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EXISTENCE AND ITERATIVE ALGORITHMS OF NONOSCILLATORY SOLUTIONS FOR A CLASS OF HIGHER ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract

This paper deals with the following higher order neutral delay differential equation:

$$[x(t)+cx(t-\tau)]^{(n)}$$
$$+(-1)^{n+1}Q(t,\ x(t-\sigma_1),\ x(t-\sigma_2),\ \ldots,\ x(t-\sigma_k))=0, \quad t\geq t_0,$$

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where n is a positive integer, $c \in \mathbb{R}$, $\tau > 0$, $\sigma_i > 0$ for $i = 1, ..., k$, and $Q \in C([t_0, \infty) \times \mathbb{R}^k, \mathbb{R})$. By employing the contraction mapping principle, we dicuss several existence results on nonoscillatory solutions for the above equation, establish a few Mann type iterative approximation algorithms for these nonoscillatory solutions and construct several error estimates between the approximate solutions and the nonoscillatory solutions. These results presented, in this paper, extend, improve and unify many known results due to Cheng and Annie [3], Graef et al. [6], Kulenović and Hadžiomerspahić [8, 9], Liu et al. [10], Zhang and Yu [14], Zhang [16] and Zhou and Zhang [18] and others.

1. Introduction and Preliminaries

In this paper, we study the higher order neutral delay differential equation:

$$[x(t) + cx(t-\tau)]^{(n)} + (-1)^{n+1} Q(t, x(t-\sigma_1), x(t-\sigma_2), ..., x(t-\sigma_k)) = 0, \quad t \geq t_0, \tag{1.1}$$

where n is a positive integer, $c \in \mathbb{R}$, $\tau > 0$, $\sigma_i > 0$ for $i = 1, ..., k$, and $Q \in C([t_0, \infty) \times \mathbb{R}^k, \mathbb{R})$ satisfy the following assumptions:

(H) there exist functions $p, q \in C([t_0, \infty), \mathbb{R}^+)$ and constants $M > N > 0$ satisfying:

$$| Q(t, w_1, ..., w_k) - Q(t, \overline{w}_1, ..., \overline{w}_k) |$$
$$\leq p(t) \max\{| w_i - \overline{w}_i | : 1 \leq i \leq k\}, \; t \in [t_0, \infty), \; w_i, \overline{w}_i \in [N, M], \; 1 \leq i \leq k$$

and

$$| Q(t, w_1, ..., w_k) | \leq q(t), \quad t \in [t_0, \infty), \quad w_i \in [N, M], \quad 1 \leq i \leq k.$$

In recent years, several authors have discussed the existence of nonoscillatory and oscillatory solutions for various kinds of neutral delay differential equations, see [1-18]. In particular, Kulenović and

Hadžiomerspahić [8, 9] studied the existence of nonoscillatory solutions for the first and second order neutral delay differential equations:

$$[x(t) + cx(t - \tau)]' + Q_1(t)x(t - \sigma_1) - Q_2(t)x(t - \sigma_2) = 0, \quad t \geq t_0, \quad (1.2)$$

$$[x(t) + cx(t - \tau)]'' + Q_1(t)x(t - \sigma_1) - Q_2(t)x(t - \sigma_2) = 0, \quad t \geq t_0. \quad (1.3)$$

Cheng and Annie [3] continued to study the existence of nonoscillatory solutions for equation (1.3) by omitting the conditions $c \neq 1$ and $aQ_1(t) \geq Q_2(t)$, which were used by Kulenović and Hadžiomerspahić [8]. Zhou and Zhang [18] extended the result in [8] to higher order neutral functional differential equation:

$$\begin{aligned}&[x(t) + cx(t - \tau)]^{(n)} \\ &+ (-1)^{n+1}[Q_1(t)x(t - \sigma_1) - Q_2(t)x(t - \sigma_2)] = 0, \quad t \geq t_0. \quad (1.4)\end{aligned}$$

To the best of the authors' knowledge, there is no literature referred to the iterative approximations of these nonoscillatory solutions for equations (1.2)-(1.4).

This paper is aimed to study the existence of nonoscillatory solutions for equation (1.1) to develop several Mann iterative approximation algorithms for these nonoscillatory solutions and to obtain some error estimates between the approximate solutions and the nonoscillatory solutions. These results presented, in this paper, extend, improve and unify many known results due to Cheng and Annie [3], Graef et al. [6], Kulenović and Hadžiomerspahić [8, 9], Zhang and Yu [14], Zhang [16] and Zhou and Zhang [18] and others.

Let $\gamma = \max\{\tau, \sigma_i : i = 1, \ldots, k\}$. By a solution of equation (1.1), we mean a function $x \in C([t_1 - \gamma, \infty), \mathbb{R})$ for some $t_1 \geq t_0$ such that $x(t) + cx(t - \tau)$ is n times continuously differentiable on $[t_1, \infty)$ and such that equation (1.1) is satisfied for $t \geq t_1$. As is customary, a solution of equation (1.1) is said to be *oscillatory* if it has arbitrarily large zeros and nonoscillatory otherwise.

2. Main Results

Throughout this paper, we assume that X denotes the Banach space of all continuous and bounded functions on $[t_0, \infty)$ with norm $\| x \| = \sup_{t \geq t_0} | x(t) |$, and

$$A(N, M) = \{x \in X : N \leq x(t) \leq M, t \geq t_0\} \text{ for } M > N > 0.$$

It is easy to see that $A(N, M)$ is a bounded closed and convex subset of X.

Theorem 2.1. *Let* (H) *hold and*

$$\int_{t_0}^{\infty} s^{n-1} \max\{p(s), q(s)\} ds < \infty. \tag{2.1}$$

Assume that $\{\lambda_n\}_{n \geq 0}$ *is an arbitrary sequence in* $[0, 1]$ *satisfying*

$$\sum_{m=0}^{\infty} \lambda_m = \infty. \tag{2.2}$$

If $| c | < 1$, *then there exist* $\theta_1 \in (0, 1)$ *and* $T > t_0 + \gamma$ *such that for any* $x_0 \in A(N, M)$, *the Mann iterative sequence* $\{x_m\}_{m \geq 0}$ *generated by the following algorithm*:

$$x_{m+1}(t) =$$

$$\begin{cases} (1 - \lambda_m) x_m(t) \\ + \lambda_m \Big\{ \frac{1}{2}(1 + c)(M + N) - c x_m(t - \tau) \\ + \frac{1}{(n-1)!} \int_t^{\infty} (s - t)^{n-1} Q(s, x_m(s - \sigma_1), \ldots, x_m(s - \sigma_k)) ds \Big\}, \quad t \geq T, m \geq 0, \\ (1 - \lambda_m) x_m(T) \\ + \lambda_m \Big\{ \frac{1}{2}(1 + c)(M + N) - c x_m(T - \tau) \\ + \frac{1}{(n-1)!} \int_T^{\infty} (s - T)^{n-1} Q(s, x_m(s - \sigma_1), \ldots, x_m(s - \sigma_k)) ds \Big\}, \quad t_0 \leq t < T, m \geq 0 \end{cases} \tag{2.3}$$

converges to a nonoscillatory solution $x \in A(N, M)$ *of equation* (1.1) *and* *has the following error estimate*:

$$\| x_{m+1} - x \| \le e^{-(1-\theta_1)\sum_{i=0}^{m} \lambda_i} \| x_0 - x \|, \quad m \ge 0. \tag{2.4}$$

Proof. It follows from $|c| < 1$, (H) and (2.1) that there exist constants $\theta_1 \in (0, 1)$ and $T > t_0 + \gamma$ satisfying

$$|c| + \frac{1}{(n-1)!} \int_T^{\infty} (s - T)^{n-1} p(s) ds = \theta_1 \tag{2.5}$$

and

$$\frac{1}{(n-1)!} \int_T^{\infty} (s - T)^{n-1} q(s) ds \le \frac{1}{2} (1 - |c|)(M - N). \tag{2.6}$$

Define a mapping $G : A(N, M) \to X$ by

$$Gx(t)$$

$$= \begin{cases} \frac{1}{2}(1+c)(M+N) - cx(t-\tau) \\ + \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} Q(s, x(s-\sigma_1), ..., x(s-\sigma_k)) ds, & t \ge T, \\ Gx(T), & t_0 \le t < T. \end{cases} \tag{2.7}$$

Clearly, Gx is continuous. For arbitrary $x, y \in A(N, M)$ and $t \ge T$, by (2.5), (2.7) and (H), we know that

$$|Gx(t) - Gy(t)|$$

$$\le |c| |x(t-\tau) - y(t-\tau)|$$

$$+ \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} | Q(s, x(s-\sigma_1), ..., x(s-\sigma_k))$$

$$- Q(s, y(s-\sigma_1), ..., y(s-\sigma_k)) | ds$$

$$\leq |c| \| x - y \| + \frac{\| x - y \|}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} p(s) ds$$

$$\leq \theta_1 \| x - y \|,$$

that is,

$$\| Gx - Gy \| \leq \theta_1 \| x - y \|, \quad x, y \in A(N, M). \tag{2.8}$$

Now we consider the following two cases :

Case 1. Suppose that $c \in [0, 1)$. According to (2.6) and (2.7), we deduce that for any $x \in A(N, M)$ and $t \geq T$,

$$Gx(t) = \frac{1}{2}(1+c)(M+N) - cx(t-\tau)$$

$$+ \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} Q(s, x(s-\sigma_1), \ldots, x(s-\sigma_k)) ds$$

$$\leq \frac{1}{2}(1+c)(M+N) - cN + \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} q(s) ds$$

$$\leq \frac{1}{2}(1+c)(M+N) - cN + \frac{1}{2}(1-c)(M-N)$$

$$= M$$

and

$$Gx(t) = \frac{1}{2}(1+c)(M+N) - cx(t-\tau)$$

$$+ \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} Q(s, x(s-\sigma_1), \ldots, x(s-\sigma_k)) ds$$

$$\geq \frac{1}{2}(1+c)(M+N) - cM - \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} q(s) ds$$

$$\geq \frac{1}{2}(1+c)(M+N) - cM - \frac{1}{2}(1-c)(M-N)$$

$$= N,$$

which mean that $S(A(N, M)) \subset A(N, M)$.

Case 2. Suppose that $c \in (-1, 0)$. In view of (2.6) and (2.7), we obtain that for any $x \in A(N, M)$ and $t \geq T$,

$$Gx(t) \leq \frac{1}{2}(1+c)(M+N) - cM + \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}q(s)ds$$

$$\leq \frac{1}{2}(1+c)(M+N) - cM + \frac{1}{2}(1+c)(M-N)$$

$$= M$$

and

$$Gx(t) \geq \frac{1}{2}(1+c)(M+N) - cN - \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}q(s)ds$$

$$\geq \frac{1}{2}(1+c)(M+N) - cN - \frac{1}{2}(1+c)(M-N)$$

$$= N,$$

which signify that $S(A(N, M)) \subset A(N, M)$.

It follows from (2.8), Case 1 and Case 2 that S is a contraction mapping and it has a unique fixed point $x \in A(N, M)$, which is a nonoscillatory solution of equation (1.1). For any $m \geq 0$ and $t \geq T$, in the light of (2.3), (2.7) and (2.8), we get that

$$| x_{m+1}(t) - x(t) |$$

$$= \left| (1-\lambda_m)x_m(t) + \lambda_m \left\{ \frac{1}{2}(1+c)(M+N) - cx_m(t-\tau) \right.\right.$$

$$\left.\left. + \frac{1}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}Q(s, x_m(s-\sigma_1), \ldots, x_m(s-\sigma_k))ds \right\} - x(t) \right|$$

$$\leq (1-\lambda_m)| x_m(t) - x(t) | + \lambda_m | Gx_m(t) - Gx(t) |$$

$$\leq (1-(1-\theta_1)\lambda_m)| x_m(t) - x(t) |$$

$$\leq e^{-(1-\theta_1)\sum_{i=0}^{m}\lambda_i} \| x_0 - x \|,$$

which yields that (2.4) holds. From (2.2) and (2.4), we conclude that $x_m \to x$ as $m \to \infty$. This completes the proof.

Theorem 2.2. *Let* (H), (2.1) *and* (2.2) *hold. If* $|c| > 1$, *then there exist* $\theta_2 \in (0, 1)$ *and* $T > t_0 + \gamma$ *such that for each* $x_0 \in A(N, M)$, *the Mann iterative sequence* $\{x_m\}_{m \geq 0}$ *generated by the following iterative algorithm*:

$$x_{m+1}(t) =$$

$$\begin{cases} (1-\lambda_m)x_m(t) \\ +\lambda_m\left\{\frac{1}{2}\left(1+\frac{1}{c}\right)(M+N)-\frac{1}{c}x_m(t+\tau)\right. \\ \left.+\frac{1}{c(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}Q(s, x_m(s-\sigma_1), \ldots, x_m(s-\sigma_k))ds\right\}, & t \geq T, m \geq 0, \\ (1-\lambda_m)x_m(T) \\ +\lambda_m\left\{\frac{1}{2}\left(1+\frac{1}{c}\right)(M+N)-\frac{1}{a}x_m(T+\tau)\right. \\ \left.+\frac{1}{c(n-1)!}\int_{T+\tau}^{\infty}(s-T-\tau)^{n-1}Q(s, x_m(s-\sigma_1), \ldots, x_m(s-\sigma_k))ds\right\}, & t_0 \leq t < T, m \geq 0 \end{cases} \tag{2.9}$$

converges to a nonoscillatory solution $x \in A(N, M)$ *of equation* (1.1) *and has the following error estimate*:

$$\| x_{m+1} - x \| \leq e^{-(1-\theta_2)\sum_{i=0}^{m}\lambda_i} \| x_0 - x \|, \quad m \geq 0. \tag{2.10}$$

Proof. It follows from $|c| > 1$, (H) and (2.1) that there exist constants $\theta_2 \in (0, 1)$ and $T > t_0 + \gamma$ satisfying

$$\frac{1}{|c|} + \frac{1}{|c|(n-1)!}\int_{T+\tau}^{\infty}(s - T - \tau)^{n-1} p(s) ds = \theta_2 \tag{2.11}$$

and

$$\frac{1}{|c|(n-1)!}\int_{T+\tau}^{\infty}(s-T-\tau)^{n-1}q(s)ds \leq \frac{1}{2}\left(1-\frac{1}{|c|}\right)(M-N). \quad (2.12)$$

Define a mapping $G : A(N, M) \to X$ by

$$Gx(t) = \begin{cases} \frac{1}{2}\left(1+\frac{1}{c}\right)(M+N) - \frac{1}{c}x(t+\tau) \\ + \frac{1}{c(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}Q(s, x(s-\sigma_1), \ldots, x(s-\sigma_k))ds, & t \geq T, \\ Gx(T), & t_0 \leq t < T. \end{cases}$$

(2.13)

Clearly, Gx is continuous. For every $x, y \in A(N, M)$ and $t \geq T$, by (2.11), (2.13) and (H), we conclude that for any $x, y \in A(N, M)$ and $t \geq T$,

$$\begin{aligned} & |Gx(t) - Gy(t)| \\ & \leq \frac{1}{|c|}|x(t+\tau) - y(t+\tau)| \\ & \quad + \frac{1}{|c|(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}|Q(s, x(s-\sigma_1), \ldots, x(s-\sigma_k)) \\ & \quad - Q(s, y(s-\sigma_1), \ldots, y(s-\sigma_k))|ds \\ & \leq \frac{1}{|c|}\|x-y\| + \frac{\|x-y\|}{|c|(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}p(s)ds \\ & \leq \theta_2\|x-y\|, \end{aligned}$$

which lead to

$$\|Gx - Gy\| \leq \theta_2\|x-y\|, \quad x, y \in A(N, M). \quad (2.14)$$

The following proof involves two cases:

Case 1. Assume that $c > 1$. Using (2.11) and (2.13), we infer that for any $x \in A(N, M)$ and $t \geq T$,

$$Gx(t) \leq \frac{1}{2}\left(1+\frac{1}{c}\right)(M+N) - \frac{1}{c}N + \frac{1}{c(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}q(s)ds$$

$$\leq \frac{1}{2}\left(1+\frac{1}{c}\right)(M+N) - \frac{1}{c}N + \frac{1}{2}\left(1-\frac{1}{c}\right)(M-N)$$

$$= M$$

and

$$Gx(t) \geq \frac{1}{2}\left(1+\frac{1}{c}\right)(M+N) - \frac{1}{c}M - \frac{1}{c(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}q(s)ds$$

$$\geq \frac{1}{2}\left(1+\frac{1}{c}\right)(M+N) - \frac{1}{c}M - \frac{1}{2}\left(1-\frac{1}{c}\right)(M-N)$$

$$= N,$$

which yield that $S(A(N, M)) \subset A(N, M)$.

Case 2. Assume that $c < -1$. Notice that (2.12) and (2.13) imply that for any $x \in A(N, M)$ and $t \geq T$,

$$Gx(t) \leq \frac{1}{2}\left(1+\frac{1}{c}\right)(M+N) - \frac{1}{c}M - \frac{1}{c(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}q(s)ds$$

$$\leq \frac{1}{2}\left(1+\frac{1}{c}\right)(M+N) - \frac{1}{c}M + \frac{1}{2}\left(1+\frac{1}{c}\right)(M-N)$$

$$= M$$

and

$$Gx(t) \geq \frac{1}{2}\left(1+\frac{1}{c}\right)(M+N) - \frac{1}{c}N + \frac{1}{c(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}q(s)ds$$

$$\geq \frac{1}{2}\left(1+\frac{1}{c}\right)(M+N)-\frac{1}{c}N-\frac{1}{2}\left(1+\frac{1}{c}\right)(M-N)$$

$$= N,$$

which give that $S(A(N, M)) \subset A(N, M)$.

Thus, (2.14), Case 1 and Case 2 ensure that S is a contraction mapping and it possesses a unique fixed point $x \in A(N, M)$, which is a nonoscillatory solution of equation (1.1). For any $m \geq 0$ and $t \geq T$, it follows from (2.9), (2.13) and (2.14) that

$$|x_{m+1}(t) - x(t)|$$

$$= \Bigg|(1-\lambda_m)x_m(t) + \lambda_m\left\{\frac{1}{2}\left(1+\frac{1}{c}\right)(M+N) - \frac{1}{c}x_m(t+\tau)\right.$$

$$\left. + \frac{1}{c(n-1)!}\int_{t+\tau}^{\infty}(s-t-\tau)^{n-1}Q(s, x_m(s-\sigma_1), \ldots, x_m(s-\sigma_k))ds\right\} - x(t)\Bigg|$$

$$\leq (1-\lambda_m)|x_m(t)-x(t)| + \lambda_m|Gx_m(t) - Gx(t)|$$

$$\leq (1-(1-\theta_2)\lambda_m)|x_m(t) - x(t)|$$

$$\leq e^{-(1-\theta_2)\sum_{i=0}^{m}\lambda_i}\|x_0 - x\|,$$

which means that (2.10) holds. By (2.2) and (2.10), we get that $x_n \to x$ as $n \to \infty$. This completes the proof.

Theorem 2.3. *Let* $c = 1$, (H) *and* (2.2) *hold. If* (2.1) *is satisfied for* $n = 1$, *then there exist* $\theta_3 \in (0, 1)$ *and* $T > t_0 + \gamma$ *such that for every* $x_0 \in A(N, M)$, *the Mann iterative sequence* $\{x_m\}_{m\geq 0}$ *generated by the following iterative algorithm*:

$$x_{m+1}(t) =$$

$$\begin{cases} (1-\lambda_m)x_m(t) \\ +\lambda_m\left\{\dfrac{M+N}{2}+\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau} Q(s, x_m(s-\sigma_1), \dots, x_m(s-\sigma_k))ds\right\}, \quad t \geq T, m \geq 0, \\ (1-\lambda_m)x_m(T) \\ +\lambda_m\left\{\dfrac{M+N}{2}+\sum_{j=1}^{\infty}\int_{T+(2j-1)\tau}^{T+2j\tau} Q(s, x_m(s-\sigma_1), \dots, x_m(s-\sigma_k))ds\right\}, \\ \qquad\qquad\qquad\qquad t_0 \leq t < T, m \geq 0 \end{cases} \tag{2.15}$$

converges to a nonoscillatory solution $x \in A(N, M)$ *of equation* (1.1) *and has the following error estimate*:

$$\| x_{m+1} - x \| \leq e^{-(1-\theta_3)\sum_{i=0}^{m} \lambda_i} \| x_0 - x \|, \quad m \geq 0. \tag{2.16}$$

Proof. Noting that $c = 1$, (H) and (2.1), we deduce that there exist constants $T > t_0 + \gamma$ and $\theta_3 \in (0, 1)$ satisfying

$$\int_T^{\infty} q(s)ds \leq \frac{M - N}{2} \tag{2.17}$$

and

$$\int_T^{\infty} p(s)ds = \theta_3. \tag{2.18}$$

Define a mapping $G : A(N, M) \to X$ by

$$Gx(t) = \begin{cases} \dfrac{M+N}{2} + \sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau} Q(s, x(s-\sigma_1), \dots, x(s-\sigma_k))ds, & t \geq T, \\ x(T), & t_0 \leq t < T. \end{cases} \tag{2.19}$$

Clearly, Gx is continuous. For every $x, y \in A(N, M)$ and $t \geq T$, together with (2.18), (2.19) and (H), we infer that for any $x, y \in A(N, M)$ and

$t \geq T$,

$$| Gx(t) - Gy(t) | \leq \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} | Q(s,\ x(s - \sigma_1),\ ...,\ x(s - \sigma_k))$$

$$- Q(s,\ y(s - \sigma_1),\ ...,\ y(s - \sigma_k)) \,| ds$$

$$\leq \| x - y \| \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} p(s) ds$$

$$\leq \theta_3 \| x - y \|,$$

which yields that

$$\| Gx - Gy \| \leq \theta_3 \| x - y \|, \quad x,\ y \in A(N,\ M). \tag{2.20}$$

Using (2.17) and (2.19), we know that for any $x \in A(N,\ M)$ and $t \geq T$,

$$Gx(t) \leq \frac{M + N}{2} + \int_t^{\infty} | Q(s,\ x(s - \sigma_1),\ ...,\ x(s - \sigma_k)) | ds$$

$$\leq \frac{M + N}{2} + \int_t^{\infty} q(s) ds$$

$$\leq M$$

and

$$Gx(t) \geq \frac{M + N}{2} - \int_t^{\infty} | Q(s,\ x(s - \sigma_1),\ ...,\ x(s - \sigma_k)) | ds$$

$$\geq \frac{M + N}{2} - \int_t^{\infty} q(s) ds$$

$$\geq N,$$

which imply that $S(A(N,\ M)) \subset A(N,\ M)$ and S is a contraction mapping and it possesses a unique fixed point $x \in A(N,\ M)$, which is a nonoscillatory solution of equation (1.1). For any $m \geq 0$ and $t \geq T$, it follows from (2.15), (2.19) and (2.20) that

$$
\begin{aligned}
&| x_{m+1}(t) - x(t) | \\
&= \Bigg| (1 - \lambda_m) x_m(t) + \lambda_m \Bigg\{ \frac{M + N}{2} \\
&\quad + \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} Q(s,\ x_m(s - \sigma_1),\ \ldots,\ x_m(s - \sigma_k)) ds \Bigg\} - x(t) \Bigg| \\
&\le (1 - \lambda_m) | x_m(t) - x(t) | + \lambda_m | Gx_m(t) - Gx(t) | \\
&\le (1 - (1 - \theta_3)\lambda_m) | x_m(t) - x(t) | \\
&\le e^{-(1-\theta_3)\sum_{i=0}^{m} \lambda_i} \| x_0 - x \|,
\end{aligned}
$$

which means that (2.16) holds. By (2.2) and (2.16), we see that $x_m \to x$ as $m \to \infty.$ This completes the proof.

Theorem 2.4. *Let* $c = 1,$ (H) *and* (2.2) *hold. If* (2.1) *is satisfied for* $n \ge 2,$ *then there exist* $\theta_4 \in (0, 1)$ *and* $T > t_0 + \gamma$ *such that for arbitrary* $x_0 \in A(N, M),$ *the Mann iterative sequence* $\{x_m\}_{m \ge 0}$ *generated by the following iterative algorithm*:

$$
x_{m+1}(t) = \begin{cases}
(1 - \lambda_m) x_m(t) + \lambda_m \Bigg\{ \dfrac{M + N}{2} \\
+ \dfrac{1}{(n-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{\infty} (s - u)^{n-2} \\
\times Q(s,\ x_m(s - \sigma_1),\ \ldots,\ x_m(s - \sigma_k)) dsdu \Bigg\}, & t \ge T,\ m \ge 0, \\
(1 - \lambda_m) x_m(T) + \lambda_m \Bigg\{ \dfrac{M + N}{2} \\
+ \dfrac{1}{(n-2)!} \sum_{j=1}^{\infty} \int_{T+(2j-1)\tau}^{T+2j\tau} \int_u^{\infty} (s - u)^{n-2} \\
\times Q(s,\ x_m(s - \sigma_1),\ \ldots,\ x_m(s - \sigma_k)) dsdu \Bigg\}, & t_0 \le t < T,\ m \ge 0
\end{cases}
\tag{2.21}
$$

converges to a nonoscillatory solution $x \in A(N, M)$ *of equation* (1.1) *and has the following error estimate*:

$$\| x_{m+1} - x \| \le e^{-(1-\theta_4)\sum_{i=0}^{m} \lambda_i} \| x_0 - x \|, \quad m \ge 0. \tag{2.22}$$

Proof. It follows from $c = 1$, (H) and (2.1) that there exist constants $\theta_4 \in (0, 1)$ and $T > t_0 + \gamma$ satisfying

$$\frac{1}{(n-1)!}\int_T^\infty (s - T)^{n-1} p(s) ds = \theta_4 \tag{2.23}$$

and

$$\frac{1}{(n-1)!}\int_T^\infty (s - T)^{n-1} q(s) ds \le \frac{M - N}{2}. \tag{2.24}$$

Define a mapping $G : A(N, M) \to X$ by

$$Gx(t) = \begin{cases} \dfrac{M + N}{2} + \dfrac{1}{(n-2)!}\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_u^\infty (s-u)^{n-2} & \\ \cdot\, Q(s, x(s - \sigma_1), \dots, x(s - \sigma_k)) ds du, & t \ge T, \\ Gx(T), & t_0 \le t < T. \end{cases} \tag{2.25}$$

Clearly, Gx is continuous. For every $x, y \in A(N, M)$ and $t \ge T$, by virtue of (2.23), (2.25) and (H), we derive that for any $x, y \in A(N, M)$ and $t \ge T$,

$$\begin{aligned} &| Gx(t) - Gy(t) | \\ &\le \frac{1}{(n-2)!}\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_u^\infty (s-u)^{n-2} | Q(s, x(s - \sigma_1), \dots, x(s - \sigma_k)) \\ &\quad - Q(s, y(s - \sigma_1), \dots, y(s - \sigma_k)) | ds du \end{aligned}$$

$$\leq \frac{\| x - y \|}{(n-2)!} \int_t^\infty \int_u^\infty (s-u)^{n-2} p(s) ds du$$

$$\leq \frac{\| x - y \|}{(n-1)!} \int_t^\infty (s-t)^{n-1} p(s) ds$$

$$\leq \theta_4 \| x - y \|,$$

which yields that

$$\| Gx - Gy \| \leq \theta_4 \| x - y \|, \quad x, y \in A(N, M). \tag{2.26}$$

In the light of (2.24) and (2.25), we conclude that for any $x \in A(N, M)$ and $t \geq T$,

$$Gx(t) \leq \frac{M+N}{2}$$

$$+ \frac{1}{(n-2)!} \int_t^\infty \int_u^\infty (s-u)^{n-2} | Q(s, x(s-\sigma_1), ..., x(s-\sigma_k)) | ds du$$

$$\leq \frac{M+N}{2} + \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1} q(s) ds$$

$$\leq M$$

and

$$Gx(t) \geq \frac{M+N}{2}$$

$$- \frac{1}{(n-2)!} \int_t^\infty \int_u^\infty (s-u)^{n-2} | Q(s, x(s-\sigma_1), ..., x(s-\sigma_k)) | ds du$$

$$\geq \frac{M+N}{2} - \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1} q(s) ds$$

$$\geq N,$$

which mean that $S(A(N, M)) \subset A(N, M)$ and S is a contraction mapping and it has a unique fixed point $x \in A(N, M)$, which is a nonoscillatory solution of equation (1.1). For any $m \geq 0$ and $t \geq T$, it follows from (2.21), (2.25) and (2.26) that

$$
\begin{aligned}
&| x_{m+1}(t) - x(t) | \\
&= \Bigg| (1 - \lambda_m) x_m(t) + \lambda_m \Bigg\{ \frac{M + N}{2} + \frac{1}{(n-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{\infty} (s-u)^{n-2} \\
&\quad \cdot Q(s, x_m(s - \sigma_1), \ldots, x_m(s - \sigma_k)) ds du \Bigg\} - x(t) \Bigg| \\
&\leq (1 - \lambda_m) | x_m(t) - x(t) | + \lambda_m | Gx_m(t) - Gx(t) | \\
&\leq (1 - (1 - \theta_4)\lambda_m) | x_m(t) - x(t) | \\
&\leq e^{-(1-\theta_4)\sum_{i=0}^{m} \lambda_i} \| x_0 - x \|,
\end{aligned}
$$

which shows that (2.22) holds. It follows from (2.2) and (2.22) that $x_m \to x$ as $m \to \infty$. This completes the proof.

Remark 2.1. Theorems 2.1-2.4 extend, improve and unify many known results due to Cheng and Annie [3], Graef et al. [6], Kulenović and Hadžiomerspahić [8, 9], Liu et al. [10], Zhang and Yu [14], Zhang [16] and Zhou and Zhang [18] and others.

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