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SEQUENT CALCULUS FOR THE INTERSECTION OF LK AND THE REVERSED

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Abstract

A cut-free sequent calculus such that $\Gamma \to \Delta$ is provable there, iff both $\Gamma \to \Delta$ and $\Delta \to \Gamma$ are provable in Gentzen's LK for the classical predicate logic, is given. This exemplifies how to make a sequent calculus for the intersection of two sequent calculi, and how to show the completeness of such a calculus.

The purpose of this note is to give a cut-free sequent calculus, which we call *LKKL*, such that $\Gamma \to \Delta$ is provable in LKKL, iff both $\Gamma \to \Delta$ and $\Delta \to \Gamma$ are provable in Gentzen's sequent calculus LK for the classical predicate logic. This exemplifies how to make a sequent calculus for the intersection of two sequent calculi, and how to show the completeness of such a calculus.

We mention only $\neg$, $\wedge$ and $\forall$ as the logical symbols, for simplicity. A formula with the logical symbol $\circ$ as its outermost one is called a $\circ$*-formula*.

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For sequent calculi, consult Takeuti [3], for example.

Greek capital letters $\Gamma, \Delta, \Pi, \Lambda, \Phi, \Psi, \ldots$ denote finite (possibly empty) sequences of formulas separated by commas, while Greek lower-case letters $\alpha, \beta, \ldots$ (finite or infinite) sets of formulas.

1. The Sequent Calculus LKKL

In this section, our sequent calculus LKKL is introduced and the main theorem is formulated.

Definition 1.1. The *sequent calculus LKKL* consists of the following beginning sequents and inference rules:

(1) Beginning sequents:

$$A \to A$$

(2) Inference rules:

Structural rules:

$$\text{Weakening } \frac{\Gamma \to \Delta}{A, \Gamma \to \Delta} \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}$$

$$\text{Exchange } \frac{\Gamma, A, B, \Pi \to \Delta}{\Gamma, B, A, \Pi \to \Delta} \qquad \frac{\Gamma \to \Delta, A, B, \Lambda}{\Gamma \to \Delta, B, A, \Lambda}$$

$$\text{Contraction } \frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta} \qquad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A}$$

$$\text{Cut } \frac{\Gamma \to \Delta, A \quad A, \Pi \to \Lambda}{\Gamma, \Pi \to \Delta, \Lambda}$$

Logical rules:

$$(\neg \to)\frac{\Gamma \to \Delta, A}{\neg A, \Gamma \to \Delta} \qquad (\to \neg)\frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, \neg A}$$

$$(\wedge \to)\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta} \qquad (\to \wedge)\frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}$$

$$(\forall \to)\frac{F(a),\ \Gamma \to \Delta}{\forall x F(x),\ \Gamma \to \Delta} \qquad (\to \forall)\frac{\Gamma \to \Delta,\ F(a)}{\Gamma \to \Delta,\ \forall x F(x)}$$

$$(\wedge \to \wedge)\frac{A,\ B,\ \Gamma \to \Delta \quad \Gamma \to \Delta,\ C,\ D}{A \wedge B,\ \Gamma \to \Delta,\ C \wedge D}$$

$$(\forall \to \forall)\frac{F(b),\ \Gamma \to \Delta \quad \Gamma \to \Delta,\ G(c)}{\forall x F(x),\ \Gamma \to \Delta,\ \forall y G(y)}$$

$$(\wedge \to \forall)\frac{A,\ B,\ \Gamma \to \Delta \quad \Gamma \to \Delta,\ F(b)}{A \wedge B,\ \Gamma \to \Delta,\ \forall x F(x)}$$

$$(\forall \to \wedge)\frac{F(b),\ \Gamma \to \Delta \quad \Gamma \to \Delta,\ A,\ B}{\forall x F(x),\ \Gamma \to \Delta,\ A \wedge B}.$$

Restriction on variables: In the rules $(\forall \to)$ and $(\to \forall)$, the free variable a must not occur in the lower sequent; while in the rules $(\forall \to \forall)$, $(\wedge \to \forall)$ and $(\forall \to \wedge)$, the free variables b and c are arbitrary.

We must show the following theorem, and the proof is given in the next section.

Theorem 1.2. *The following properties are mutually equivalent for any sequent* $\Gamma \to \Delta$:

(a) $\Gamma \to \Delta$ *is provable in LKKL.*

(b) $\Gamma \to \Delta$ *is cut-free provable in LKKL.*

(c) *Both* $\Gamma \to \Delta$ *and* $\Delta \to \Gamma$ *are provable in LK.*

2. Proof of Theorem 1.2

Among the equivalency of (a), (b) and (c) of our theorem, "(b) implies (a)" is evident, and the inductive proof of "(a) implies (c)" is routine. So, we conclude the proof of Theorem 1.2 by showing "(c) implies (b)".

Incidentally, although "(a) implies (b)" (the cut-elimination property for LKKL) is obtained along the above line, one can prove it directly by the usual way, namely by eliminating the Mix rule through the double induction on the grade and rank.

For the ease of description of the proof of "(c) implies (b)", some promises are made.

Promise 2.1.

(1) Provability means provability in LKKL, unless specified otherwise.

(2) The notion of sequent is modified as follows: The antecedent and succedent of a sequent are (finite or infinite) sets of formulas, but not finite sequences of formulas. If both the antecedent and succedent are finite, the sequent is called a *finite sequent*.

(3) Thus-modified sequent $\alpha \to \beta$ is (cut-free) provable, iff $\Gamma \to \Delta$ is (cut-free) provable for some Γ and Δ such that every constituent of Γ and Δ belongs to α and β, respectively.

(4) Each finite sequence of formulas is identified with the set of all its constituents.

(5) In the antecedents and succedents of sequents, commas are used to denote the unions of sets.

Note that Promise 2.1(4) together with (3) causes no trouble in the (cut-free) provability of finite sequents, owing to the three structural rules: Weakening, Exchange and Contraction.

Definition 2.2. The pair $\langle \alpha, \beta \rangle$ of sets α and β of formulas is called a *Hintikka pair*, if the seven properties $P(\alpha, \beta)$, $P_\neg(\alpha, \beta)$, $P_\neg(\beta, \alpha)$, $P_\wedge^+(\alpha)$, $P_\wedge^-(\beta)$, $P_\forall^+(\alpha)$ and $P_\forall^-(\beta)$ hold, where the properties P and $P_\neg$ on a pair of sets of formulas as well as $P_\wedge^+$, $P_\wedge^-$, $P_\forall^+$ and $P_\forall^-$ on a set of formulas are defined as follows:

(1) $P(\alpha, \beta)$, iff $\alpha \cap \beta = \varnothing$.

(2) $P_\neg(\alpha, \beta)$, iff for every $\neg$-formula $\neg A$, if $\neg A \in \alpha$, then $A \in \beta$.

(3) $P^{+}_{\wedge}(\alpha)$, iff for every $\wedge$-formula $A \wedge B$, if $A \wedge B \in \alpha$, then $A \in \alpha$ and $B \in \alpha$.

(4) $P^{-}_{\wedge}(\alpha)$, iff for every $\wedge$-formula $A \wedge B$, if $A \wedge B \in \alpha$, then $A \in \alpha$ or $B \in \alpha$.

(5) $P^{+}_{\forall}(\alpha)$, iff for every $\forall$-formula $\forall x F(x)$, if $\forall x F(x) \in \alpha$, then $F(a) \in \alpha$ for every free variable *a*.

(6) $P^{-}_{\forall}(\alpha)$, iff for every $\forall$-formula $\forall x F(x)$, if $\forall x F(x) \in \alpha$, then $F(a) \in \alpha$ for some free variable *a*.

Note that ‘Hintikka pair’ forms the sequent version of ‘Hintikka set’ in Smullyan [2], and the first order version of ‘semi-valuation’ in Schütte [1], as well.

Recall the semantical characterization of LK that $\Gamma \to \Delta$ is (cut-free) provable there, iff for every interpretation *I* (for the classical predicate logic), either some constituents of Γ are falsified by *I* or some of Δ are satisfied.

The semantical proof of the cut-elimination property for LK appearing in the literature always depends, explicitly or implicitly, on the following lemma, and so the proof is omitted (see Smullyan [2], Takeuti [3], for example).

Lemma 2.3. *Let* $\langle \alpha, \beta \rangle$ *be a Hintikka pair. Then, an interpretation by which every formula in* α *is satisfied, but every formula in* β *is falsified, can be constructed; so, if* $\Gamma \subset \alpha$ *and* $\Delta \subset \beta$, *then* $\Gamma \to \Delta$ *is unprovable in LK.*

Definition 2.4. A sequent $\alpha \to \beta$ is *saturated*, if the following conditions hold:

(i) $\alpha \to \beta$ is cut-free unprovable.

(ii) For every Φ and Ψ, if $\Phi, \alpha \to \beta, \Psi$ is cut-free unprovable, then $\Phi \subset \alpha$ and $\Psi \subset \beta$.

(iii) The property $P_{\forall}^{-}(\alpha)$ holds.

(iv) The property $P_{\forall}^{-}(\beta)$ holds.

Lemma 2.5. *If* $\Gamma \rightarrow \Delta$ *is cut-free unprovable*, *then there is a saturated sequent* $\alpha \rightarrow \beta$ *such that* $\Gamma \subset \alpha$ *and* $\Delta \subset \beta$.

Proof. Suppose that $\Gamma \rightarrow \Delta$ is cut-free unprovable, and let

$$\Phi_0 \rightarrow \Psi_0,\ \ \Phi_1 \rightarrow \Psi_1, ..., \Phi_n \rightarrow \Psi_n, ... \tag{$*$}$$

be an enumeration of all the finite sequents.

We will define Γ_n and Δ_n by induction on *n* as follows:

First, define Γ_0 and Δ_0 to be Γ and Δ, respectively. Suppose next, that Γ_n and Δ_n have been defined, and define Γ_{n+1} and Δ_{n+1} by cases.

Case 1. $\Phi_n, \Gamma_n \rightarrow \Delta_n, \Psi_n$ is cut-free provable. Define Γ_{n+1} and Δ_{n+1} to be Γ_n and Δ_n, respectively.

Case 2. $\Phi_n, \Gamma_n \rightarrow \Delta_n, \Psi_n$ is cut-free unprovable.

Subcase 2.1. Φ_n consists solely of a $\forall$-formula, say $\forall x F(x)$, while Ψ_n is empty. Let *a* be a free variable which does not occur in any formula of $\Gamma_n, \Delta_n, \forall x F(x)$. Define Γ_{n+1} and Δ_{n+1} to be $F(a), \forall x F(x), \Gamma_n$ and Δ_n, respectively.

Subcase 2.2. Φ_n is empty, while Ψ_n consists solely of a $\forall$-formula, say $\forall x F(x)$. Let *a* be a free variable which does not occur in any formula of $\Gamma_n, \Delta_n, \forall x F(x)$. Define Γ_{n+1} and Δ_{n+1} to be Γ_n and $\Delta_n, \forall x F(x), F(a)$, respectively.

Subcase 2.3. Otherwise. Define Γ_{n+1} and Δ_{n+1} to be Φ_n, Γ_n and Δ_n, Ψ_n, respectively.

After defining all Γ_n's and Δ_n's, let α and β be the unions $\bigcup_n \Gamma_n$

and $\bigcup_n \Delta_n$, respectively. It is clear that $\Gamma \subset \alpha$ and $\Delta \subset \beta$. Let us show that $\alpha \to \beta$ is saturated by checking (i)-(iv) of Definition 2.4 one by one.

Proof of (i). It is an easy induction to show that $\Gamma_n \to \Delta_n$ is cut-free unprovable for every n; in examining Subcases 2.1 and 2.2, the rules $(\forall \to)$ and $(\to \forall)$ are essential, respectively. It follows that $\alpha \to \beta$ is cut-free unprovable; for, otherwise $\Gamma \to \Delta$ would be cut-free provable for some $\Gamma \subset \alpha$ and $\Delta \subset \beta$, and then $\Gamma \subset \Gamma_n$ and $\Delta \subset \Delta_n$ for some n, and so $\Gamma_n \to \Delta_n$ would be cut-free provable, which is a contradiction.

Proof of (ii). Suppose that $\Phi, \alpha \to \beta, \Psi$ is cut-free unprovable. Let $\Phi \to \Psi$ be the nth term in $(*)$. Then, $\Phi_n, \Gamma_n \to \Delta_n, \Psi_n$ is cut-free unprovable, since $\Phi_n \cup \Gamma_n \subset \Phi \cup \alpha$ and $\Delta_n \cup \Psi_n \subset \beta \cup \Psi$. So, $\Phi_n \subset \Gamma_{n+1}$ and $\Psi_n \subset \Delta_{n+1}$ in any subcases, and so $\Phi \subset \alpha$ and $\Psi \subset \beta$.

Proof of (iii). Suppose $\forall x F(x) \in \alpha$. Let $\forall x F(x) \to$ be the nth term in $(*)$. Then, $\Phi_n, \Gamma_n \to \Delta_n, \Psi_n$ is cut-free unprovable by (i), since $\Phi_n \cup \Gamma_n \subset \alpha$ and $\Delta_n \cup \Psi_n \subset \beta$. So, Subcase 2.1 works, and so $F(a) \in \Gamma_{n+1}$ and hence $F(a) \in \alpha$ for some a.

Proof of (iv). Similar to the proof of (iii).

Thus, we have completed the proof that $\alpha \to \beta$ is saturated. □

Lemma 2.6. *Suppose that* $\alpha \to \beta$ *is saturated*.

(1) *The property* $P(\alpha, \beta)$ *holds*.

(2) *The property* $P_{\neg}(\alpha, \beta)$ *holds*.

(3) *The property* $P_{\neg}(\beta, \alpha)$ *holds*.

(4) *The property* $P_{\wedge}^{-}(\alpha)$ *holds*.

(5) *The property* $P_{\wedge}^{-}(\beta)$ *holds*.

(6) *Either the property* $P_{\wedge}^{+}(\alpha)$ *or* $P_{\wedge}^{+}(\beta)$ *holds*.

(7) *Either the property* $P_{\wedge}^{+}(\alpha)$ *or* $P_{\forall}^{+}(\beta)$ *holds*.

(8) *Either the property* $P_{\forall}^{+}(\alpha)$ *or* $P_{\wedge}^{+}(\beta)$ *holds*.

(9) *Either the property* $P_{\forall}^{+}(\alpha)$ *or* $P_{\forall}^{+}(\beta)$ *holds*.

Proof. (1) If $\alpha \cap \beta \neq \varnothing$, then $\alpha \to \beta$ would be cut-free provable, which contradicts Definition 2.4(i). So, $\alpha \cap \beta = \varnothing$, namely $P(\alpha, \beta)$.

(2) Suppose $\neg A \in \alpha$. Then, $\alpha \to \beta, A$ is cut-free unprovable; for, otherwise $\neg A, \alpha \to \beta$ and so $\alpha \to \beta$ would be cut-free provable by the rule $(\neg \to)$, which is a contradiction. So, $A \in \beta$ by Definition 2.4(ii).

(3)-(5) Similar to (2) by the rules $(\to \neg)$, $(\wedge \to)$ and $(\to \wedge)$, respectively.

(6) Suppose that neither $P_{\wedge}^{+}(\alpha)$ nor $P_{\wedge}^{+}(\beta)$ holds. It follows that $A \wedge B \in \alpha$ but not $\{A, B\} \subset \alpha$ for some $A \wedge B$, and $C \wedge D \in \beta$ but not $\{C, D\} \subset \beta$ for some $C \wedge D$. Then, either $A, B, \alpha \to \beta$ or $\alpha \to \beta, C, D$ is cut-free unprovable; for, otherwise $A \wedge B, \alpha \to \beta, C \wedge D$ and so $\alpha \to \beta$ would be cut-free provable by the rule $(\wedge \to \wedge)$, which is a contradiction. So, either $\{A, B\} \subset \alpha$ or $\{C, D\} \subset \beta$ by Definition 2.4(ii), which is a contradiction in both cases. Hence, either $P_{\wedge}^{+}(\alpha)$ or $P_{\wedge}^{+}(\beta)$ holds.

(7)-(9) Similar to (6) by the rules $(\wedge \to \forall)$, $(\forall \to \wedge)$ and $(\forall \to \forall)$, respectively. □

Lemma 2.7. *If* $\alpha \to \beta$ *is saturated*, *then either* $\langle \alpha, \beta \rangle$ *or* $\langle \beta, \alpha \rangle$ *forms a Hintikka pair*.

Proof. Suppose that $\alpha \to \beta$ is saturated. We must show that, either

$$P(\alpha, \beta),\ P_{\neg}(\alpha, \beta),\ P_{\neg}(\beta, \alpha),\ P_{\wedge}^{+}(\alpha),\ P_{\wedge}^{-}(\beta),\ P_{\forall}^{+}(\alpha) \text{ and } P_{\forall}^{-}(\beta)$$

or

$$P(\alpha, \beta),\ P_{\neg}(\beta, \alpha),\ P_{\neg}(\alpha, \beta),\ P_{\wedge}^{+}(\beta),\ P_{\wedge}^{-}(\alpha),\ P_{\forall}^{+}(\beta) \text{ and } P_{\forall}^{-}(\alpha).$$

Among these properties, $P(\alpha, \beta)$, $P_{\neg}(\alpha, \beta)$, $P_{\neg}(\beta, \alpha)$, $P_{\wedge}^{-}(\alpha)$ and $P_{\wedge}^{-}(\beta)$ hold by Lemma 2.6(1)-(5), while $P_{\forall}^{-}(\alpha)$ and $P_{\forall}^{-}(\beta)$ hold by Definition 2.4(iii) and (iv). So, it is left to show that

$$\text{either } P_{\wedge}^{+}(\alpha) \text{ and } P_{\forall}^{+}(\alpha), \text{ or } P_{\wedge}^{+}(\beta) \text{ and } P_{\forall}^{+}(\beta).$$

By the 'distributive law', this is equivalent to the 'conjunction' of the following four 'disjunctions':

$$P_{\wedge}^{+}(\alpha) \text{ or } P_{\wedge}^{+}(\beta);\ P_{\wedge}^{+}(\alpha) \text{ or } P_{\forall}^{+}(\beta);\ P_{\forall}^{+}(\alpha) \text{ or } P_{\wedge}^{+}(\beta);\ P_{\forall}^{+}(\alpha) \text{ or } P_{\forall}^{+}(\beta).$$

These 'disjunctions' are nothing but Lemma 2.6(6)-(9), and so hold certainly. □

Now, we are in a position to give a postponed proof of "(c) implies (b)" of Theorem 1.2.

To show the contraposition, suppose that $\Gamma \to \Delta$ is cut-free unprovable (in LKKL). By Lemma 2.5, there is a saturated sequent $\alpha \to \beta$ such that $\Gamma \subset \alpha$ and $\Delta \subset \beta$. Then, by Lemma 2.7, either $\langle \alpha, \beta \rangle$ or $\langle \beta, \alpha \rangle$ forms a Hintikka pair. It follows by Lemma 2.3 that either $\Gamma \to \Delta$ or $\Delta \to \Gamma$ is unprovable in LK.

This ends the proof of "(c) implies (b)", and hence concludes the proof of our theorem.

References

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[2] R. M. Smullyan, First-order Logic, Dover, New York, 1995.

[3] G. Takeuti, Proof Theory, 2nd ed., North-Holland, Amsterdam, 1987.