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LINEAR DYNAMICAL SYSTEMS WITH ERGODIC SHADOWING

Manseob Lee

Department of Mathematics
Mokwon University
Daejeon, 302-729, Korea
e-mail: lmsds@mokwon.ac.kr

Abstract

Let $f(x) = Ax$ of $\mathbb{C}^n$. Then f has the ergodic shadowing property if and only if the matrix A is hyperbolic.

1. Introduction

Let (X, d) be a compact metric space with a metric d, and $f : X \to X$ be a homeomorphism. The notion of the shadowing property is one of the most important notions in dynamical systems [1]. For $\delta > 0$, a sequence $\{x_i\}_{i=0}^{\infty}$ is called a δ-*pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in \mathbb{Z}$. We say that f has the *shadowing property* if for every $\varepsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i=a}^{b}$ of f $(-\infty \le a < b \le \infty)$, there is a point

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$y \in M$ such that $d(f^i(y), x_i) < \varepsilon$ for all $a \le i \le b-1$. The notion of pseudo orbits often appears in several methods of the modern theory of dynamical system ([4]). Moreover, the pseudo orbit shadowing property usually plays an important role in the investigation of stability theory and ergodic theory. The notion of ergodic shadowing property for continuous onto maps over compact metric spaces was defined by Fakhari and Ghane in [2]. For any $\delta > 0$, a sequence $\xi = \{x_i\}_{i \in \mathbb{Z}}$ is δ-*ergodic pseudo orbit* of *f* if for

$$Np_n^+(\xi, f, \delta) = \{i : d(f(x_i), x_{i+1}) \ge \delta\} \cap \{0, 1, ..., n-1\}$$

and

$$Np_n^-(\xi, f, \delta) = \{-i : d(f^{-1}(x_{-i}), x_{-i-1}) \ge \delta\} \cap \{-n+1, ..., -1, 0\},$$

$$\lim_{n \to \infty} \frac{\# Np_n^+(\xi, f, \delta)}{n} = 0 \text{ and } \lim_{n \to -\infty} \frac{\# Np_n^-(\xi, f, \delta)}{n} = 0.$$

We say that *f* has the *ergodic shadowing property* if for any $\varepsilon > 0$, there is a $\delta > 0$ such that for every δ-ergodic pseudo orbit $\xi = \{x_i\}_{i \in \mathbb{Z}}$ of *f*, there is a point $z \in M$ satisfying that whenever

$$Ns_n^+(\xi, f, z, \varepsilon) = \{i : d(f^i(z), x_i) \ge \varepsilon\} \cap \{0, 1, ..., n-1\}$$

and

$$Ns_n^-(\xi, f, z, \varepsilon) = \{-i : d(f^{-i}(z), x_{-i}) \ge \varepsilon\} \cap \{-n+1, ..., -1, 0\},$$

$$\lim_{n \to \infty} \frac{\# Ns_n^+(\xi, f, z, \varepsilon)}{N} = 0 \text{ and } \lim_{n \to -\infty} \frac{\# Ns_n^-(\xi, f, z, \varepsilon)}{N} = 0.$$

In [2], the authors showed that the ergodic shadowing property has the shadowing property, and studied chaotic behavior and specification for maps having ergodic shadowing property. In this paper, we study the ergodic shadowing property in linear dynamical systems. The following is the main result of the paper.

Theorem 1.1. *Let* $f(x) = Ax$ *of* $\mathbb{C}^n$ *be a linear dynamical system. If f has the ergodic shadowing property, then the matrix A is hyperbolic.*

2. Proof of Theorem 1.1

Let (X, d) be as before, and let $f : X \to X$ be a homeomorphism. Let A be a nonsingular matrix $\mathbb{C}^n$, and let $f(x) = Ax$ of $\mathbb{C}^n$. We say that the matrix A is called *hyperbolic* if the spectrum does not intersect the circle $\{\lambda : |\lambda| = 1\}$.

Lemma 2.1. *Let* $f, g : X \to X$ *be homeomorphisms, and there exists a homeomorphism* $h : X \to X$ *such that* $f \circ h = h \circ g$. *Then f has the ergodic shadowing property if and only if g has the ergodic shadowing property.*

Proof. For $\delta > 0$, let $\xi = \{x_i\}_{i \in \mathbb{Z}}$ be a δ-ergodic pseudo orbit of *f*. Then $Np(\xi, \delta) = \{i : d(f(x_i), x_{i+1}) \geq \delta\}$. Therefore,

$$Np_n^+(\xi, f, \delta) = \{i : d(f(x_i), x_{i+1}) \geq \delta\} \cap \{0, 1, ..., n-1\}$$

and

$$Np_n^-(\xi, f, \delta) = \{-i : d(f^{-1}(x_{-i}), x_{-i-1}) \geq \delta\} \cap \{-n+1, ..., -1, 0\}.$$

Thus

$$\lim_{n\to\infty} \frac{\# Np_n^+(\xi, f, \delta)}{n} = 0 \text{ and } \lim_{n\to-\infty} \frac{\# Np_n^-(\xi, f, \delta)}{n} = 0,$$

where $\#$ denotes the number of the elements in the set. Since $f = h \circ g \circ h^{-1}$,

$$d(f(x_i), x_{i+1}) = d(h \circ g \circ h^{-1}(x_i), x_{i+1}) = d(g(h^{-1}(x_i), h^{-1}(x_{i+1}))).$$

Thus

$$Np(\xi, \delta) = \{i : d(f(x_i), x_{i+1}) \geq \delta\} = \{i : d(g(h^{-1}(x_i), h^{-1}(x_{i+1}))) \geq \delta\}.$$

Then

$$Np_n^+(\xi, f, \delta) = \{i : d(f(x_i), x_{i+1}) \geq \delta\} \cap \{0, 1, ..., n-1\}$$

$$= \{i : d(g(h^{-1}(x_i), h^{-1}(x_{i+1}))) \geq \delta\} \cap \{0, 1, ..., n-1\}$$

$$= Np_n^+(\xi', g, \delta)$$

and

$$Np_n^-(\xi, f, \delta) = \{-i : d(f^{-1}(x_{-i}), x_{-i-1}) \geq \delta\} \cap \{-n+1, ..., -1, 0\}$$

$$= \{-i : d(g(h^{-1}(x_i), h^{-1}(x_{i+1})) \geq \delta\} \cap \{-n+1, ..., -1, 0\}$$

$$= Np_n^-(\xi', g, \delta),$$

where $\xi' = \{h^{-1}(x_i)\}_{i\in\mathbb{Z}}$. Thus we see that

$$\lim_{n\to\infty} \frac{\# Np_n^+(\xi, f, \delta)}{n} = \lim_{n\to\infty} \frac{\# Np_n^+(\xi', g, \delta)}{n} = 0$$

and

$$\lim_{n\to-\infty} \frac{\# Np_n^-(\xi, f, \delta)}{n} = \lim_{n\to-\infty} \frac{\# Np_n^-(\xi', g, \delta)}{n} = 0.$$

Therefore, $\{h^{-1}(x_i)\}_{i\in\mathbb{Z}}$ is a δ-ergodic pseudo orbit of g. Since f has the ergodic shadowing property, there is a point $y \in M$ such that

$$Ns_n^+(\xi, f, y, \varepsilon) = \{i : d(f^i(y), x_i) \geq \varepsilon\} \cap \{0, 1, ..., n-1\},$$

$$Ns_n^-(\xi, f, y, \varepsilon) = \{-i : d(f^{-i}(y), x_{-i}) \geq \varepsilon\} \cap \{-n+1, ..., -1, 0\}$$

and

$$\lim_{n\to\infty} \frac{\# Ns_n^+(\xi, f, y, \varepsilon)}{n} = 0 \text{ and } \lim_{n\to-\infty} \frac{\# Ns_n^-(\xi, f, y, \varepsilon)}{n} = 0.$$

Since $f \circ h = h \circ g$,

$$Ns_n^+(\xi, f, y, \varepsilon) = \{i : d(f^i(y), x_i) \geq \varepsilon\} \cap \{0, 1, ..., n-1\}$$

$$= \{i : d(g^i(h^{-1})(y), h^{-1}(x_i)) \geq \varepsilon\} \cap \{0, 1, ..., n-1\},$$

$$Ns_n^-(\xi, f, y, \varepsilon) = \{-i : d(f^{-i}(y), x_{-i}) \geq \varepsilon\} \cap \{-n+1, ..., -1, 0\}$$

$$= \{i : d(g^i(h^{-1})(y), h^{-1}(x_i)) \geq \varepsilon\} \cap \{-n+1, ..., -1, 0\}$$

and

$$\lim_{n\to\infty} \frac{\# Ns_n^+(\xi, f, y, \varepsilon)}{n} = \lim_{n\to\infty} \frac{\# Ns_n^+(\xi', g, h^{-1}(y), \varepsilon)}{n} = 0$$

and

$$\lim_{n\to-\infty} \frac{\# Ns_n^-(\xi, f, y, \varepsilon)}{n} = \lim_{n\to-\infty} \frac{\# Ns_n^-(\xi', g, h^{-1}(y), \varepsilon)}{n} = 0.$$

Thus g has the ergodic shadowing property. □

Note that f has the ergodic shadowing property if and only if f^n has the ergodic shadowing property for $n \in \mathbb{Z}$ (see [2]).

Lemma 2.2 [4]**.** *Let A be a nonhyperbolic matrix, and λ be an eigenvalue of A with $|\lambda| = 1$. Then there is a nonsingular matrix T such that $J = T^{-1}AT$ is a Jordan form of A and the matrix J has the form* $\begin{pmatrix} B & O \\ O & D \end{pmatrix}$, *where B is the hyperbolic matrix, and D is the nonsingular $m \times m$ complex matrix with the form* $\begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda \end{pmatrix}$.

Lemma 2.3. *Let $f(x) = Ax$ of $\mathbb{C}^n$. If $f(x) = Ax$ has the ergodic shadowing property, then the matrix A is hyperbolic.*

Proof. We will derive a contradiction. Suppose that the matrix A is nonhyperbolic. Then by Lemma 2.2, there is the nonsingular matrix T such that $J = T^{-1}AT$. Let $g(x) = Jx$ of $\mathbb{C}^n$. Then as in the proof of Theorem 3.2.1 in [4], g does not have the shadowing property. Thus g does not have the ergodic shadowing property. □

Remark 2.4 [2]**.** Let $f : X \to X$ be a homeomorphism. For the dynamical systems (X, f), f has the ergodic shadowing property if and only if f has the pseudo-orbital specification property.

Note that the pseudo-orbital specification property is a kind of shadowing property. If the matrix A is hyperbolic, then f has the shadowing property. However, the converse of Theorem 1.1 is not true. Indeed, we can construct an automorphism σ of the n-dimensional torus $\mathbb{T}^n$ which satisfies the specification property but is not hyperbolic (see [3]).

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