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ON THE LEFSCHETZ DIRECT STABILITY CRITERION FOR AN IMPLICIT EVOLUTION PROBLEM, WITH A DYNAMIC BOUNDARY CONDITION

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Abstract

We consider a nonlinear dynamic problem comprising a system of nonlinear parabolic equations. The second of these equations is a nonlinear dynamic boundary condition to the problem. The derivation of the system is through the heat energy conservation laws [4]. Problems of this type occur in heat energy absorption and release through the surfaces of solids. The second equation to the system describes surface radiation itself. We rewrite the system as an implicit evolution equation, thus exposing trace-like canonical operators. These operators have been studied and characterized in [3] and [4]. Subsequent to that, we study the stability of the null solution to the implicit evolution problem using the modified Lefschetz [6] system for the direct stability criterion. We show that, even though the modified Lefschetz system leads to a new Lyapunov function for the problem, the Lefschetz direct stability criterion itself is invariant. We test the

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modified Lefschetz system on the cooling problem in [3], by constructing the corresponding Lyapunov function and confirming its known properties.

Symbols used:

1. α, β, α_0 and β_0 are positive real constants;

2. Ω is an open bounded domain in $\Re^3$, where $\partial\Omega \not\subset \Omega$;

3. $\Delta := \nabla \cdot \nabla$;

4. $\Delta_s = \nabla_s \cdot \nabla_s$; the Beltrammi-Laplace operator; with

$$\nabla_s := \frac{\partial}{\partial s_1}\tau_1 + \frac{\partial}{\partial s_2}\tau_2 \text{ for an arbitrary point } (\tau_1, \tau_2) \text{ on } \partial\Omega.$$

1. Introduction

We wish to investigate the application of the Lefschetz direct stability criterion to an implicit evolution equation. For that purpose, we consider the following nonlinear dynamic problem:

$$\begin{cases} \text{We look for } u(x, t) \text{ such that} \\ \text{(a) } \alpha\partial_t u(x, t) = \beta\Delta u(x, t) + f(u(x, t));\ x \in \Omega,\ u(x, t) \in L^2((0, t), H^2(\Omega)) \\ \text{Subject to:} \\ \text{(b) } u(x, 0) = u^0(x), \\ \text{(c) } \gamma_1 u = 0, \text{ and,} \\ \text{(d) } \alpha_0\partial_t[\gamma_0 u(y, t)] = \beta_0\Delta_s[\gamma_0 u(y, t)] + \gamma_0 f([\gamma_0 u(y, t)]), \\ \qquad\qquad \gamma_0 u(y, t) \in L^2\left((0, t), H^{\frac{3}{2}}(\partial\Omega)\right),\ y \in \partial\Omega. \end{cases} \tag{1}$$

2. Some Basic Assumptions on the Problem

In this paper, we assume that

(a) the surface $\partial\Omega$ is smooth enough for the trace operator, $\gamma : H^2(\Omega) \to H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, to exist so that we can define its restriction, $\gamma_0 : H^2(\Omega) \to H^{\frac{3}{2}}(\partial\Omega)$ by $u \mapsto \gamma_0 u$; mapping the solution to 1(a) to the solution to 1(d) (see [5, Theorem 8.3, p. 39]);

(b) the weak solution $\langle u(x, t), \gamma_0 u(y, t)\rangle$ to (1) exists and is unique in

$$\Theta = \left\{\langle u(x, t), \gamma_0 u(x, t)\rangle \in L^2\left((0, t), H^2(\Omega) \times H^{\frac{3}{2}}(\partial\Omega)\right)\right.$$

$$\left. : \gamma_1 u = 0; u(0) = u^0(x)\right\}.$$

3. The Problem as an Implicit Evolution Equation

We rewrite the problem (1) in the form,

$$\partial_t Bu(t) = Lu(t) + Fu(t), \tag{2}$$

$$\text{Subject to: } Bu(0) = Bu^0(x),$$

$$\gamma_1 u = 0,$$

where

$$Bu(t) = \langle \alpha u(t), \alpha_0[\gamma_0 u(t)]\rangle \in Y;$$

$$Lu(t) = \langle \beta\Delta u(t), \beta_0\Delta_s[\gamma_0 u(t)]\rangle \in Y;$$

$$Fu(t) = \langle f(u(t)), \gamma_0 f[\gamma_0 u(t)]\rangle \in Y;$$

$$Y := L^2\left((0, t), H^2(\Omega) \times H^{\frac{3}{2}}(\partial\Omega)\right).$$

Remark 3.1. (a) While the operators B and L are linear, the operator F is nonlinear.

(b) All the canonical operators are “related” to the trace operator through the trace extension operator, $\gamma_0 : H^2(\Omega) \to H^{\frac{3}{2}}(\partial\Omega)$.

(c) It can be shown that the space $H^2(\Omega) \times H^{\frac{3}{2}}(\partial\Omega)$ is a finite dimensional Hilbert space.

(d) The trace theorem ([5, Theorem 8.3, p. 39]) ensures the existence of a surjection, $\chi : L^2([0, T), H^2(\Omega)) \to L^2\left([0, T), H^2(\Omega) \times H^{\frac{3}{2}}(\partial\Omega)\right)$, so that the solution to (2) is also the solution to (1).

(e) The eigenvalue of the linear operator L is $\lambda = -1$ (see [4, p. 32]).

4. The Lefschetz System for the Direct Stability Criterion

We present the following Lefschetz system (see [6]) on which the direct stability criterion would be based:

$$x^{/}(t) = Ax(t) + b\phi(\alpha(t)), \tag{3}$$

where

$\sigma(t) := c^T x(t);$ with the corresponding Lyapunov function:

$$V(x(t)) := (x(t))^T B(x(t)) + \int_0^{\sigma(t)} \phi(\sigma(t)) d\sigma(t).$$

The basic assumptions associated with this system are:

(a) x, b, c are real n vectors; with $n = 2$ in our case;

(b) A is a real $n \times n$ matrix with eigenvalues whose real parts are negative;

(c) B is a positive definite symmetric matrix satisfying condition to be specified later;

(d) ϕ is a continuous function on σ;

(e) $\sigma(0) = 0;$ for $\sigma \neq 0,$ $\sigma\phi(\sigma) > 0.$

To conclude the direct stability criterion, Aizerman and Gantmacher [1], for $\alpha > 0,$ subtracted and added $\alpha\sigma\phi(\sigma) > 0$ from $-V^{/}$ thus, obtaining,

$$-c^T b > \left(Bb + \frac{1}{2}A^T c + \frac{1}{2}\alpha c\right)^T C^{-1}\left(Bb + \frac{1}{2}A^T c + \frac{1}{2}\alpha c\right); \qquad (4)$$

where $-C = A^T B + BA$ is an arbitrary $n \times n$ positive definite symmetric matrix calculated from the relation: $B = -\int_0^{\infty} e^{At} C e^{At} dt.$

In the next section, we define a mathematical transformation to modify (3) for the problem (2). In the process, we also generate the corresponding Lyapunov function.

5. Transformation of the Lefschetz System for the Implicit Evolution Equation

For $u(t) \in \Theta,$ we put:

$$\partial_t Bu(t) = A(Bu(t)) + bF(\sigma(t)), \qquad (5)$$

where

(a) $\sigma(t) = c_1 u(t), \gamma_0\sigma(t) = c_2\gamma_0 u(t);$ where $\gamma_0 : \sigma(t) \to \gamma_0\sigma(t);$

(b) $Bu(t) = \langle \alpha u(t), \alpha_0\gamma_0 u(t)\rangle;$

(c) $B^*\sigma(t) = Bu(t) \cdot c^T;$ $c = \langle c_1, c_2\rangle;$ $b = \langle b_1, b_2\rangle;$ $Bu(0) = Bu^0(x, 0);$ $B^*\sigma(0) = Bu^0 \cdot c^T;$

(d) $F(\sigma(t)) := \begin{pmatrix} f(\sigma(t)) & 0 \\ 0 & \gamma_0 f(\gamma_0\sigma(t)) \end{pmatrix},$

(e) $A := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$ $\lambda_1 = \lambda_2 = -1,$ the eigenvalues of the operator L.

The corresponding Lyapunov function is given by

$$V(u(t)) = (Bu(t))^T D(Bu(t)) + \int_{B^*\sigma(0)}^{B^*\sigma} F(\sigma(t)) dB^*\sigma. \qquad (6)$$

Remark 5.1. (a) We observe that $\partial_t B^* \sigma = c^T \partial_t Bu.$

(b) Choosing $c_1, c_2 > 0$ for $f(\sigma), \gamma_0 f(\gamma_0 \sigma) > 0,$ we have

$$B^* \sigma F(\sigma) = c^T F(\sigma)(Bu)^T = \alpha c_1 f(\sigma) u + \alpha_0 c_2 \gamma_0 f(\gamma_0 \sigma) \gamma_0 u > 0;$$

similar to condition 3(c) in the Lefschetz system (3).

(c) The flexibility in the choice of $b = \langle b_1, b_2 \rangle$ will be illustrated when the criterion derived through (4) is applied to a cooling problem.

(d) $A := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix};$ $\lambda_1 = \lambda_2 = -1$ is the corresponding matrix for the linear operator L under a different basis.

(e) As in the Lefschetz system (3), matrix D is symmetric and positive definite and can be calculated from the relation: $D = -\int_0^\infty e^{At} C e^{At} dt.$

6. Confirmation of the Direct Stability Criterion for the Implicit Evolution Problem

Main Theorem 6.1. *For* $u(t) \in \Theta,$ *and under the system* (5) *for an implicit evolution problem, the Lefschetz direct stability criterion* (4) *is invariant to the transformation* (5).

Proof. We differentiate (6) to obtain

$$V^/(u) = 2BuD\partial_t Bu + F(B\sigma)\partial_t B^* \sigma. \tag{7}$$

However, by Remark 5.1(a),

$$\partial_t B^* \sigma(t) = c\partial_t Bu(t).$$

Therefore,

$$V^/(u) = [2BuD + cF(\sigma)]\partial_t Bu$$

$$= [2BuD + cF(\sigma)][A(Bu) + bF(\sigma)], \text{ by (4);}$$

$$= 2BuDA(Bu) + F(\sigma)Ac(Bu) + 2BuDbF(\sigma) + c[F(\sigma)]^2$$

$$= (Bu)^T A^T D(Bu) + (Bu)^T DA(Bu) + F(\sigma)AcF(\sigma)(Bu)$$

$$+ 2BuDbF(\sigma) + cb[F(\sigma)]^2$$

$$= (Bu)^T [A^T D + DA](Bu) + F(\sigma)Ac(Bu) + 2BuDbF(\sigma) + cb[F(\sigma)]^2.$$

Hence,

$$-V^{/}(u) = (Bu)^T(-C)(Bu) - F(\sigma)AcF(\sigma)(Bu) - 2BuDF(\sigma) - cb[F(\sigma)]^2, \tag{8}$$

where $-C = A^T D + DA$ is a positive definite matrix.

Rewriting (7), we have

$$-V^{/}(u) = (Bu)^T(-C)(Bu) - 2\left(Db + \frac{1}{2}Ac\right)BuF(\sigma) - cb[F(\sigma)]^2, \tag{9}$$

where $-C = A^T D + DA$ is a positive definite matrix.

We add and subtract $\lambda B\sigma(t)F(\sigma(t))$ on (9) (in the sense of Aizerman and Gantmacher; see [6, p. 436]), to obtain

$$-V^{/}(u) = (Bu)^T(-C)(Bu) - 2\left(Db + \frac{1}{2}Ac + \frac{1}{2}\lambda c\right)BuF(\sigma)$$

$$- cb[F(\sigma)]^2 + \lambda B\sigma(t)F(\sigma(t)). \tag{10}$$

For the direct stability criterion, we demand that

$$-V(u(t)) > 0, \quad \text{for} \quad u(t) \neq 0.$$

Since $\lambda B^*\sigma(t)F(\sigma(t)) > 0$ (by Remark 5.1(b)), this implies that

$$(Bu)^T(-C)(Bu) - 2\left(Db + \frac{1}{2}Ac + \frac{1}{2}\lambda c\right)BuF(\sigma) - cb[F(\sigma)]^2 > 0,$$

a quadratic inequality in $F(\sigma)$. For this inequality, the discriminant takes the form:

$$\left[-2\left(Db + \frac{1}{2}Ac + \frac{1}{2}\lambda c\right)Bu\right]^2 - 4(-cb)(Bu)^T(-C)(Bu) < 0,$$

which implies that

$$\left(Db + \frac{1}{2}Ac + \frac{1}{2}\lambda\right)^2 - cb(-C) < 0,$$

that is,

$$-cb > \left(Db + \frac{1}{2}Ac + \frac{1}{2}\lambda c\right)^T C^{-1}\left(Db + \frac{1}{2}Ac + \frac{1}{2}\lambda c\right), \text{ similar to (4).}$$

Thus, the Lefschetz direct stability criterion (4) remains invariant under the transformation (5). □

7. On the Lyapunov Function for a Surface Radiative Cooling Problem

We consider a cooling problem, proposed in [3]. The mathematical model for the problem is:

Find: $u(x, t) \in L^2([0, T), H^2(\Omega))$, such that

$$\begin{cases} c_h \rho \partial_t u(x, t) = \kappa \Delta u(x, t) - \eta(u(x, t) - u_e)^m; \; 1 \le m \le 3, \; x \in \Omega, \\ \text{Subject to}: u(x, 0) = u^0(x), \\ \gamma_0 u = 0, \text{ and} \\ c_h \rho \partial_t [\gamma_0 u(y, t)] = \kappa \Delta_s [\gamma_0 u(y, t)] - k(\gamma_0 u(y, t) - u_e)^m; \; y \in \partial\Omega, \end{cases} \tag{11}$$

with $k > 0$ as the Stefan-Boltzmann constant and $\eta > 0$. The two constants are comparable in magnitude $(\eta \approx k \approx 10^{-8})$.

As an implicit evolution equation, the problem is:

$$\partial_t Bu(t) = Lu(t) + N(u(t)), \tag{12}$$

where

$$Bu(t) = \langle c_h \rho u(t),\ c_h \rho[\gamma_0 u(t)]\rangle;$$

$$Lu(t) = \langle \kappa\Delta u(t),\ \kappa\Delta_s[\gamma_0 u(t)]\rangle;$$

$$N(u(t)) = \langle -\eta(u(t) - u_e)^m,\ -k(\gamma_0 u(t) - u_e)^m\rangle;\ 1 \le m \le 3.$$

We construct (5) for (12) as follows:

(a) $\sigma(t) = c_1 u(t),\ \gamma_0\sigma(t) = c_2\gamma_0 u(t);$ where $\gamma_0 : \sigma(t) \to \gamma_0\sigma(t)$.

(b) $B^*\sigma(t) = Bu(t)\cdot c^T$.

(c) Put

$$b^T = \begin{pmatrix} -1 \\ -1 \end{pmatrix};\quad f(\sigma(t)) = \eta|\,\sigma(t) - u_e\,|^m;\quad \gamma_0 f(\gamma_0\sigma(t)) = k|\,\gamma_0\sigma(t) - u_e\,|^m.$$

(d) Then, $F(\sigma(t)) = \begin{pmatrix} \eta|\,\sigma(t) - u_e\,|^m & 0 \\ 0 & k|\,\gamma_0\sigma(t) - u_e\,|^m \end{pmatrix}.$

(e) For this problem, we choose: $c_1, c_2 \ge 1$, with the following consequences:

(i) By Remark 5.1(b), $B^*\sigma F(\sigma) > 0$; a requirement by Aizerman and Gantmacher [1].

(ii)

$$\begin{pmatrix} \eta|\,\sigma(t) - u_e\,|^m & 0 \\ 0 & k|\,\gamma_0\sigma(t) - u_e\,|^m \end{pmatrix} = \begin{pmatrix} \eta|\,c_1 u(t) - u_e\,|^m & 0 \\ 0 & k|\,c_2\gamma_0 u(t) - u_e\,|^m \end{pmatrix}.$$

(f) For the current problem, we still choose:

$$A := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix};\quad \lambda_1 = \lambda_2 = -1.$$

(g) By Remark 5.1(e), $D = -\int_0^{\infty} e^{At} C e^{At} dt$. We arbitrarily put $-C = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$; $r, s > 0$ so that it is symmetric and positive definite. We then obtain $D = \frac{1}{2}\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$.

Thus, the Lyapunov function for the current problem is given by

$$V(u(t)) = \langle c_h \rho u(t), c_h \rho \gamma_0 u(t) \rangle \frac{1}{2}\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \cdot \begin{pmatrix} c_h \rho u(t) \\ c_h \rho \gamma_0 u(t) \end{pmatrix}$$

$$+ \int_{B_* \sigma(0)}^{B^* \sigma(t)} \begin{pmatrix} \eta | \sigma(t) - u_e |^m & 0 \\ 0 & k| \gamma_0 \sigma(t) - u_e |^m \end{pmatrix} dB^* \sigma. \qquad (13)$$

We have

$$\int_{B^* \sigma(0)}^{B^* \sigma(t)} \begin{pmatrix} \eta | \sigma(t) - u_e |^m & 0 \\ 0 & k| \gamma_0 \sigma(t) - u_e |^m \end{pmatrix} dB^* \sigma$$

$$= \int_{B^* \sigma(0)}^{B^* \sigma(t)} \begin{pmatrix} \eta | c_1 u(t) - u_e |^m & 0 \\ 0 & k| c_2 \gamma_0 u(t) - u_e |^m \end{pmatrix} dB^* \sigma$$

$$= \int_{B^* \sigma(0)}^{B^* \sigma(t)} \begin{pmatrix} \eta | c_1 u(t) - u_e |^m & 0 \\ 0 & k| c_2 \gamma_0 u(t) - u_e |^m \end{pmatrix} dB \cdot c^T$$

$$= \int_{c_1 u^0}^{c_1 u} c_h \rho \eta c_1 | c_1 u(t) - u_e |^m du + \int_{c_2 \gamma_0 u^0}^{c_2 \gamma_0 u} k c_h \rho c_2 | c_2 \gamma_0 u(t) - u_e |^m d\gamma_0 u$$

$$= \frac{1}{m+1}[\eta c_h \rho | (c_1 u(t) - u_e)^{m+1} - (c_1 u^0 - u_e)^{m+1} |]$$

$$+ \frac{1}{m+1}[k c_h \rho | (c_2 \gamma_0 u(t) - u_e)^{m+1} - (c_2 \gamma_0 u^0 - u_e)^{m+1} |],$$

since $c_1 u(t) - u_e,\ c_1\gamma_0 u(t) - u \geq 0,$ for the cooling problem

$$(|\, c_1 u(t) - u_e\,| = c_1 u(t) - u_e;\ |\, c_1\gamma_0 u(t) - u\,| = c_1\gamma_0 u(t) - u),$$

$$V(u(t)) = \frac{1}{2}[c_h^2\rho^2 r(u(t))^2 + c_h^2\rho^2 s(\gamma_0 u(t))^2]$$

$$+ \frac{1}{m+1}[\eta c_h\rho|(c_1 u(t) - u_e)^{m+1}(c_1 u^0 - u_e)^{m+1}|]$$

$$+ \frac{1}{m+1}[k c_h\rho|(c_2\gamma_0 u(t) - u_e)^{m+1} - (c_2\gamma_0 u^0 - u_e)^{m+1}|]. \quad (14)$$

Remark 7.1. (a) From (14), we have

$$E(t) = \frac{1}{2}[c_h^2\rho^2 r(u(t))^2 + c_h^2\rho^2 s(\gamma_0 u(t))^2],$$

the energy for the cooling model.

(b) For $u(t) = 0,\ V(0) = 0,$ and for $u(t) \neq 0,\ V(u(t)) > 0.$

By [2, Definition 5-2-1, p. 71], V is positive definite in Θ.

(c)

$$\frac{dE(t)}{dt} = -\kappa c_h\rho\| c_1\nabla u(t)\|^2_{L^2(\Omega)} - \kappa c_h\rho\| c_2\nabla_s(\gamma_0 u(t))\|^2_{H^{\frac{3}{2}}(\Gamma)}$$

$$- \eta c_h\rho(c_1 u(t) - u_e)^m u - \eta c_h\rho(c_2\gamma_0 u(t) - u_e)^m \gamma_0 u$$

(see [3, p. 698]).

We are now in a position to determine $V^{/}(u(t))$:

$$V^{/}(u(t)) = \frac{d}{dt}\frac{1}{2}[c_h^2\rho^2 r(u(t))^2 + c_h^2\rho^2 s(\gamma_0 u(t))^2]$$

$$+ \frac{1}{m+1}\frac{d}{dt}[\eta c_h\rho|(c_1 u(t) - u_e)^{m+1} - (c_1 u^0 - u_e)^{m+1}|]$$

$$+\frac{1}{m+1}\frac{d}{dt}[kc_h\rho|(c_2\gamma_0 u(t)-u_e)^{m+1}-(c_2\gamma_0 u^0-u_e)^{m+1}|]$$

$$= -\kappa c_h\rho\| c_1\nabla u(t)\|^2_{L^2(\Omega)} - \kappa c_h\rho\| c_2\nabla_s(\gamma_0 u(t))\|^2_{H^{\frac{3}{2}}(\Gamma)}$$

$$- \eta c_h\rho(c_1 u(t)-u_e)^m u - kc_h\rho(c_2\gamma_0 u(t)-u_e)^m\gamma_0 u$$

$$+ \eta c_1 c_h\rho|(c_1 u(t)-u_e)^m| + kc_2 c_h\rho|(c_2\gamma_0 u(t)-u_e)^m|.$$

For the cooling model, we have that $c_1 u(t) \geq u_e$ and $c_2\gamma_0 u(t) \geq u_e$.

Hence,

$$V^{/}(u(t)) = -\kappa c_h\rho\| c_1\nabla u(t)\|^2_{L^2(\Omega)} - \kappa c_h\rho\| c_2\nabla_s(\gamma_0 u(t))\|^2_{H^{\frac{3}{2}}(\Gamma)}$$

$$- \eta c_h\rho(c_1 u(t)-u_e)^m u - \kappa c_h\rho(c_2\gamma_0 u(t)-u_e)^m\gamma_0 u$$

$$+ \eta c_1(c_1 u(t)-u_e)^m + kc_2(c_2\gamma_0 u(t)-u_e)^m < 0,$$

since $c_h\rho(c_1 u(t)-u_e)^m u \geq c_1(c_1 u(t)-u_e)^m$.

Thus, the preceding analysis confirms what is already known about a Lyapunov function.

8. Conclusion

Although our analysis has involved 'weak' solutions (in the sense of distributions) (see assumption 2(b)), the validity of the transformation (5) is based on the uniqueness of those solutions as guarantee in [3]. The restrictions of b to $b = \langle -1, -1\rangle$ and $c = \langle c_1, c_2\rangle$, with $c_1, c_2 \geq 1$, although looking artificial, will work specifically for the cooling problem only [3]. Further, applications of the transformation (5) in problems of permeable boundary Navier-Stokes flows would enhance a "similarity" between "heat transfer through surface radiation" and "fluid flow through a permeable boundary". The question is: can we apply the same stability criterion for the two time-dependent phenomena?

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