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ON BARNES MULTIPLE HURWITZ ZETA FUNCTIONS AND DIRICHLET-TYPE MULTIPLE *L*-FUNCTIONS RELATED TO A UNIFIED CLASS OF POLYNOMIALS

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Abstract

In this paper, we define a unified-Barnes multiple Hurwitz zeta function and a Dirichlet-type multiple *L*-function which interpolate a unified class of polynomials and numbers at negative integer values. This unified class of polynomials contains as special case the celebrated Apostol-Bernoulli, Euler and Genocchi polynomials of higher order as well as their χ-extended versions. Many relationships are given.

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1. Introduction, Definitions and Notation

In the last years, Luo and Srivastava [12, 13] introduced the generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x)$ of order α, Luo [11] investigated the generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x)$ of order α and the generalized Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x)$ of order α. Ozden [14, 15] and Ozden et al. [17] have investigated an interesting unification of the Apostol-Bernoulli, Euler and Genocchi polynomials. Explicitly, Ozden studied the following generating function:

Definition 1.1. Let $k \in \mathbb{N}_0$, $a, b \in \mathbb{R}^+$ and $\alpha, \beta \in \mathbb{C}$,

$$\left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}^{(\alpha)}(x;\, k,\, a,\, b)\frac{t^n}{n!} \tag{1.1}$$

$$\left(\left|\, t + b\log\left(\frac{\beta}{a}\right)\right| < 2\pi,\; x \in \mathbb{R};\; 1^{\alpha} := 1\right).$$

This family of polynomials includes as special cases, the well-known Apostol-Bernoulli, Euler and Genocchi polynomials. These polynomials are defined, respectively, as follows:

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x;\, \lambda)\frac{t^n}{n!} \quad (|\, t + \log\lambda\,| < 2\pi;\; 1^{\alpha} := 1), \tag{1.2}$$

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\, \lambda)\frac{t^n}{n!} \quad (|\, t + \log\lambda\,| < \pi;\; 1^{\alpha} := 1), \tag{1.3}$$

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\, \lambda)\frac{t^n}{n!} \quad (|\, t + \log\lambda\,| < \pi;\; 1^{\alpha} := 1). \tag{1.4}$$

We can see that

$$\mathcal{Y}_{n,\lambda}^{(\alpha)}(x;\, 1,\, 1,\, 1) = \mathcal{B}_n^{(\alpha)}(x;\, \lambda), \tag{1.5}$$

$$\mathcal{Y}_{n,\lambda}^{(\alpha)}(x;\, 0,\, -1,\, 1) = \mathcal{E}_n^{(\alpha)}(x;\, \lambda), \tag{1.6}$$

$$\mathcal{Y}_{n,\lambda}^{(\alpha)}(x; 1, -1, 1) = \frac{1}{2^{\alpha}} \mathcal{G}_n^{(\alpha)}(x; \lambda). \tag{1.7}$$

Moreover, Ozden et al. in [17] have extended and investigated the generating function (1.1) $(\alpha = 1)$ in terms of a Dirichlet character χ of conductor $f \in \mathbb{N}$ since these polynomials are very important in several fields of mathematics and physics. They proposed the following χ-extension of the generating function for the generalized Apostol-Bernoulli, Euler and Genocchi polynomials. Let χ be a Dirichlet character of conductor $f \in \mathbb{N}$. Then

$$2^{1-k} t^k \sum_{j=0}^{f-1} \frac{\chi(j) \left(\frac{\beta}{a}\right)^{bj} e^{jt}}{\beta^{bf} e^{ft} - a^{bf}} e^{xt} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\chi,\beta}(x; k, a, b) \frac{t^n}{n!} \tag{1.8}$$

$$\left(\left| ft + bf \log\left(\frac{\beta}{a}\right)\right| < 2\pi,\ x \in \mathbb{R};\ k \in \mathbb{N}_0;\ f \in \mathbb{N},\ a, b \in \mathbb{R}^+;\ \beta \in \mathbb{C}\right).$$

Recently, the authors in [7] have studied symmetry properties of the χ-extended unified polynomials of higher order denoted by $\mathcal{Y}_{n,\chi,\beta}^{(\alpha)}(x; k, a, b)$ and defined as follows:

Definition 1.2. Let χ be a Dirichlet character of conductor $f \in \mathbb{N}$. Let $k \in \mathbb{N}_0$, $a, b \in \mathbb{R}^+$ and $\alpha, \beta \in \mathbb{C}$. Then the χ-extended unified polynomials of higher order $\mathcal{Y}_{n,\chi,\beta}^{(\alpha)}(x; k, a, b)$ are given by

$$\left(2^{1-k} t^k \sum_{j=0}^{f-1} \frac{\chi(j) \left(\frac{\beta}{a}\right)^{bj} e^{jt}}{\beta^{bf} e^{ft} - a^{bf}}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\chi,\beta}^{(\alpha)}(x; k, a, b) \frac{t^n}{n!} \tag{1.9}$$

$$\left(\left| ft + bf \log\left(\frac{\beta}{a}\right)\right| < 2\pi,\ x \in \mathbb{R};\ k \in \mathbb{N}_0;\ f \in \mathbb{N},\ a, b \in \mathbb{R}^+;\ \alpha, \beta \in \mathbb{C}\right).$$

The case $x = 0$, $\mathcal{Y}_{n,\chi,\beta}^{(\alpha)}(0; k, a, b) = \mathcal{Y}_{n,\chi,\beta}^{(\alpha)}(k, a, b)$ gives the χ-extended unified numbers of higher order.

Remark 1.3. If we set $\chi \equiv 1$ in (1.9), then we obtain the generating function (1.1). The χ-extended versions $\mathcal{B}_{n,\chi}^{(\alpha)}(x, \beta)$ of the Apostol-Bernoulli polynomials of higher order, $\mathcal{E}_{n,\chi}^{(\alpha)}(x, \beta)$ of the Apostol-Euler polynomials of higher order and $\mathcal{G}_{n,\chi}^{(\alpha)}(x, \beta)$ of the Apostol-Genocchi polynomials of higher order are given, respectively, by

$$\mathcal{B}_{n,\chi}^{(\alpha)}(x, \beta) = \mathcal{Y}_{n,\chi,\beta}^{(\alpha)}(x; 1, 1, 1), \tag{1.10}$$

$$\mathcal{E}_{n,\chi}^{(\alpha)}(x, \beta) = \mathcal{Y}_{n,\chi,\beta}^{(\alpha)}(x; 0, -1, 1), \tag{1.11}$$

$$\mathcal{G}_{n,\chi}^{(\alpha)}(x, \beta) = 2^{\alpha}\mathcal{Y}_{n,\chi,\beta}^{(\alpha)}(x; 1, -1, 1). \tag{1.12}$$

Moreover, if we put $\alpha = 1$ and $\beta = 1$ in (1.10)-(1.12), then we get the following χ-extended version of the classical Bernoulli, Euler and Genocchi polynomials [20, 22]:

$$B_{n,\chi}(x) = \mathcal{Y}_{n,\chi,1}^{(1)}(x; 1, 1, 1), \tag{1.13}$$

$$E_{n,\chi}(x) = \mathcal{Y}_{n,\chi,1}^{(1)}(x; 0, -1, 1), \tag{1.14}$$

$$G_{n,\chi}(x) = 2\mathcal{Y}_{n,\chi,1}^{(1)}(x; 1, -1, 1), \tag{1.15}$$

respectively.

Next, Ozden et al. [17] constructed a unification of the Hurwitz zeta functions by making use of the Mellin transformation applied to the generating function of the unification of Apostol-Bernoulli, Euler and Genocchi polynomials. Precisely, they defined the following family of Hurwitz zeta functions:

Definition 1.4. For $s, \beta \in \mathbb{C}$, $|\beta| < 1$, $k \in \mathbb{N}_0$ and $x, a, b \in \mathbb{R}^+$, we have for $\operatorname{Re}(s) > 1$:

$$\zeta_\beta(s,\, x;\, k,\, a,\, b) = \frac{(-1)^{k-1}2^{(1-k)}}{a^b}\sum_{n=0}^{\infty}\frac{\left(\frac{\beta^b}{a^b}\right)^n}{(n+x)^s}, \tag{1.16}$$

where $\zeta_\beta(s,\, x;\, k,\, a,\, b)$ is called the *unified Hurwitz zeta function*.

A large number of works have been done recently concerning the construction of Hurwitz zeta functions that interpolate special classes of Bernoulli, Euler, Genocchi polynomials and numbers. In addition, numerous papers treat the construction of Dirichlet-type L-functions for the special classes of Bernoulli, Euler, Genocchi numbers attached to a primitive Dirichlet character (see [3, 8-10, 16, 18, 19, 21]).

In this paper, we define, in Section 2, a unified-Barnes multiple Hurwitz zeta function $\zeta_\beta^{(l)}(s,\, x;\, k,\, a,\, b)$ which interpolates the unified class of polynomials and numbers of higher order $\mathcal{Y}_{n,\beta}^{(l)}(x;\, k,\, a,\, b)$ at negative integer values of s. A multiplication formula and some special cases are given. In Section 3, we construct a Dirichlet-type multiple L-function $L_\beta^{(l)}(s,\, \chi;\, k,\, a,\, b)$ which interpolates the χ version of the unified class of numbers of higher order $\mathcal{Y}_{n,\chi,\beta}^{(l)}(k,\, a,\, b)$ at negative integer values s. A relationship between $L_\beta^{(l)}(s,\, \chi;\, k,\, a,\, b)$ and $\zeta_\beta^{(l)}(s,\, x;\, k,\, a,\, b)$ is established and some examples are computed.

2. The Unified-Barnes Multiple Hurwitz Zeta Function

This section aims at providing a unified-Barnes multiple Hurwitz zeta function which interpolates the unified class of polynomials of higher order $\mathcal{Y}_{n,\beta}^{(l)}(x;\, k,\, a,\, b)$ $(l \in \mathbb{N})$ and the unified numbers at negative integers. We first begin by giving a multiplication formula for this unified class of polynomials.

Theorem 2.1. *Let* $\alpha, \beta \in \mathbb{C}$, $l, m \in \mathbb{N}$, $k \in \mathbb{N}_0$ *and* $x, a, b \in \mathbb{R}^+$. *Then the following formula holds*:

$$\mathcal{Y}_{n,\beta}^{(\alpha)}(x;\, k,\, a,\, b) = a^{\alpha b(m-1)} m^{n-k\alpha} \sum_{n_1, \dots, n_{m-1} \geq 0}^{\infty} \binom{\alpha}{n_1, \dots, n_{m-1}} \left(\frac{\beta}{a}\right)^{br}$$

$$\times \mathcal{Y}_{n,\beta^m}^{(\alpha)}\left(\frac{x+r}{m};\, k,\, a^m,\, b\right), \tag{2.1}$$

where $r = n_1 + 2n_2 + \cdots + (m-1)n_{m-1}$.

Proof. Using the generating function (1.1), we have

$$\sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}^{(\alpha)}(x;\, k,\, a,\, b)\frac{t^n}{n!}$$

$$= \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}$$

$$= \frac{(2^{1-k}t^k)^{\alpha}}{a^{\alpha b}\left(\left(\frac{\beta}{a}\right)^{bm} e^{mt} - 1\right)^{\alpha}} \left(\frac{\left(\frac{\beta}{a}\right)^{bm} e^{mt} - 1}{\left(\frac{\beta}{a}\right)^{b} e^{t} - 1}\right)^{\alpha} e^{xt}$$

$$= a^{\alpha b(m-1)} m^{-k\alpha} \left(\frac{2^{1-k}(mt)^k}{\beta^{bm}e^{mt} - a^{bm}}\right)^{\alpha} \left(\sum_{j=0}^{m-1} \left(\frac{\beta}{a}\right)^{bj} e^{jt}\right)^{\alpha} e^{xt}. \tag{2.2}$$

Let us recall the generalized multinomial theorem (see [5, p. 41]). If $x_1, x_2, \dots, x_r$ are commuting elements of a ring, then for $\alpha \in \mathbb{C}$, we have

$$(1 + x_1 + \cdots + x_m)^{\alpha} = \sum_{n_1, \dots, n_m \geq 0}^{\infty} \binom{\alpha}{n_1, \dots, n_m} x_1^{n_1} \cdots x_m^{n_m}, \tag{2.3}$$

where summation takes place over all integers $n_i \geq 0$, with

$$\binom{\alpha}{n_1, \ldots, n_m} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n_1-n_2-\cdots-n_m+1)}{n_1!n_2!\cdots n_m!}. \tag{2.4}$$

By applying this formula to (2.2), we have

$$\sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}^{(\alpha)}(x;\, k,\, a,\, b)\frac{t^n}{n!}\, a^{\alpha b(m-1)} m^{-k\alpha}\left(\frac{2^{1-k}(mt)^k}{\beta^{bm}e^{mt} - a^{bm}}\right)^{\alpha}$$

$$\times \sum_{n_1, \ldots, n_{m-1} \geq 0}^{\infty} \binom{\alpha}{n_1, \ldots, n_{m-1}}\left(\frac{\beta}{a}\right)^{br} e^{(x+r)t}$$

$$= \sum_{n=0}^{\infty} a^{\alpha b(m-1)} m^{n-k\alpha} \sum_{n_1, \ldots, n_{m-1} \geq 0}^{\infty} \binom{\alpha}{n_1, \ldots, n_{m-1}}\left(\frac{\beta}{a}\right)^{br}$$

$$\times \mathcal{Y}_{n,\beta^m}^{(\alpha)}\left(\frac{x+r}{m};\, k,\, a^m,\, b\right)\frac{t^n}{n!}, \tag{2.5}$$

where $r = n_1 + 2n_2 + \cdots + (m-1)n_{m-1}$. Equating the coefficients of $\frac{t^n}{n!}$, the result follows. □

Remark 2.2. Setting $\alpha = 1$ in (2.1), we get the result obtained by Ozden et al. [17], namely:

$$\mathcal{Y}_{n,\beta}(x;\, k,\, a,\, b) = a^{b(m-1)} m^{n-k} \sum_{l=0}^{m-1}\left(\frac{\beta}{a}\right)^{bl} \mathcal{Y}_{n,\beta^m}\left(\frac{x+l}{m};\, k,\, a^m,\, b\right). \tag{2.6}$$

Our objective is now to apply the Mellin transformation to the generating function (1.1) for the unified polynomials of higher order $\mathcal{Y}_{n,\beta}^{(\alpha)}(x;\, k,\, a,\, b)$ in order to construct the unified-Barnes multiple Hurwitz zeta function and, by the way, to interpolate $\mathcal{Y}_{n,\beta}^{(\alpha)}(x;\, k,\, a,\, b)$ for negative integer values of n and for $\alpha \in \mathbb{N}$.

Consider the following function:

$$f_{a,b}^{(l)}(x,\ t;\ k,\ \beta) = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^l e^{xt}$$

$$= \frac{2^{l(1-k)}t^{lk}}{a^{lb}} \underbrace{\left(\frac{1}{\left(\frac{\beta}{a}\right)^b e^t - 1}\right)\cdots\left(\frac{1}{\left(\frac{\beta}{a}\right)^b e^t - 1}\right)}_{l\text{-times}} e^{xt}$$

$$= \frac{(-1)^l\, 2^{l(1-k)}t^{lk}}{a^{lb}} \sum_{n_1,\, n_2,\, \ldots,\, n_l=0}^{\infty} \left(\frac{\beta^b}{a^b}\right)^{n_1+n_2+\cdots+n_l} e^{(n_1+\cdots+n_l+x)t}$$

$$= \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}^{(\alpha)}(x;\ k,\ a,\ b)\frac{t^n}{n!}. \tag{2.7}$$

By applying the Mellin transformation to (2.7), we find for $\operatorname{Re}(s) > lk$:

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-lk-1} f_{a,b}^{(l)}(x,\ -t;\ k,\ \beta)dt$$

$$= \frac{(-1)^{l(k+1)}2^{l(1-k)}}{a^{lb}} \sum_{n_1,\, n_2,\, \ldots,\, n_l=0}^{\infty} \left(\frac{\beta^b}{a^b}\right)^{n_1+n_2+\cdots+n_l}$$

$$\cdot \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(n_1+\cdots+n_l+x)t}dt$$

$$= \frac{(-1)^{l(k+1)}2^{l(1-k)}}{a^{lb}} \sum_{n_1,\, n_2,\, \ldots,\, n_l=0}^{\infty} \frac{\left(\frac{\beta^b}{a^b}\right)^{n_1+n_2+\cdots+n_l}}{(n_1+n_2+\cdots+n_l+x)^s}. \tag{2.8}$$

Note that the Mellin transformation of function of the type e^{-at} is given by (see [6])

$$\mathcal{M}\{e^{-at}\} := \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}dt = a^{-s} \quad (\operatorname{Re}(s) > 0,\ \operatorname{Re}(a) > 0). \tag{2.9}$$

We thus define the unification $\zeta_{\beta}^{(l)}(s, x; k, a, b)$ of the Barnes multiple Hurwitz zeta functions as follows.

Definition 2.3. For $s, \beta \in \mathbb{C}$, $|\beta| < 1$, $l \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $x, a, b \in \mathbb{R}^+$, we have for $\operatorname{Re}(s) > l$:

$$\zeta_{\beta}^{(l)}(s, x; k, a, b) = \frac{(-1)^{l(k+1)} 2^{l(1-k)}}{a^{lb}} \sum_{n_1, n_2, \ldots, n_l = 0}^{\infty} \frac{\left(\frac{\beta^b}{a^b}\right)^{n_1 + n_2 + \cdots + n_l}}{(n_1 + n_2 + \cdots + n_l + x)^s}, \tag{2.10}$$

where $\zeta_{\beta}^{(l)}(s, x; k, a, b)$ is called the *unified-Barnes multiple Hurwitz zeta function*.

We know (see [1, 2]) that the Barnes-type Hurwitz zeta functions can be continued meromorphically to the whole complex s-plane with simple poles at $s = 1, 2, \ldots, l$. The unified-Barnes multiple Hurwitz zeta function $\zeta_{\beta}^{(l)}(s, x; k, a, b)$ interpolates the polynomials $\mathcal{Y}_{n,\beta}^{(\alpha)}(x; k, a, b)$ for negative integer values of s. Substituting $s = -j$, $j \in \mathbb{N}_0$ in (2.10), we obtain the following theorem:

Theorem 2.4. *Let $j, k \in \mathbb{N}_0$ and $x, a, b \in \mathbb{R}^+$. Then*

$$\zeta_{\beta}^{(l)}(-j, x; k, a, b) = \frac{(-1)^{lk} j! \mathcal{Y}_{j+lk,\beta}^{(l)}(x; k, a, b)}{(j + lk)!}. \tag{2.11}$$

Proof. We know from (2.7) that

$$f_{a,b}^{(l)}(x, t; k, \beta)$$

$$= \sum_{j=0}^{\infty} \mathcal{Y}_{j,\beta}^{(l)}(x; k, a, b) \frac{t^j}{j!}$$

$$= \frac{(-1)^l 2^{l(1-k)} t^{lk}}{a^{lb}} \sum_{n_1, n_2, \ldots, n_l = 0}^{\infty} \left(\frac{\beta^b}{a^b}\right)^{n_1+n_2+\cdots+n_l} e^{(n_1+\cdots+n_l+x)t}$$

$$= \frac{(-1)^l 2^{l(1-k)} t^{lk}}{a^{lb}}$$

$$\cdot \sum_{j=0}^{\infty} \sum_{n_1, n_2, \ldots, n_l = 0}^{\infty} \left(\frac{\beta^b}{a^b}\right)^{n_1+n_2+\cdots+n_l} (n_1 + \cdots + n_l + x)^j \frac{t^{j+lk}}{j!}. \tag{2.12}$$

Replacing j by $j + lk$ in the left hand side of (2.12) and equating the coefficients on both side yield

$$\frac{j!\mathcal{Y}_{j+lk,\beta}^{(l)}(x; k, a, b)}{(j + lk)!}$$

$$= \frac{(-1)^l 2^{l(1-k)}}{a^{lb}} \sum_{n_1, n_2, \ldots, n_l = 0}^{\infty} \left(\frac{\beta^b}{a^b}\right)^{n_1+n_2+\cdots+n_l} (n_1 + \cdots + n_l + x)^j. \tag{2.13}$$

Finally, by substituting $s = -j,\ j \in \mathbb{N}_0$ in (2.10), the result follows. □

Remark 2.5. The proof of Theorem 2.4 could also be done by using the same method as Srivastava et al. [21], the Cauchy residue theorem and the equation (2.8).

Remark 2.6. If we set $l = 1$ in Theorem 2.4, then we rediscover a result given recently by Ozden et al. [17]:

$$\zeta_\beta(-j, x; k, a, b) = \frac{(-1)^k j!\mathcal{Y}_{j+k,\beta}(x; k, a, b)}{(j + k)!}. \tag{2.14}$$

Remark 2.7. Putting $k = 0,\ a = -1$ and $b = \beta = 1$ in Theorem 2.4 gives the Euler-Barnes multiple Hurwitz zeta function obtained by Kim [8],

$$\zeta_E^{(l)}(-j, x) = E_j^{(l)}(x). \tag{2.15}$$

Theorem 2.8. *For* $s \in \mathbb{C},\ \beta \in \mathbb{C}\ (|\beta| < 1),\ x, a, b \in \mathbb{R}^+,\ k \in \mathbb{N}_0$ *and*

$l,\ m \in \mathbb{N},$ *the following formula holds*:

$$\zeta_{\beta}^{(l)}(s,\ x;\ k,\ a,\ b)$$

$$= \frac{a^{lb(m-1)}}{m^s} \sum_{\substack{0 \le n_1, \ldots, n_{m-1} \le i \\ n_1 + \cdots + n_{m-1} = l}} \binom{l}{n_1,\ \ldots,\ n_{m-1}} \left(\frac{\beta}{a}\right)^{br} \zeta_{\beta^m}^{(l)}\left(s,\ \frac{x+r}{m};\ k,\ a^m,\ b\right), \tag{2.16}$$

where $r = n_1 + 2n_2 + \cdots + (m-1)n_{m-1}.$

Proof. We have seen in the proof of Theorem 2.1 that we can write the unified function $f_{a,b}^{(l)}(x,\ t;\ k,\ \beta)$ in the following way

$$f_{a,b}^{(l)}(x,\ t;\ k,\ \beta)$$

$$= \frac{(2^{1-k}t^k)^l}{a^{bl}\left(\left(\frac{\beta}{a}\right)^{bm} e^{mt} - 1\right)^l} \left(\sum_{j=0}^{m-1} \left(\frac{\beta}{a}\right)^{bj} e^{jt}\right)^l e^{xt}$$

$$= \frac{a^{-bl}\, 2^{l(1-k)} t^{lk}}{\left(\left(\frac{\beta}{a}\right)^{bm} e^{mt} - 1\right)^l} \sum_{\substack{0 \le n_1, \ldots, n_{m-1} \le l \\ n_1 + \cdots + n_{m-1} = l}} \binom{l}{n_1,\ \ldots,\ n_{m-1}} \left(\frac{\beta}{a}\right)^{br} e^{(x+r)t}$$

$$= \frac{(-1)^l a^{lb(m-1)} 2^{l(1-k)}}{a^{lbm}} \sum_{\substack{0 \le n_1, \ldots, n_{m-1} \le l \\ n_1 + \cdots + n_{m-1} = l}} \binom{l}{n_1,\ \ldots,\ n_{m-1}} \left(\frac{\beta}{a}\right)^{br}$$

$$\times \sum_{s_1, \ldots, s_l \ge 0}^{\infty} \left(\frac{\beta^{bm}}{a^{bm}}\right)^{s_1 + s_2 + \cdots + s_l} e^{\left(s_1 + \cdots + s_l + \frac{x+r}{m}\right)mt} t^{lk}. \tag{2.17}$$

Applying the Mellin transformation on both sides of (2.17) as done in (2.8) gives the result. □

We end this section by computing some special cases of this last theorem. These special cases are given in the next corollaries.

If we set $a = b = \beta = k = 1$ in Theorem 2.8, we obtain a relation given by Choi [4] involving a particular Barnes-type Hurwitz zeta function called the *Bernoulli Barnes Hurwitz zeta function* (since $\mathcal{Y}_{n,1}^{(l)}(x; 1, 1, 1) = B_n^{(l)}(x)$).

Corollary 2.9. *For* $s \in \mathbb{C},\ x \in \mathbb{R}^+$ *and* $l,\ m \in \mathbb{N}$, *the following formula holds*

$$\zeta_1^{(l)}(s,\ x; 1, 1, 1) = \frac{1}{m^s} \sum_{\substack{0 \le n_1, \ldots, n_{m-1} \le l \\ n_1 + \cdots + n_{m-1} = l}} \binom{l}{n_1,\ \ldots,\ n_{m-1}} \zeta_1^{(l)}\left(s,\ \frac{x+r}{m}; 1, 1, 1\right), \tag{2.18}$$

where $r = n_1 + 2n_2 + \cdots + (m-1)n_{m-1}$.

In addition, substituting $l = 1$ in (2.18) provides the following Bernoulli Hurwitz zeta function:

Corollary 2.10. *Let* $s \in \mathbb{C},\ x \in \mathbb{R}^+$ *and* $m \in \mathbb{N}$. *Then we have*

$$\zeta_1(s,\ x; 1, 1, 1) = \frac{1}{m^s} \sum_{j=0}^{m-1} \zeta_1\left(s,\ \frac{x+j}{m}; 1, 1, 1\right). \tag{2.19}$$

It is easy to see that

$$\zeta_1(s,\ x; 1, 1, 1) = -\sum_{n=0}^{\infty} \frac{1}{(n+x)^s} = -\zeta(s,\ x), \tag{2.20}$$

where $\zeta(s,\ x)$ is the well-known classical Hurwitz zeta function. Thus, we can rewrite (2.19) as

$$\zeta(s,\ x) = \frac{1}{m^s} \sum_{j=0}^{m-1} \zeta\left(s,\ \frac{x+j}{m}\right) = \frac{1}{m^s} \sum_{j=1}^{m} \zeta\left(s,\ \frac{x+j-1}{m}\right) \tag{2.21}$$

which is a classical result for the Hurwitz zeta function (see [19]).

3. Dirichlet-type Multiple *L*-function

In this section, we define the Dirichlet-type multiple *L*-function $L_{\beta}^{(l)}(s, \chi; k, a, b)$ which function interpolates the numbers $\mathcal{Y}_{r, \chi, \beta}^{(l)}(k, a, b)$ for negative integer values of s. We exhibit a relation between $L_{\beta}^{(l)}(s, \chi; k, a, b)$ and $\zeta_{\beta}^{(l)}(s, x; k, a, b)$. We also give some special cases.

Let us recall Definition 1.2 for the χ-extended unified numbers of higher order

$$\mathcal{Y}_{n, \chi, \beta}^{(l)}(0, k, a, b) = \mathcal{Y}_{n, \chi, \beta}^{(l)}(k, a, b).$$

Let χ be a primitive Dirichlet character of conductor $f \in \mathbb{N}$. Let $k \in \mathbb{N}_0$, $a, b \in \mathbb{R}^+$, $l \in \mathbb{N}$ and $\beta \in \mathbb{C}$. Then

$$f_{a, b}^{(l)}(\chi, t; k, \beta) = \left(2^{1-k} t^k \sum_{j=0}^{f-1} \frac{\chi(j)\left(\frac{\beta}{a}\right)^{bj} e^{jt}}{\beta^{bf} e^{ft} - a^{bf}} \right)^l = \sum_{r=0}^{\infty} \mathcal{Y}_{r, \chi, \beta}^{(l)}(k, a, b) \frac{t^r}{r!}. \tag{3.1}$$

Let us now rewrite the function $f_{a, b}^{(l)}(\chi, t; k, \beta)$ in the following way:

$$\begin{aligned}
&f_{a, b}^{(l)}(\chi, t; k, \beta) \\
&= \left(2^{1-k} t^k \sum_{j=0}^{f-1} \frac{\chi(j)\left(\frac{\beta}{a}\right)^{bj} e^{jt}}{\beta^{bf} e^{ft} - a^{bf}} \right)^l \\
&= \frac{2^{(1-k)l} t^{kl}}{a^{bfl}} \sum_{n_1, \dots, n_l = 0}^{f-1} \frac{\left(\frac{\beta^b}{a^b}\right)^{n_1 + \cdots + n_l} e^{(n_1 + \cdots + n_l)t} \prod_{i=1}^{l} \chi(n_i)}{\left(\left(\frac{\beta^b}{a^b}\right)^f e^{ft} - 1\right)^l}
\end{aligned}$$

$$= \frac{(-1)^l 2^{(1-k)l} t^{kl}}{a^{bfl}} \sum_{n_1, \ldots, n_l = 0}^{f-1} \left(\frac{\beta^b}{a^b}\right)^{n_1 + \cdots + n_l} e^{(n_1 + \cdots + n_l)t} \prod_{i=1}^{l} \chi(n_i)$$

$$\times \sum_{x_1, \ldots, x_l = 0}^{\infty} \left(\frac{\beta^b}{a^b}\right)^{(x_1 + \cdots + x_l) f} e^{(x_1 + \cdots + x_l) ft}$$

$$= \frac{(-1)^l 2^{(1-k)l} t^{kl}}{a^{bfl}} \sum_{x_1, \ldots, x_l = 0}^{\infty} \sum_{n_1, \ldots, n_l = 0}^{f-1} \left(\frac{\beta^b}{a^b}\right)^{\sum_{i=1}^{l} n_i + x_i f}$$

$$\times \prod_{i=1}^{l} \chi(n_i + x_i f) e^{t \sum_{i=1}^{l} n_i + x_i f}$$

$$= \frac{(-1)^l 2^{(1-k)l} t^{kl}}{a^{bfl}} \sum_{n_1, \ldots, n_l = 0}^{\infty} \left(\frac{\beta^b}{a^b}\right)^{n_1 + \cdots + n_l} \prod_{i=1}^{l} \chi(n_i) e^{(n_1 + \cdots + n_l)t}. \qquad (3.2)$$

Since $\chi(0) = 0$, we can start at $n_1 = 1, n_2 = 1, \ldots, n_l = 1$. Thus, we have

$$f_{a,b}^{(l)}(\chi, t; k, \beta)$$

$$= \frac{(-1)^l 2^{(1-k)l} t^{kl}}{a^{bfl}} \sum_{n_1, \ldots, n_l = 1}^{\infty} \left(\frac{\beta^b}{a^b}\right)^{n_1 + \cdots + n_l} \prod_{i=1}^{l} \chi(n_i) e^{(n_1 + \cdots + n_l)t}$$

$$= \sum_{r=0}^{\infty} \mathcal{Y}_{r,\chi,\beta}^{(l)}(k, a, b) \frac{t^r}{r!}. \qquad (3.3)$$

From (3.3), we can easily derive the next relation by equating the coefficient of $\frac{t^r}{r!}$ on both sides of the previous relation after having expand $e^{(n_1 + \cdots + n_l)t}$ in power series. Hence, we obtain

$$\mathcal{Y}_{r+kl,\chi,\beta}^{(l)}(k, a, b)$$

$$= \frac{(-1)^l (r + kl)!}{2^{(k-1)l} a^{bfl} r!} \sum_{n_1, \ldots, n_l = 1}^{\infty} \left(\frac{\beta^b}{a^b}\right)^{n_1 + \cdots + n_l} \prod_{i=1}^{l} \chi(n_i) (n_1 + \cdots + n_l)^j. \qquad (3.4)$$

Now, by applying the Mellin transformation to (3.3), we have

$$\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-lk-1} f_{a,b}^{(l)}(\chi, -t; k, \beta)dt$$

$$= \frac{(-1)^{l(k+1)}2^{(1-k)l}}{a^{bfl}} \sum_{n_1, \dots, n_l=1}^{\infty} \left(\frac{\beta^b}{a^b}\right)^{n_1+\cdots+n_l}$$

$$\cdot \prod_{i=1}^{l} \chi(n_i) \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(n_1+\cdots+n_l)t} dt$$

$$= \frac{(-1)^{l(k+1)}2^{(1-k)l}}{a^{bfl}} \sum_{n_1, \dots, n_l=1}^{\infty} \left(\frac{\beta^b}{a^b}\right)^{n_1+\cdots+n_l} \prod_{i=1}^{l} \chi(n_i)(n_1+\cdots+n_l)^{-s}. \quad (3.5)$$

We define the Dirichlet-type multiple unified L-function as follows:

Definition 3.1. Let χ be a primitive Dirichlet character of conductor $f \in \mathbb{N}$. Let $s \in \mathbb{C}$, $k \in \mathbb{N}_0$, $a, b \in \mathbb{R}^+$, $l \in \mathbb{N}$ and $\beta \in \mathbb{C}$. Then

$$L_{\beta}^{(l)}(s, \chi; k, a, b) = \frac{(-1)^{l(k+1)}2^{(1-k)l}}{a^{bfl}} \sum_{n_1, \dots n_l=1}^{\infty} \frac{\left(\frac{\beta^b}{a^b}\right)^{n_1+\cdots+n_l} \prod_{i=1}^{l} \chi(n_i)}{(n_1+\cdots+n_l)^s}. \quad (3.6)$$

Remark 3.2. The function $L_{\beta}^{(l)}(s, \chi; k, a, b)$ is an analytic function in the whole complex s-plane.

The Dirichlet-type multiple unified L-function $L_{\beta}^{(l)}(s, \chi; k, a, b)$ interpolates the numbers $\mathcal{Y}_{r,\chi,\beta}^{(l)}(k, a, b)$ for negative integer values of s. Substituting $s = -r$, $r \in \mathbb{N}_0$ in (3.6), we obtain the following theorem:

Theorem 3.3. *Let χ be a primitive Dirichlet character of conductor $f \in \mathbb{N}$. Let $r, k \in \mathbb{N}_0$, $a, b \in \mathbb{R}^+$, $l \in \mathbb{N}$ and $\beta \in \mathbb{C}$. Then*

$$L_{\beta}^{(l)}(-r, \chi; k, a, b) = \frac{(-1)^{lk} r!}{(r+lk)!} \mathcal{Y}_{r+kl,\chi,\beta}^{(l)}(k, a, b). \quad (3.7)$$

Proof. By comparing (3.4) and (3.6), the result follows easily. □

Remark 3.4. Setting $l = 1$, $k = 0$, $a = -1$, $b = \beta = 1$ in Theorem 3.3 and let f be an odd conductor of χ, we find

$$L_1^{(1)}(-r, \chi; 0, -1, 1) = E_{r,\chi}^{(l)}, \tag{3.8}$$

where $E_{r,\chi}^{(l)}$ are the classical χ-extended Euler numbers of higher order [20, 22].

Theorem 3.5. *Let χ be a primitive Dirichlet character of conductor $f \in \mathbb{N}$. Let $r, k \in \mathbb{N}_0$, $a, b \in \mathbb{R}^+$, $l \in \mathbb{N}$ and $\beta \in \mathbb{C}$. Then*

$$L_\beta^{(l)}(s, \chi; k, a, b) = \frac{1}{f^s} \sum_{a_1, \ldots, a_l = 1}^{f-1} \left(\frac{\beta^a}{a^b}\right)^{a_1 + \cdots + a_l}$$
$$\times \prod_{i=1}^{l} \chi(a_i) \zeta_{\beta^f}^{(l)}\left(s, \frac{a_1 + \cdots + a_l}{f}; k, a^f, b\right). \tag{3.9}$$

Proof. Substituting $n_j = a_j + m_j f$, where $m_j = 0, 1, 2, \ldots, \infty$, $a_j = 1, 2, \ldots, f$, where $\chi(a_j + m_j f) = \chi(a_j)$ and f is conductor of χ, $1 \le j \le l$ into (3.6), we have

$$L_\beta^{(l)}(s, \chi; k, a, b)$$
$$= \frac{(-1)^{l(k+1)} 2^{(1-k)l}}{a^{bfl}} \sum_{a_1, \ldots, a_l = 1}^{f-1} \sum_{m_1, \ldots, m_l = 0}^{\infty} \frac{\left(\frac{\beta^b}{a^b}\right)^{a_1 + m_1 f + \cdots + a_l + m_l f}}{(a_1 + m_1 f + \cdots + a_l + m_l f)^s}$$
$$\times \prod_{i=1}^{l} \chi(a_i + m_i f)$$
$$= \frac{(-1)^{l(k+1)} 2^{(1-k)l} f^{-s}}{a^{bfl}} \sum_{a_1, \ldots, a_l = 1}^{f-1} \left(\frac{\beta^b}{a^b}\right)^{a_1 + \cdots + a_l} \prod_{i=1}^{l} \chi(a_i)$$

$$\times \sum_{m_1, \ldots, m_l = 0}^{\infty} \frac{\left(\frac{\beta^{bf}}{a^{bf}}\right)^{m_1 + \cdots + m_l}}{\left(m_1 + \cdots + m_l + \frac{a_1 + \cdots + a_l}{f}\right)^s}. \tag{3.10}$$

Making use of (2.10), we arrive to (3.9). □

Replacing $s = -r,\ r \in \mathbb{N}_0$ in (3.9) and with the help of (2.11) and (3.7), we can also deduce the following theorem:

Theorem 3.6. *Let* χ *be a primitive Dirichlet character of conductor* $f \in \mathbb{N}$. *Let* $r, k \in \mathbb{N}_0,\ a, b \in \mathbb{R}^+,\ l \in \mathbb{N}$ *and* $\beta \in \mathbb{C}$. *Then*

$$\mathcal{Y}^{(l)}_{r+kl, \chi, \beta}(k, a, b) = f^r \sum_{a_1, \ldots, a_l = 1}^{f-1} \left(\frac{\beta^b}{a^b}\right)^{a_1 + \cdots + a_l}$$

$$\times \prod_{i=1}^{l} \chi(a_i) \mathcal{Y}^{(l)}_{r+lk, \beta^f}\left(\frac{a_1 + \cdots + a_l}{f}; k, a^f, b\right). \tag{3.11}$$

Remark 3.7. If we let $a = b = \beta = k = 1$ in Theorem 3.6, then we rediscover a result given by Cenkci et al. [3] for the χ-extended Bernoulli numbers of higher order, namely:

$$\mathcal{B}^{(l)}_{r+l, \chi} = f^r \sum_{a_1, \ldots, a_l = 1}^{f-1} \chi(a_i + \cdots + a_l) \mathcal{B}^{(l)}_{r+l}\left(\frac{a_1 + \cdots + a_l}{f}\right). \tag{3.12}$$

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