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ANTI-TRIANGULAR MAPS WITH CLOSED PERIODIC SETS

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Abstract

In this paper, some dynamical properties of the anti-triangular maps are given. Moreover, some sufficient and necessary conditions for anti-triangular maps on unit square with closed periodic sets are given.

1. Introduction

In what follows we denote by N the positive integers. Let (X, d) be a metric space and $C^0(X)$ be the set of continuous self-maps on X.

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Let (X, d) be a metric space and $f \in C^0(x)$. For any $x \in X$, the orbit of x under f is $orb(x, f) = \{f^n(x) : n = 0, 1, 2, ...\}$, and is called the *orbit* of x for short. We say that x is an *n-periodic point* of f if $f^n(x) = x$ and $f^n(x) \neq x$ for $1 \leq i < n$. A point $x \in X$ is said to be a *fixed point* of f if $f(x) = x$. Denote by $P(f)$, $Fix(f)$, $P_n(f)$ the sets of periodic, fixed and n-periodic points of f, respectively. We say that $x \in X$ is an *eventually periodic point* if $f^j(x) = f^k(x)$ for some pair j, k with $j \neq k$. Denote by $EP(f)$ the set of all eventually periodic points.

For any $x \in X$, let $\omega(x) = \omega(x, f) = \bigcap_{n \geq 0} \overline{\{f^n(x) : n \geq m\}}$. Then we say that $\omega(x)$ is the ω-*limit set* of x and every point of $\omega(x)$ is ω-limit point of x. A point $x \in X$ is said to be *recurrent* if $x \in \omega(x)$. x is non-wandering if $f^{n_k}(x_k) \to x$ for some sequence of points $x_k \to x$ and some sequence of integers $n_k \to \infty$. Let $R(f)$, $\Omega(f)$ denote, respectively, the sets of all recurrent and non-wandering points. A point $x \in X$ is said to be *almost periodic* if x is recurrent and for every open set U containing x, there exists a positive integer m such that $x \in \bigcap_{k \geq 0} f^{-km}(U)$. We denote by $AP = AP(f)$ or $RR = RR(f)$ the set of all almost periodic points. A point $x \in X$ is said to be *strong recurrent* if x is recurrent and for every open set U containing x, there exists a positive integer $N = N(U)$ such that if $f^m(x) \in U$, where $m \geq 0$, then $x \in \bigcup_{k \geq 1}^{N} f^{-(m+k)(U)}$ for some k with $0 \leq k \leq N$. Denote by $UR = UR(f)$ the set of all strong recurrent points.

Let $\varepsilon > 0$ be given and x, y be any points of X. A finite sequence $x_0, x_1, ..., x_n$ $(n \geq 1)$ of points of X is said to be ε-*chain* or *pseudo-orbit* from x to y of f if $x_0 = x$, $x_n = y$ and $d(x_i, f(x_{i-1})) < \varepsilon$ for $i = 1, 2, ..., n$. Let

$$W_\varepsilon(x, f) = \{w \in X : \text{there exists an } \varepsilon\text{-chain from } x \text{ to } w\} \qquad (1.1)$$

and

$$W(x,\ f) = \bigcap\{W_{\varepsilon}(x,\ f) : \varepsilon > 0\}. \tag{1.2}$$

$W(x,\ f)$ is said to be a *chain-available set* of *f*, and every point of $W(x,\ f)$ is said to be a *chain-available point* of *x*. A point $x \in X$ is said to be *chain-recurrent point* if $x \in W(x,\ f)$. Denote $CR(f)$ the set of all chain recurrent points.

In [1], we know that

$$P(f) \subset AP(f) \subset UR(f) \subset R(f) \subset \omega(f) \subset \Omega(f) \subset CR(f) \tag{1.3}$$

and there exists a map f such that $P(f) \subset AP(f) \subset UR(f) \subset R(f) \subset \omega(f) \subset \Omega(f) \subset CR(f)$ (see [2]).

Let X be a compact metric space, $f,\ g \in C^0(X)$. Then a map F is said to be *anti-triangular map* if it has the form $F(x,\ y) = (g(y),\ f(x)),\ (x,\ y) \in X^2$. It has been concerned for the study of the anti-triangular maps in recent years (see [3-5] et al). The aim of this paper is to study the properties of the anti-triangular maps, and drive some conditions for periodic set of anti-triangular maps on I^2 being closed.

2. Fundamental Results

Lemma 2.1. *Let X be a compact metric space,* $f,\ g \in C^0(X)$, $F(x,\ y) = (g(y),\ f(x))$ *be an anti-triangular map. Then*

(1) $P(F^2) = P(F)$; $F^{2n}(x,\ y) = ((g \circ f)^n(x), (f \circ g)^n(y))$;

$P(F) = P(g \circ f) \times P(f \circ g)$;

(2) $AP(F^2) = AP(F)$; $AP(F) = AP(g \circ f) \times AP(f \circ g)$;

(3) $CR(F^2) = CR(F)$; $CR(F) = CR(g \circ f) \times CR(f \circ g)$;

(4) $R(F^2) = R(F)$; $R(F) \subset R(g \circ f) \times R(f \circ g)$;

(5) $UR(F^2) = UR(F)$; $UR(F) \subset UR(g \circ f) \times UR(f \circ g)$;

(6) $\omega(F^2) = \omega(F)$; $\omega(F) \subset \omega(g \circ f) \times \omega(f \circ g)$;

(7) $\Omega(F^2) \subset \Omega(F)$; $\Omega(F^2) \subset \Omega(g \circ f) \times \Omega(f \circ g)$.

Lemma 2.2. *Let X be a compact metric space,* $f, g \in C^0(X)$, $F(x, y) = (g(y), f(x))$ *be an anti-triangular map. Then* $EP(F^2) = EP(F)$ *and* $EP(F) = EP(g \circ f) \times EP(f \circ g)$.

Proof. It is obvious that $EP(F^2) = EP(F)$. Now we prove $EP(F) = EP(g \circ f) \times EP(f \circ g)$.

Let $(x, y) \in EP(F)$. Then from the definition of $EP(F)$, there exists a positive integer *m* such that $F^m(x, y) \in P(F)$, thus $F^{2m}(x, y) \in P(F)$. By Lemma 2.1, we obtain that $(g \circ f)^m(x) \in P(g \circ f)$, $(f \circ g)^m(x) \in P(f \circ g)$, that is $x \in EP(g \circ f)$, $y \in EP(f \circ g)$. Then $EP(F) \subset EP(g \circ f) \times EP(f \circ g)$.

On the other hand, let $(x, y) \in EP(g \circ f) \times EP(f \circ g)$. Then there exist some pair *m*, *n* such that $(g \circ f)^m(x) \in P(g \circ f)$, $(f \circ g)^n(y) \in (f \circ g)$. Then $(g \circ f)^{mn}(x) \in P(g \circ f)$, $(f \circ g)^{mn}(y) \in P(f \circ g)$. Hence $((g \circ f)^{mn}(x), (f \circ g)^{mn}(y)) \in P(g \circ f) \times P(f \circ g) = P(F)$. Therefore, $(x, y) \in EP(F)$. Then $EP(F) \supset EP(g \circ f) \times EP(f \circ g)$.

We conclude the proof.

Corollary 2.3. *Let* $X = [0, 1]$ *be a compact metric space,* $f, g \in C^0(X)$, $F(x, y) = (g(y), f(x))$ *be an anti-triangular map. Then* $\Omega(F^2) \subset \overline{EP(F)}$.

Proof. Let $F_1 = g \circ f$, $F_2 = f \circ g$. By Lemma 2.1, we have $\Omega(F^2) \subset \Omega(F_1) \times \Omega(F_2)$. According to [1, Proposition 4.18], $\Omega(F^2) \subset \Omega(F_1) \times \Omega(F_2)$

$\subset \overline{EP(F_1)} \times \overline{EP(F_2)}$. By Lemma 2.2, $\overline{EP(F_1)} \times \overline{EP(F_2)} = \overline{EP(F_2)}$. Therefore, $\Omega(F^2) \subset \overline{EP(F)}$.

Proposition 2.4. *Let* $X = [0, 1]$ *be a compact metric space,* $f, g \in C^0(X)$, $F(x, y) = (g(y), f(x))$ *be an anti-triangular map. Then any isolated point of* $P(F)$ *is also an isolated point of* $R(F)$.

Proof. Let (x, y) be an isolated point of $P(F)$. Then (x, y) is also an isolated point of $P(g \circ f) \times P(f \circ g)$. Thus x is an isolated point of $P(g \circ f)$ and y is an isolated point of $P(f \circ g)$. According to [1, Proposition 4.35], x is an isolated point of $R(g \circ f)$, y is an isolated point of $R(f \circ g)$. Thus (x, y) is an isolated point of $R(g \circ f) \times R(f \circ g)$. Obviously, $(x, y) \in R(F)$. By Lemma 2.1, (x, y) is an isolated point of $R(F)$.

3. Main Results

Proposition 3.1. *Let X be a compact metric space,* $f, g \in C^0(X)$, *and* $f \circ g = g \circ f$, $F(x, y) = (g(y), f(x))$ *be an anti-triangular map. Then* $(x, x) \in A(F)$ *if and only if* $x \in A(g \circ f)$, *where* $A(\bullet) \in \{P(\bullet), AP(\bullet), R(\bullet), \omega(\bullet), \Omega(\bullet), CR(\bullet)\}$.

Proof. If $A(\bullet) \in \{P(\bullet), AP(\bullet), CR(\bullet)\}$, then the conclusion is obvious by Lemma 2.1. Now we consider the other cases:

(1) Since $\omega(F) \subset \omega(g \circ f) \times \omega(f \circ g)$,

$$(x, x) \in \omega(F) \Rightarrow (x, x) \in \omega(g \circ f) \times \omega(f \circ g) \Rightarrow x \in \omega(g \circ f).$$

On the other hand, since $x \in \omega(g \circ f)$, there exist a point $y \in X$ and integers $n_k \to \infty$ such that $(g \circ f)^{n_k}(y) \to x$. Then $(g \circ f)^{n_k} \times (g \circ f)^{n_k} \cdot (y, y) \to (x, x)$, and jointly with $F^2 = (g \circ f) \times (g \circ f)$, we obtain $(x, x) \in \omega(F^2)$. Additionally, $\omega(F^2) = \omega(F)$, thus $(x, x) \in \omega(F)$.

(2) Since $R(F) \subset R(g \circ f) \times R(f \circ g)$,

$$(x, x) \in R(F) \Rightarrow (x, x) \in R(g \circ f) \times R(f \circ g) \Rightarrow x \in R(g \circ f).$$

On the other hand, since $x \in R(g \circ f)$, there exist integers $n_k \to \infty$ such that $(g \circ f)^{n_k}(x) \to x$. Then $(g \circ f)^{n_k} \times (g \circ f)^{n_k}(x, x) \to (x, x)$, and jointly with $F^2 = (g \circ f) \times (g \circ f)$, we obtain $(x, x) \in R(F^2)$. Additionally, $R(F^2) = R(F)$, thus $(x, x) \in R(F)$.

(3) Let $(x, x) \in \Omega(F)$ and $U \subset X$ be an open neighborhood of x. Since $(x, x) \in \Omega(F)$, there exists a positive integer m such that $F^m(U \times U) \cap (U \times U) \neq \Phi$. We have two possibilities: (1) $m = 2n$, for $n \in N$; (2) $m = 2n + 1$, for $n \in N$. If (1) happens, then

$$F^{2n}(U \times U) \cap (U \times U) = ((g \circ f)^n(U) \times (g \circ f)^n(U)) \cap (U \times U) \neq \Phi,$$

thus $(g \circ f)^n(U) \cap U \neq \Phi$. If (2) happens, then

$$F^{2n+1}(U \times U) \cap (U \times U) = (f \circ (g \circ f)^n(U) \times (g \circ f)^n(U)) \cap (U \times U) \neq \Phi,$$

thus we have $(g \circ f)^n(U) \cap U \neq \Phi$. In both cases, $x \in \Omega(g \circ f)$. On the other hand, let $x \in \Omega(g \circ f)$. For any open set $U_1 \times U_2$ containing (x, x), let $U = U_1 \cap U_2$. Then U is an open set and $x \in U$. Since $x \in \Omega(g \circ f)$, there exists a positive integer n such that $(g \circ f)^n(U) \cap U \neq \Phi$. Then $((g \circ f)^n(U) \times (g \circ f)^n(U)) \cap (U \times U) \neq \Phi$. Since $U \times U \subset U_1 \times U_2$ and

$$(g \circ f)^n(U) \times (g \circ f)^n(U) \subset (g \circ f)^n(U_1) \times (g \circ f)^n(U_2),$$

$$((g \circ f)^n(U_1) \times (g \circ f)^n(U_2)) \cap (U_1 \times U_2) \neq \Phi.$$

Thus $(x, x) \in \Omega((g \circ f) \times (g \circ f))$. And jointly with $F^2 = (g \circ f) \times (g \circ f)$, we obtain $(x, x) \in \Omega(F^2)$. Additionally, $\Omega(F^2) \subset \Omega(F)$, thus $(x, x) \in \Omega(F)$.

Corollary 3.2 [4, Lemma 3.5]**.** *Let* $F(x, y) = (y, f(x))$ *be an anti-*

triangular map defined on I^2. *Let* $x \in I$. *Then* $x \in A(g \circ f)$ *if and only if* $(x, x) \in A(F)$, *where* $A(\bullet) \in \{P(\bullet), AP(\bullet), R(\bullet), \omega(\bullet), \Omega(\bullet), CR(\bullet)\}$.

In [5], the author studied the relation between $Per(F)$ and $Per(g \circ f)$, and have proved $Per(g \circ f) = Per(f \circ g)$ and $Per(F) = \wp$ or $Per(F) = \wp \cup \{2\}$, where $\wp = 2(Per(g \circ f) - \{1\}) \cup \{k \in Per(g \circ f) : k \text{ is odd}, k \geq 1\}$.

In [4], Balibrea et al. have proved that if $Per(F)$ is of type 2^∞, then the following result is right:

Lemma 3.3 [4, Lemma 3.4]. *Let* $F(x, y) = (g(y), f(x))$ *be an anti-triangular map such that* $Per(F) \subset \{2^n : n \in N \cup \{0\}\}$. *Assume* $P(F)$ *is closed. If* $A(\bullet), B(\bullet) \in \{P(\bullet), AP(\bullet), UR(\bullet), \omega(\bullet), \Omega(\bullet), CR(\bullet)\}$, *then the following two statements hold*:

(1) $A(F) = B(F)$;

(2) $A(F) = A(g \circ f) \times A(f \circ g)$, *and* $\Omega(F^2) = \Omega(F)$.

Lemma 3.4. *Let* $I = [0, 1]$ *be a compact metric space,* $f, g \in C^0(I)$, *and* $F(x, y) = (g(y), f(x))$ *be an anti-triangular map. Assume* $\overline{P(F)} = P(F)$. *Then* $P(F) = CR(F)$.

Proof. First, we prove $P(F) = \Omega(F)$. Obviously, $P(F) \subset \Omega(F)$.

Now we prove $\Omega(F) \subset P(F)$.

Since $P(F) = P(g \circ f) \times (f \circ g)$, $\overline{P(F)} = \overline{P(g \circ f)} \times \overline{P(f \circ g)}$. So $P(F) = \overline{P(F)}$ implies $P(g \circ f) = \overline{P(g \circ f)}$ and $P(f \circ g) = \overline{P(f \circ g)}$. According to [6], $P(g \circ f) = \Omega(g \circ f)$, $P(f \circ g) = \Omega(f \circ g)$. Therefore, $\Omega(F) \subset \Omega(g \circ f) \times \Omega(f \circ g) \subset P(g \circ f) \times P(f \circ g) = P(F)$. By [1, Proposition 6.36], $P(g \circ f) = CR(g \circ f)$, and $P(f \circ g) = CR(f \circ g)$. According to Lemma 2.1, $CR(F) = CR(g \circ f) \times CR(f \circ g) = P(g \circ f) \times P(f \circ g) = P(F)$.

Theorem 3.5. *Let* $I = [0, 1]$ *be a compact metric space,* $f, g \in C^0(I)$, *and* $F(x, y) = (g(y), f(x))$ *be an anti-triangular map. If* $\overline{P(F)} = P(F)$, *then the following three statements hold*:

(1) $P(F) = AP(F) = UR(F) = R(F) = \omega(F) = \Omega(F) = CR(F)$;

(2) $UR(F) = UR(g \circ f) \times UR(f \circ g)$; $R(F) = R(g \circ f) \times R(f \circ g)$;

$\omega(F) = \omega(g \circ f) \times \omega(f \circ g)$; $\Omega(F) = \Omega(g \circ f) \times \Omega(f \circ g)$;

(3) $\Omega(F^2) = \Omega(F)$.

Proof. (1) It is obvious according to Lemma 3.4 and (1.3).

(2) From the proof of Lemma 3.4, $P(g \circ f) = \Omega(g \circ f)$, and $P(f \circ g) = \Omega(f \circ g)$. By Lemma 2.1,

$$P(F) \subset UP(F) \subset UR(g \circ f) \times UR(f \circ g) \subset \Omega(g \circ f) \times \Omega(f \circ g)$$

$$= P(g \circ f) \times P(f \circ g) = P(F).$$

Therefore, $UP(F) = UR(g \circ f) \times UR(f \circ g)$.

Another three equalities can be proved similarly.

(3) Obviously, from $P(F) \subset \Omega(F^2) \subset \Omega(F) = P(F)$, we end the proof.

Corollary 3.6. *Let* $I = [0, 1]$ *be a compact metric space,* $f, g \in C^0(I)$, *and* $F(x, y) = (g(y), f(x))$ *be an anti-triangular map. If* $P(F) = \overline{P(F)}$, *then the following statements hold*:

(1) $P(F) = \{(x, y) : \exists n = n(x, y), s.t. F^{2^n}(x, y) = (x, y)\}$;

(2) $AP(F) = \{(x, y) : \exists n = n(x, y), s.t. F^{2^n}(x, y) = (x, y)\}$;

(3) $UR(F) = \{(x, y) : \exists n = n(x, y), s.t. F^{2^n}(x, y) = (x, y)\}$;

(4) $R(F) = \{(x, y) : \exists n = n(x, y), \ s.t. \ F^{2^n}(x, y) = (x, y)\}$;

(5) $\omega(F) = \{(x, y) : \exists n = n(x, y), \ s.t. \ F^{2^n}(x, y) = (x, y)\}$;

(6) $\Omega(F) = \{(x, y) : \exists n = n(x, y), \ s.t. \ F^{2^n}(x, y) = (x, y)\}$;

(7) $CR(F) = \{(x, y) : \exists n = n(x, y), \ s.t. \ F^{2^n}(x, y) = (x, y)\}$.

Proof. It is necessary to prove (1). The others can be obtained by Theorem 3.5 and (1).

Now we prove (1).

Since $P(F) = \overline{P(F)}$, we have $P(g \circ f) = P\overline{(g \circ f)}$. By [1, Proposition 6.36], $g \circ f$ is strong non-chaotic. So we have

$$P(g \circ f) = \{x : \exists n = n(x), \text{ s.t. } (g \circ f)^{2^n}(x) = (x)\}.$$

Thus $P(F) = \{(x, y) : \exists n = n(x, y), \text{ s.t. } F^{2^n}(x, y) = (x, y)\}$.

Lemma 3.7. *Let* $I = [0, 1]$ *be a compact metric space,* $f, g \in C^0(I)$, *and* $F(x, y) = (g(y), f(x))$ *be an anti-triangular map. If* $P(F) = AP(F)$, *then* $P(F) = \overline{P(F)}$.

Proof. Since $P(F) = P(g \circ f) \times P(f \circ g)$, $AP(F) = AP(g \circ f) \times AP(f \circ g)$ and $P(F) = AP(F)$, we obtain that $P(g \circ f) \times P(f \circ g) = AP(g \circ f) \times AP(f \circ g)$. So we have $P(g \circ f) = AP(g \circ f)$ and $P(f \circ g) = AP(f \circ g)$. By Lemma 2.1,

$$\begin{aligned} P(F) \subset \overline{P(F)} \subset \Omega(F) \subset CR(F) &= CR(g \circ f) \times CR(f \circ g) \\ &= P(g \circ f) \times P(f \circ g) = P(F). \end{aligned}$$

Therefore, $P(F) = \overline{P(F)}$.

Lemma 3.8. *Let* $I = [0, 1]$ *be a compact metric space,* $f, g \in C^0(I)$, *and* $F(x, y) = (g(y), f(x))$ *be an anti-triangular map. If one of the following conditions holds, then* $P(F) = \overline{P(F)}$:

(1) $AP(F) = \{(x, y) : \exists n = n(x, y), \text{ s.t. } F^{2^n}(x, y) = (x, y)\}$;

(2) $UR(F) = \{(x, y) : \exists n = n(x, y), \text{ s.t. } F^{2^n}(x, y) = (x, y)\}$;

(3) $R(F) = \{(x, y) : \exists n = n(x, y), \text{ s.t. } F^{2^n}(x, y) = (x, y)\}$;

(4) $\omega(F) = \{(x, y) : \exists n = n(x, y), \text{ s.t. } F^{2^n}(x, y) = (x, y)\}$;

(5) $\Omega(F) = \{(x, y) : \exists n = n(x, y), \text{ s.t. } F^{2^n}(x, y) = (x, y)\}$;

(6) $CR(F) = \{(x, y) : \exists n = n(x, y), \text{ s.t. } F^{2^n}(x, y) = (x, y)\}$.

Proof. Since (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1), it is necessary to prove that when (1) holds, $P(F) = \overline{P(F)}$.

If (1) holds, then by Lemma 2.1,

$$AP(g \circ f) = \{x : \exists n = n(x), \text{ s.t. } (g \circ f)^{2^n}(x) = x\};$$

$$AP(f \circ g) = \{y : \exists n = n(y), \text{ s.t. } (f \circ g)^{2^n}(y) = y\}.$$

So $g \circ f$ and $f \circ g$ both are strong non-chaotic. According to [1, Proposition 6.10],

$$P(g \circ f) = \overline{P(g \circ f)}, \quad P(f \circ g) = \overline{P(f \circ g)}.$$

Therefore,

$$P(F) = P(g \circ f) \times P(f \circ g) = \overline{P(g \circ f)} \times \overline{P(f \circ g)} = \overline{P(F)}.$$

From Theorem 3.5, Corollary 3.6, Lemmas 3.7-3.8, the following result holds:

Theorem 3.9. *Let* $I = [0, 1]$ *be a compact metric space,* $f, g \in C^0(I)$, *and* $F(x, y) = (g(y), f(x))$ *be an anti-triangular map. Then the following three conditions are equivalent*:

(1) $P(F) = \overline{P(F)}$;

(2) $P(F) = CR(F)$;

(3) $CR(F) = \{(x, y) : \exists n = n(x, y), \text{ s.t. } F^{2^n}(x, y) = (x, y)\}$.

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