8 satisfies that $K_{2p} \leq (2p)^{2p}$.

2.3. Large and moderate deviations for F

We use the following convention: For any function $\varphi : M \to \mathbb{R}$, let $\tilde{\varphi}$ be the lift of φ to Δ defined by $\tilde{\varphi} = \varphi \circ \pi$, where $\pi : \Delta \to M$ is the projection which satisfies (2.1), and if $\tilde{\varphi}$ is a constant on γ^s disks, then we will confuse it with the function on $\overline{\Delta}$ called $\overline{\varphi}$. Let ν be the measure as in Remark 1.3. We define an F-invariant probability measure $\tilde{\nu}$ by $\pi_* \tilde{\nu} = \nu$. Then we have that for any $\varphi \in L^1(\nu)$, and an interval $I \subset \mathbb{R}$, $\nu(\{x \in M \,|\, \varphi(x) \in I\}) = \tilde{\nu}(\{x \in \Delta \,|\, \tilde{\varphi}(x) \in I\})$. So to prove Theorem A, it suffices to establish the upper bounds on large and moderate deviations of $(F, \tilde{\nu})$. For any $\tilde{\varphi} : \Delta \to \mathbb{R}$ and $n \in \mathbb{N}$, let $S_n\tilde{\varphi} := \sum_{i=0}^{n-1} \tilde{\varphi} \circ F^i$.

Let $\varphi \in \mathcal{H}_\eta$. Without loss of generality, we may assume that $\int \varphi d\nu = 0$. Let $\tilde{\psi}$ be a function of Lemma 2.1 associated with φ. Then by Lemma 2.1(1), $\tilde{\psi}$ satisfies that $\int \tilde{\psi} d\tilde{\nu} = 0$ since $\tilde{\nu}$ is F-invariant. Let $\tau \in \left(\frac{1}{2}, 1\right]$ and $\varepsilon > 0$. By Lemma 2.1, there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$\{x \in \Delta \,|\, |S_n\tilde{\varphi}(x)| > n^\tau \varepsilon\} \subset \{x \in \Delta \,|\, |S_n\tilde{\psi}(x)| > n^\tau \varepsilon/2\},$$

and so

$$\tilde{\nu}(\{x \in \Delta \,|\, |S_n\tilde{\varphi}(x)| \geq n^\tau \varepsilon\}) \leq \tilde{\nu}(\{x \in \Delta \,|\, |S_n\tilde{\psi}(x)| \geq n^\tau \varepsilon/2\}).$$

On the other hand, it follows from the same arguments as in [31] that $\overline{\nu} = \overline{\pi}_* \tilde{\nu}$. Here $\tilde{\nu}$ is a measure as in Lemma 2.4 and $\tilde{\pi}$ is the natural projection from Δ to $\overline{\Delta}$. Since $\tilde{\psi}$ depends only on future coordinates by Lemma 2.1, we have that

$$\tilde{\nu}(\{x \in \Delta \,|\, |S_n\tilde{\psi}(x)| \geq n^\tau \varepsilon/2\}) \leq \overline{\nu}(\{\overline{x} \in \overline{\Delta} \,|\, |S_n\overline{\psi}(\overline{x})| \geq n^\tau \varepsilon/2\}).$$

So, for any $p \geq 1$,

$$\tilde{\nu}(\{x \in \Delta \,|\, |S_n\tilde{\psi}(x)| \geq n^\tau \varepsilon/2\}) \leq \left(\frac{2}{n^\tau \varepsilon}\right)^{2p} (\| S_n\overline{\psi} \|_{2p})^{2p}, \tag{2.5}$$

where $\|\cdot\|_{2p}$ denotes the norm on $L^{2p}(\overline{\nu})$.

Proof of Theorem A. Let $p \geq 1$. Since $\overline{\psi}$ is bounded by (C6)(a) and (2.2) in the proof of Lemma 2.1, we have that $\| \overline{\psi} \|_{2p} < \infty$. Then by Theorem 2.8, for any $n \in \mathbb{N}$,

$$\| S_n\overline{\psi} \|_{2p} \leq (K_{2p})^{\frac{1}{2p}} n^{\frac{1}{2}} \left(\| \overline{\psi} \circ \overline{F} \|_{2p} + C_5 \sum_{k=1}^{n} k^{-\frac{1}{2}} \| E(\overline{\psi} \circ \overline{F}^k \,|\, \overline{\mathcal{B}}) \|_{2p} \right). \tag{2.6}$$

We note that $\| E(\overline{\psi} \circ \overline{F}^k \,|\, \overline{\mathcal{B}}) \|_{2p} = \| E(\overline{\psi} \,|\, \overline{F}^{-k}\overline{\mathcal{B}}) \|_{2p}$, since $\overline{\nu}$ is $\overline{F}$ invariant. Using the same argument in [31, p. 611], we have that

$$\int | E(\overline{\psi} \,|\, \overline{F}^{-k}\mathcal{B}) |^2 d\overline{\nu} \le \| \overline{\psi} \|_\infty \int | \overline{\mathcal{L}}^k (\overline{\psi}\, \overline{\varrho}) \,|\, d\overline{m}.$$

Thus we estimate that

$$\| E(\overline{\psi} \circ \overline{F}^k \,|\, \overline{\mathcal{B}}) \|_{2p} \le \left((\| \overline{\psi} \|_\infty)^{2p-2} \int | E(\overline{\psi} \,|\, \overline{F}^{-k}\overline{\mathcal{B}}) |^2 \, d\overline{\nu} \right)^{\frac{1}{2p}}$$

$$\le \left((\| \overline{\psi} \|_\infty)^{2p-1} \int | \overline{\mathcal{L}}^k (\overline{\psi}\, \overline{\varrho}) \,|\, d\overline{m} \right)^{\frac{1}{2p}}. \qquad (2.7)$$

Let $v : \mathbb{R}^+ \to \mathbb{R}$ be such that (i) $\sum_{\ell=1}^{\infty} v(\ell) m_{\gamma^u}(\{R > \ell\}) < \infty$ for some $\gamma \in \Gamma^u$ and (ii) the sequence $\left\{ \frac{v(\ell)}{v(\ell+1)} \right\}_{\ell=1}^{\infty}$ is also increasing, and $p \in [1, \infty)$ be such that $\sum_{\ell=1}^{\infty} \ell^{-\frac{1}{2}} \cdot v\left(\frac{\ell}{2}\right)^{-\frac{1}{2p}} < \infty$. We show that there exists $C_6 > 0$ such that for any $k \in \mathbb{N}$,

$$\int | \overline{\mathcal{L}}^k (\overline{\psi}\, \overline{\varrho}) \,|\, d\overline{m} \le \frac{C_6}{v\left(\frac{k}{2}\right)}. \qquad (2.8)$$

By Lemma 2.1(1), we have $\int \overline{\psi} d\overline{\nu} = 0$. Let $a_{\overline{\psi}} := 2\| \overline{\psi} \|_\infty$. Then we estimate that

$$\int | \overline{\mathcal{L}}^k (\overline{\psi}\, \overline{\varrho}) \,|\, d\overline{m} = \int \left| \overline{\mathcal{L}}^k ((\overline{\psi} + a_{\overline{\psi}})\overline{\varrho}) - \left(\int (\overline{\psi} + a_{\overline{\psi}}) d\overline{\nu} \right) \overline{\varrho} \right| d\overline{m}$$

$$\le a_{\overline{\psi}} \int \left| \overline{\mathcal{L}}^k \left(\frac{(\overline{\psi} + a_{\overline{\psi}})\overline{\varrho}}{\int (\overline{\psi} + a_{\overline{\psi}}) d\overline{\nu}} \right) - \overline{\varrho} \right| d\overline{m}$$

$$\leq a_{\overline{\psi}} \sum_{\ell=0}^{\infty} \overline{m}(\overline{\Delta}_\ell) \left\| \overline{\mathcal{L}}^k \left(\frac{(\overline{\psi} + a_{\overline{\psi}})\overline{\varrho}}{\int (\overline{\psi} + a_{\overline{\psi}}) d\overline{\nu}} \right) - \overline{\varrho} \right\|_{\infty, \ell}.$$

Since $\overline{\psi} \in C_\beta(\overline{\Delta})$ by Lemma 2.1(4), we have that $(\overline{\psi} + a_{\overline{\psi}})\overline{\varrho} \in C_\beta(\overline{\Delta})$. Since $\int \frac{(\overline{\psi} + a_{\overline{\psi}})\overline{\varrho}}{\int (\overline{\psi} + a_{\overline{\psi}}) d\overline{\nu}} d\overline{m} = 1$, we apply Theorem 2.6 to $\frac{(\overline{\psi} + a_{\overline{\psi}})\overline{\varrho}}{\int (\overline{\psi} + a_{\overline{\psi}}) d\overline{\nu}}$ in the place of $\overline{\phi}$ in Theorem 2.6, and have that for any $\ell \in \mathbb{Z}^+$,

$$\left\| \overline{\mathcal{L}}^k \left(\frac{(\overline{\psi} + a_{\overline{\psi}})\overline{\varrho}}{\int (\overline{\psi} + a_{\overline{\psi}}) d\overline{\nu}} \right) - \overline{\varrho} \right\|_{\infty, \ell} \leq C \left(\frac{(\overline{\psi} + a_{\overline{\psi}})\overline{\varrho}}{\int (\overline{\psi} + a_{\overline{\psi}}) d\overline{\nu}} \right) \frac{v(\ell)}{v\left(\frac{k}{2}\right)},$$

where $C \left(\frac{(\overline{\psi} + a_{\overline{\psi}})\overline{\varrho}}{\int (\overline{\psi} + a_{\overline{\psi}}) d\overline{\nu}} \right)$ is the constant of $\frac{(\overline{\psi} + a_{\overline{\psi}})\overline{\varrho}}{\int (\overline{\psi} + a_{\overline{\psi}}) d\overline{\nu}}$ in the place of $\overline{\phi}$ in Theorem 2.6. Thus, combining the inequalities above, we obtain (2.8) for the constant

$$C_6 := a_{\overline{\psi}} C \left(\frac{(\overline{\psi} + a_{\overline{\psi}})\overline{\varrho}}{\int (\overline{\psi} + a_{\overline{\psi}}) d\overline{\nu}} \right) \sum_{\ell=0}^{\infty} v(\ell) \overline{m}(\overline{\Delta}_\ell).$$

Substituting (2.8) into (2.7), we have that

$$\| E(\overline{\psi} \circ \overline{F}^k | \overline{\mathcal{B}}) \|_{2p} \leq \frac{C_7}{v\left(\frac{k}{2}\right)^{\frac{1}{2p}}}, \tag{2.9}$$

where $C_7 := (| \overline{\psi} |_\infty)^{\frac{2p-1}{2p}} (C_6)^{\frac{1}{2p}}$. Substituting (2.9) into (2.6), we have that

$$\| S_n \overline{\psi} \|_{2p} \leq (K_{2p})^{\frac{1}{2p}} n^{\frac{1}{2}} \left(\| \overline{\psi} \|_{2p} + C_5 C_7 \sum_{k=1}^{n} k^{-\frac{1}{2}} \cdot v\left(\frac{k}{2}\right)^{-\frac{1}{2p}} \right).$$

So the right hand side of the inequality of (2.5) is bounded above by

$$\left(\frac{2}{n^{\tau}\varepsilon}\right)^{2p} K_{2p} n^{\frac{2p}{2}} \left(\| \overline{\psi} \|_{2p} + C_5 C_7 \sum_{k=1}^{n} k^{-\frac{1}{2}} \cdot v\left(\frac{k}{2}\right)^{-\frac{1}{2p}} \right)^{2p}$$

$$\leq \frac{(4p)^{2p}}{\varepsilon^{2p}} \left(\| \overline{\psi} \|_{2p} + C_5 C_7 \sum_{k=1}^{n} k^{-\frac{1}{2}} \cdot v\left(\frac{k}{2}\right)^{-\frac{1}{2p}} \right)^{2p} n^{p(1-2\tau)}.$$

Here we used the fact that $K_{2p} \leq (2p)^{2p}$ in the last inequality. This concludes the proof of Theorem A. □

Proof of Theorem C. Assume that there exist $\lambda \in (1, 2]$ and $\gamma \in \Gamma^u$ such that $m_{\gamma^u}(\{R > n\}) = O(n^{-\lambda})$. In (2.5), if we take $p = 1$, then the right hand side of the inequality of (2.5) is bounded above by

$$\frac{4}{n^{2\tau}\varepsilon^2} \left\{ n \int \overline{\psi}^2 d\overline{\nu} + 2 \sum_{0 \leq i < j \leq n-1} \int \overline{\psi} \cdot \overline{\psi} \circ F^{j-i} d\overline{\nu} \right\} = (*).$$

Since $\overline{\psi} \in C_\beta(\overline{\Delta})$ by Lemma 2.1(4), we can apply Theorem 2.7 to $\overline{\psi}$ in the second term on the right hand side of the above inequality, and have that for $\lambda' \in (1, \lambda)$,

$$(*) \leq \frac{4}{n^{2\tau}\varepsilon^2} \left\{ n \int \overline{\psi}^2 d\nu + 2C_4 \sum_{r=1}^{n-1} \frac{n-1}{r^{\lambda'-1}} \right\}$$

$$\leq 4n^{1-2\tau}\varepsilon^{-2} \int \overline{\psi}^2 d\overline{\nu} + 8C_4 n^{1-2\tau} \varepsilon^{1-2\tau} \varepsilon^{-2} \sum_{r=1}^{n-1} r^{-\lambda'+1}$$

$$= O(n^{1-2\tau}) + O(n^{3-2\tau-\lambda'}) = O(n^{3-2\tau-\lambda'}).$$

□

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