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CONTRAVARIANT FINITENESS OF MODULES OF GORENSTEIN PROJECTIVE DIMENSION $\leq I$

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Abstract

Let A be an artin algebra. In this paper, we give a sufficient condition for the subcategory $\mathcal{GP}^i(A)$ of A-mod to be contravariantly finite in A-mod, where $\mathcal{GP}^i(A)$ is the subcategory of A-mod consisting of A-modules of Gorenstein projective dimension less than or equal to i. As an application of this condition it is shown that $\mathcal{GP}^i(A)$ is contravariantly finite in A-mod for $i \geq 1$ when A is stably equivalent to a hereditary algebra.

1. Introduction and Preliminaries

1.1. The main idea of Gorenstein homological algebra is to replace

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projective modules by Gorenstein projective modules. These modules were introduced by Enochs and Jenda [14] as a generalization of finitely generated module of G-dimension zero over a two-sided noetherian ring, in the sense of Auslander-Bridger [1]. The subject has been developed to an advanced level, see for example [2], [3], [17], [15], [11], [6], [18], [7], [12], [9], [19], [16], [10], [8], [20].

Let A be an artin algebra. Denote by A-mod the category of all finitely generated A-modules. It is an interesting problem when $\mathcal{GP}^i(A)$ is contravariantly finite in A-mod for each i, where $\mathcal{GP}^i(A)$ is the full subcategory of A-mod consisting of A-modules of Gorenstein projective dimension less than or equal to i (see [2], [3], [15], [19] for more information).

The aim of this paper is to present a condition which is sufficient for the subcategory $\mathcal{GP}^i(A)$ to be contravariantly finite in A-mod. As an application of this condition we show that $\mathcal{GP}^i(A)$ is contravariantly finite in A-mod for $i \geq 1$ when A is stably equivalent to a hereditary algebra. The main idea of the proofs is taken from [3], also [13].

1.2. Throughout this paper, A is an artin algebra, all A-modules are finitely generated left modules, and all subcategories are full subcategories. We denote by A-mod the category of all finitely generated A-modules, and $\mathcal{P}(A)$ the subcategory of A-mod consisting of all projective A-modules.

An A-module G is Gorenstein projective if there is an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective A-modules, which stays exact after applying $\mathrm{Hom}_A(-, P)$ for any $P \in \mathcal{P}(A)$, such that $G \cong \mathrm{Im}(P_0 \to P^0)$ (see [15]). Denote by $\mathcal{GP}(A)$ the full subcategory of Gorenstein projective A-modules. Note that $\mathcal{GP}(A)$ is resolving (see [18]) in the sense of [3]: it contains all projective A-modules, is closed under direct sums and direct summands, extensions, and the kernels of epimorphisms, and is a Frobenius category with projective A-modules as projective-injective objects.

The Gorenstein projective dimension Gpd$M$ of M is defined to be the smallest integer $n \geq 0$ such that there is an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ with all $G_i \in \mathcal{GP}(A)$, if it exists; and Gpd$M = \infty$ if there is no such exact sequence of finite length (see [15]). Clearly, $\mathrm{Gpd}M \leq \mathrm{pd}M$ for each A-module M, where pd$M$ denotes the projective dimension of M.

Auslander and Smalø first introduced and studied the notions of contravariantly finite subcategories of A-mod in connection with the study of the existence of almost split sequences in a subcategory of A-mod (see [4, 5]). Recall from [4] that a full subcategory $\mathcal{A}$ of $\mathcal{B}$ is said to be *contravariantly finite* in $\mathcal{B}$ if for each object X in $\mathcal{B}$, there exists a morphism $f: Y \to X$, where $Y \in \mathcal{A}$, such that for any $Y' \in \mathcal{A}$, the induced morphism $\mathrm{Hom}(Y', Y) \to \mathrm{Hom}(Y', X)$ is surjective.

2. A Sufficient Condition for $\mathcal{GP}^i(A)$ to be Contravariantly Finite

The proofs in this section are analogous to those in Section 4 in [3], but for the completeness of the article, we give here the proofs.

Proposition 2.1. *Assume that I is an ideal in A with* $\mathrm{Gpd}_A\, A/I \leq i$ *such that if M is an A-module with* $\mathrm{Gpd}_A M \leq i$, *then* M/IM *is a projective* A/I*-module. Let C be an* A/I*-module. Then we have the following.*

(1) *A map* $B \to C$ *in* A/I-mod *is a right* $\mathcal{P}(A/I)$*-approximation of C if and only if it is a right* $\mathcal{GP}^i(A)$*-approximation of C.*

(2) *If* $G \to C$ *is a right* $\mathcal{GP}^i(A)$*-approximation of C, then* $G/IG \to C$ *is a right* $\mathcal{P}(A/I)$*-approximation of C.*

Proof. (1) Let $f: B \to C$ be a right $\mathcal{P}(A/I)$-approximation of C. Since B is a projective A/I-module and $\mathrm{Gpd}_A\, A/I \leq i$, as an A-module B is in

$\mathcal{GP}^i(A)$. Let $g : X \to C$ be a morphism in A-mod with $X \in \mathcal{GP}^i(A)$. Then g is the composition of the canonical projection $\pi : X \to X/IX$ and the induced map $g_1 : X/IX \to C$. Since X/IX is a projective A/I-module, the morphism g_1 can be lifted to B, so g can be lifted to B, that is, $f : B \to C$ is a right $\mathcal{GP}^i(A)$-approximation of C.

Conversely, let $f : B \to C$ in A/I-mod be a right $\mathcal{GP}^i(A)$-approximation of C. Since $\mathrm{Gpd}_A B \leq i$, as an A/I-module $B = B/IB$ is projective. Hence, $f : B \to C$ is a right $\mathcal{P}(A/I)$-approximation of C.

(2) Trivial.

Corollary 2.2. *Let I be an ideal in A satisfying the hypothesis of Proposition* 2.1. *If $IS = 0$ for any simple A-module S with* $\mathrm{Gpd}_A S > i$, *then* $\mathcal{GP}^i(A)$ *is contravariantly finite in A*-mod.

Proof. First note that $\mathcal{GP}^i(A)$ is a resolving subcategory. Then by Proposition 3.7 in [3], $\mathcal{GP}^i(A)$ is contravariantly finite in A-mod if and only if each simple A-module has a right $\mathcal{GP}^i(A)$-approximation. Let S be a simple A-module. If $\mathrm{Gpd}_A S \leq i$, we are done. Now suppose $\mathrm{Gpd}_A S > i$, then S is an A/I-module. Hence, there is a right $\mathcal{P}(A/I)$-approximation $B \to S$ of S. By Proposition 2.1(1), $B \to S$ is also a right $\mathcal{GP}^i(A)$-approximation of S. Hence, $\mathcal{GP}^i(A)$ is contravariantly finite in A-mod.

We denote by $\Omega(A\text{-mod})$ the subcategory consisting of the syzygy modules $\Omega(C)$ of all C in A-mod. For each $i \geq 0$, we denote by I_i the trace of $\Omega(A\text{-mod}) \cap \mathcal{GP}^i(A)$ in r, i.e., $I_i = \tau_{\Omega(A\text{-mod}) \cap \mathcal{GP}^i(A)}(r)$, where r denotes the radical of A. It is obvious that $I_i \subset r$ is an ideal in A.

Proposition 2.3. *If* $\operatorname{Gpd}_A I_i \leq i$, *then* $\mathcal{GP}^{i+1}(A)$ *is contravariantly finite in A*-mod.

Proof. First note that $I_i P = \tau_{\Omega(A\text{-mod}) \cap \mathcal{GP}^i(A)}(rP)$ for any projective A-module P, where rP is the radical of P.

Let $M \in \mathcal{GP}^{i+1}(A)$. Then we have the following exact sequence:

$$0 \to \Omega(M) \to P \xrightarrow{f} M \to 0$$

with f is a projective cover of M. Since $\operatorname{Gpd}_A \Omega(M) \leq i$, it follows that $\Omega(M) \in \Omega(A\text{-mod}) \cap \mathcal{GP}^i(A)$. Therefore, we get that $\Omega(M) \subset \tau_{\Omega(A\text{-mod}) \cap \mathcal{GP}^i(A)}(rP) = I_i P$ and that $P/I_i P \cong M/I_i M$, that is, $M/I_i M$ is a projective A/I_i -module. By hypothesis we have that $\operatorname{Gpd}_A A/I_i \leq i+1$. So the ideal I_i satisfies the conditions of Proposition 2.1. Since each simple A-module is annihilated by I_i, it follows that $\mathcal{GP}^{i+1}(A)$ is contravariantly finite in A-mod by Corollary 2.2.

Corollary 2.4. *Assume that* $\operatorname{Gpd}_A I_i \leq i$. *Then the A-modules of Gorenstein projective dimension* $\leq i+1$ *are the summand of modules M which have filtrations* $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ *such that each subquotient* M_j/M_{j+1} *is an indecomposable projective* A/I_i *-module.*

Proof. Note that the A/I_i -projective covers of the simple A-modules are just the minimal right $\mathcal{GP}^i(A)$-approximations of the simple A-modules by Proposition 2.1(1). Then by Proposition 3.8 in [3] we deduce the corollary.

Now we show that $\mathcal{GP}^i(A)$ is contravariantly finite in A-mod for $i \geq 1$ when A is stably equivalent to a hereditary algebra as an application of this sufficient condition as above.

Lemma 2.5. *Assume that A satisfies the conditions*:

(i) *If a simple A-module S is a composition factor of* $rP/socP$ *for some indecomposable projective module P*, *then S is a torsion module*.

(ii) *Every indecomposable torsionless module is simple or Gorenstein projective*.

Then it holds that $\mathrm{Gpd}_A I_i \leq i$ *for each* $i \geq 0$, *where*

$$I_i = \tau_{\Omega(A\text{-mod})\cap\mathcal{GP}^i(A)}(r).$$

Proof. By hypothesis we can get that

$$I_i = G \oplus S_1 \oplus \cdots \oplus S_t,$$

where G is Gorenstein projective and S_j are simple modules.

If $\mathrm{Gpd}_A I_i > i$, then there is a simple module $S := S_j$ satisfying that $\mathrm{Gpd}_A S > i$. By the construction of I_i, there is an epimorphism $f : M \to S$ with M indecomposable in $\Omega(A\text{-mod}) \cap \mathcal{GP}^i(A)$. Note that $f(socM) = 0$. Thus, S is a composition factor of $M/socM$. Since $M \subset rP$ for some projective module P, $M/socM \subset rP/socP$. It follows that S is a composition factor of $rP/socP$. By (i) S is a torsion module. This contradicts that $S \subset I_i$. Hence, $\mathrm{Gpd}_A I_i \leq i$ for each $i \geq 0$.

Proposition 2.6. *Let A be stably equivalent to a hereditary algebra*. *Then* $\mathcal{GP}^i(A)$ *is contravariantly finite in A*-mod *for* $i \geq 1$.

Proof. By Lemma 4.12 in [3] we get that A satisfies the conditions in Lemma 3.1. Then by Proposition 2.3 we get that $\mathcal{GP}^i(A)$ is contravariantly finite in A-mod for $i \geq 1$.

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