# SEMIGROUPS OF MATRICES CLOSED UNDER CONJUGATION BY NORMAL LINEAR GROUPS 

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#### Abstract

Let $X$ be a finite set, $T(X)$ be the monoid of all transformations on $X$ and $\operatorname{Sym}(X)$ be the symmetric group on $X$. Recently Levi, McAlister and McFadden proved that if $|X|>4, G$ is a normal subgroup of $\operatorname{Sym}(X)$ and $a \in T(X) \backslash \operatorname{Sym}(X)$, then


$$
\left\langle g^{-1} a g: g \in G\right\rangle=\langle\{a\} \cup G\rangle \backslash G .
$$

The aim of this paper is to prove a linear analogue of this result.

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## 1. Introduction

Let $M_{n}(F)$ be the monoid of all $n \times n$ matrices with entries in a field $F$ and let $G L_{n}(F)$ be its group of units.

It is well known that many results about the monoid $T(X)$ of all (total) transformations on a set $X$ have analogues on $M_{n}(F)$. For example, Howie [8] proved that $T(X) \backslash \operatorname{Sym}(X)$ is generated by idempotents (when $X$ is finite) and Erdos [4] proved that every singular matrix in $M_{n}(F)$ is the product of idempotent matrices. This similarity originated the so called Independence Algebras [6] which provide a common framework where it is possible to prove results that hold for both sets and vector spaces. For example, the unified proof for Howie's and Erdos's results referred to above appears in [5] (for a direct proof see [1]).

Let $S$ be a monoid with group of units $U$, let $a \in S \backslash U$ and let $G$ be a subgroup of $U$. Denote by $\langle a: G\rangle$ the subsemigroup of $S$ generated by $\left\{g^{-1} a g: g \in G\right\}$. Denote by $\langle a, G\rangle$ the subsemigroup of $S$ generated by $\{a\} \cup G$. Throughout this paper, the brackets $\rangle$ always mean "semigroup generated by".

Symons [13] and Levi and McFadden [9] generalized Howie's result by proving that, for a finite set $X$ and $a \in T(X) \backslash \operatorname{Sym}(X)$, the semigroup $\langle a: \operatorname{Sym}(X)\rangle$ is generated by its own idempotents and we have

$$
\langle a: \operatorname{Sym}(X)\rangle=\langle a, \operatorname{Sym}(X)\rangle \backslash \operatorname{Sym}(X)
$$

In [3] it is proved that, for every $a \in M_{n}(F) \backslash G L_{n}(F)$,

$$
\begin{equation*}
\left\langle a: G L_{n}(F)\right\rangle=\left\langle a, G L_{n}(F)\right\rangle \backslash G L_{n}(F)=\mathcal{I}_{a} \tag{1}
\end{equation*}
$$

where $\mathcal{I}_{a}$ denotes the set of all the matrices $b \in M_{n}(F)$ such that $\operatorname{rank}(b) \leq \operatorname{rank}(a)$.

Recently Levi et al. [10] proposed the following problem. Let $X$ be a finite set and let $a \in T(X) \backslash \operatorname{Sym}(X)$. Describe the subgroups $G$ of $\operatorname{Sym}(X)$ such that $\langle a: G\rangle=\langle a, G\rangle \backslash G$. Their partial answer implies that,
for $|X|>4$, this equality holds for the normal subgroups of $\operatorname{Sym}(X)$. The aim of this paper is to generalize this result for the case of linear groups.

Before moving to the linear case we complete the result of [10] referred to above.

Theorem 1 [10]. Let $a \in T(X) \backslash \operatorname{Sym}(X)$ and let $G \leq \operatorname{Sym}(X)$ be a normal subgroup.
(a) If $|X| \neq 4$, then $\langle a: G\rangle=\langle a, g\rangle \backslash G$.
(b) If $|X|=4$ and $G$ is the alternating group on $X$, then $\langle a: G\rangle=$ $\langle a, G\rangle \backslash G$ if and only if $|a(X)| \in\{1,3\}$.
(c) If $|X|=4$ and $G=\{1,(12)(34),(14)(23),(13)(24)\}$, then $\langle a: G\rangle=$ $\langle a, G\rangle \backslash G$ if and only if $|a(X)|=1$ or $|a(X)|=2, a^{2} \neq a$ and each Kerclass of a has exactly 2 elements.

Proof. For $X$ such that $|X|>4$, the proof appears in [10]. It is a matter of easy calculations to check the remaining cases.

## 2. The Main Result

Let $a \in M_{n}(F) \backslash G L_{n}(F)$. For every subgroup $G$ of $G L_{n}(F)$,

$$
\langle a: G\rangle \subseteq\langle a, G\rangle \backslash G \subseteq\left\langle a, G L_{n}(F)\right\rangle \backslash G L_{n}(F)
$$

The following theorem is our main result and will be proved later.
Theorem 2. Let $G$ be a normal subgroup of $G L_{n}(F)$.
(a) If $G$ has at least one nonscalar matrix, then

$$
\langle a: G\rangle=\langle a, G\rangle \backslash G=\left\langle a, G L_{n}(F)\right\rangle \backslash G L_{n}(F)
$$

(b) If $G$ is a group of scalar matrices and $a \neq 0$, then

$$
\begin{equation*}
\langle a, G\rangle \backslash G \neq\left\langle a, G L_{n}(F)\right\rangle \backslash G L_{n}(F) \tag{2}
\end{equation*}
$$

If $G$ is a group of scalar matrices and $a \neq 0$, then $\langle a: G\rangle=\langle a, G\rangle \backslash G$ in the conditions of Section 3.

Let $S L_{n}(F)$ be the special linear group, the group of all matrices $g \in G L_{n}(F)$ such that det $g=1$. According to [12], a subgroup $G$ of $G L_{n}(F)$ is normal if and only if one of the following conditions is satisfied:

- $G$ contains $S L_{n}(F)$.
- $n=2,|F|=2$ and $G$ is generated by

$$
\left[\begin{array}{ll}
1 & 1  \tag{3}\\
1 & 0
\end{array}\right]
$$

- $n=2,|F|=3$ and $G$ is generated by

$$
\left[\begin{array}{cc}
1 & 1  \tag{4}\\
1 & -1
\end{array}\right], \quad\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right]
$$

- $G$ is a group of scalar matrices.

Lemma 3. Let $G$ be a subgroup of $G L_{n}(F)$ containing $S L_{n}(F)$. Then $\left\langle a: G L_{n}(F)\right\rangle \subseteq\langle a: G\rangle$.

Proof. First, note that, if $e=g^{-1} a g$ and $H=g^{-1} G g$, for some $g \in G L_{n}(F)$, then $\left\langle e: G L_{n}(F)\right\rangle \subseteq\langle e: H\rangle$ implies $\left\langle a: G L_{n}(F)\right\rangle \subseteq\langle a: G\rangle$. As a matter of fact, if $\left\langle e: G L_{n}(F)\right\rangle \subseteq\langle e: H\rangle$, then it is not hard to see that $\left\langle a: G L_{n}(F)\right\rangle=g\left\langle e: G L_{n}(F)\right\rangle g^{-1} \subseteq g\langle e: H\rangle g^{-1}=\langle a: G\rangle$.

Case 1. Suppose that $a=[0] \oplus a_{0}$, where $a_{0} \in M_{n-1}(F)$. For every $\lambda \in F$, let $d_{\lambda}=\operatorname{diag}(\lambda, 1, \ldots, 1) \in M_{n}(F)$. Let $b \in\left\langle a: G L_{n}(F)\right\rangle$ and suppose that $b=g_{1}^{-1} a g_{1} \cdots g_{k}^{-1} a g_{k}$, where $g_{1}, \ldots, g_{k} \in G L_{n}(F)$. Then $b=h_{1}^{-1} a h_{1} \cdots h_{k}^{-1} a h_{k}$, where $h_{i}=\left(d_{\operatorname{det}} g_{i}\right)^{-1} g_{i} \in S L_{n}(F) \subseteq G$. Therefore $b \in\langle a: G\rangle$.

Case 2. Suppose that $a$ is similar to a matrix of the form $a^{\prime}=[0] \oplus a_{0}$, where $a_{0} \in M_{n-1}(F)$. Suppose that $a^{\prime}=g^{-1} a g$, where $g \in G L_{n}(F)$. As $G$ contains $S L_{n}(F), G^{\prime}=g^{-1} G g$ also contains $S L_{n}(F)$.

According to the previous case, $\left\langle a^{\prime}: G L_{n}(F)\right\rangle \subseteq\left\langle a^{\prime}: G^{\prime}\right\rangle$. According to the remark above, $\left\langle a: G L_{n}(F)\right\rangle \subseteq\langle a: G\rangle$.

Case 3. Now we shall consider the general case. Recall that $a$ is similar to a direct sum of a nilpotent matrix and a nonsingular matrix, and a nilpotent matrix is similar to a direct sum of singular Jordan blocks. For every positive integer $p$, let $J_{p}$ be the singular Jordan block

$$
\left[\begin{array}{cc}
0 & I_{p-1} \\
0 & 0
\end{array}\right] \in M_{p}(F)
$$

Suppose that $a$ is similar to $e=J_{p_{1}} \oplus \cdots \oplus J_{p_{u}} \oplus c$, where $c$ does not exist, if $a$ is nilpotent, and $c$ is nonsingular, otherwise. Suppose that $e=g^{-1} a g$, where $g \in G L_{n}(F)$.

For every positive integer $p$, choose a $p \times p$ matrix $y_{p}$ such that all the entries $(1, p),(2, p-1), \ldots,(p, 1)$ belong to $\{1,-1\}$, all the other entries are equal to 0 and $y_{p} \in S L_{p}(F)$. Then $y_{p}^{-1} J_{p} y_{p} J_{p}$ has the form $[0] \oplus e_{p}$ and has rank $p-1$.

Let $\quad y=y_{p_{1}} \oplus \cdots \oplus y_{p_{u}} \oplus I_{n-p_{1}-\cdots-p_{u}} \in S L_{n}(F)$. Then $f=y^{-1}$ eye has the form $[0] \oplus f_{0}$ and $\operatorname{rank} f=\operatorname{rank} e=\operatorname{rank} a$. Then

$$
\left\langle f: G L_{n}(F)\right\rangle=\left\langle e: G L_{n}(F)\right\rangle=\left\langle a: G L_{n}(F)\right\rangle=\mathcal{I}_{a}
$$

As $G$ contains $S L_{n}(F), H=g^{-1} G g$ also contains $S L_{n}(F)$. According to Case 1,

$$
\left\langle f: G L_{n}(F)\right\rangle \subseteq\langle f: H\rangle
$$

As $y \in S L_{n}(F) \subseteq H$ and $f=y^{-1}$ eye,

$$
\langle f: H\rangle \subseteq\langle e: H\rangle
$$

Therefore $\left\langle e: G L_{n}(F)\right\rangle \subseteq\langle e: H\rangle$. According to the remark at the beginning of this proof, $\left\langle a: G L_{n}(F)\right\rangle \subseteq\langle a: G\rangle$.

Proof of Theorem 2. (a) When $G$ contains $S L_{n}(F)$, the proof has already been done in Lemma 3.

Suppose that $n=2,|F|=2$ and $G$ is generated by (3). It is a matter of easy calculations to check that, for all $a \in M_{2}(F) \backslash G L_{2}(F)$, $\langle a, G\rangle \backslash G=\mathcal{I}_{a}$. By [2] (or [4] together with Lemma 1 of [7]), it follows that

$$
\langle a, G\rangle \backslash G=\mathcal{I}_{a}=\left\langle E\left(\mathcal{I}_{a}\right)\right\rangle,
$$

where $E\left(\mathcal{I}_{a}\right)$ denotes the set of the idempotents of $\mathcal{I}_{a}$. Since in [11] it is proved that $\langle E(\langle a: G\rangle)\rangle=\langle E(\langle a, G\rangle \backslash G)\rangle$, it follows that

$$
\langle a: G\rangle \subseteq\langle a, G\rangle \backslash G=\langle E(\langle a, G\rangle \backslash G)\rangle=\langle E(\langle a: G\rangle)\rangle \subseteq\langle a: G\rangle
$$

As $\mathcal{I}_{a}=\langle a, G\rangle \backslash G \subseteq\left\langle a, G L_{2}(F)\right\rangle \backslash G L_{2}(F) \subseteq \mathcal{I}_{a}$, it follows that

$$
\begin{equation*}
\left\langle a, G L_{2}(F)\right\rangle \backslash G L_{2}(F)=\langle a, G\rangle \backslash G=\langle E(\langle a: G\rangle)\rangle=\langle a: G\rangle=\mathcal{I}_{a} \tag{5}
\end{equation*}
$$

Finally, suppose that $n=2,|F|=3$ and $G$ is generated by (4). Again, it is a matter of easy calculations to check that, for all $a \in M_{2}(F) \backslash G L_{2}(F), \quad\langle a, G\rangle \backslash G=\mathcal{I}_{a}, \quad$ where $\mathcal{I}_{a}$ denotes the principal ideal generated by $a$. Therefore, by repeating the arguments used above, we deduce that (5) is satisfied.
(b) Suppose that $a \neq 0$ and $G$ is a subgroup $G L_{n}(F)$ of scalar matrices. Then $G=\left\{\lambda I_{n}: \lambda \in \Lambda\right\}$, for some subgroup $\Lambda$ of $F \backslash\{0\}$ and $\langle a, G\rangle \backslash G=\langle\lambda a: \lambda \in \Lambda\rangle$.

As $a$ is singular, there exists $g \in G L_{n}(F)$ such that the last column of $a g$ is equal to zero. Then $a$ is similar to $b=g^{-1} a g$.

Suppose that (2) is false. Then

$$
\langle\lambda a: \lambda \in \Lambda\rangle=\langle a, G\rangle \backslash G=\left\langle a, G L_{n}(F)\right\rangle \backslash G L_{n}(F)=\mathcal{I}_{a}
$$

As $b=g^{-1} a g$, it is easy to deduce that $\langle\lambda b: \lambda \in \Lambda\rangle=\mathcal{I}_{b}=\mathcal{I}_{a}$, what is impossible, because all the matrices $\lambda b$ have the last column equal to zero.

## 3. The Scalar Case

It remains to study the equality

$$
\begin{equation*}
\langle a: G\rangle=\langle a, G\rangle \backslash G, \tag{6}
\end{equation*}
$$

when $G$ is a group of scalar matrices. This is the aim of this section.
Let $a \in M_{n}(F) \backslash G L_{n}(F)$. Let $G$ be a subgroup of $G L_{n}(F)$ of scalar matrices, that is,

$$
G=\left\{\lambda I_{n}: \lambda \in \Lambda\right\},
$$

for some subgroup $\Lambda$ of $F \backslash\{0\}$. Then

$$
\langle a: G\rangle=\langle a\rangle \quad \text { and } \quad\langle a, G\rangle \backslash G=\langle\lambda a: \lambda \in \Lambda\rangle .
$$

The equality (6) is trivial if either $a=0$ or $\Lambda=\{1\}$.
Let $\bar{F}$ be the algebraic closure of $F$ and let $g \in G L_{n}(\bar{F})$ such that $g^{-1} a g$ is the Jordan canonical form of $a$. Suppose that

$$
g^{-1} a g=J_{r_{1}}\left(\sigma_{1}\right) \oplus \cdots \oplus J_{r_{t}}\left(\sigma_{t}\right),
$$

where

$$
J_{r_{i}}\left(\sigma_{i}\right)=\sigma_{i} I_{r_{i}}+\left[\begin{array}{cc}
0 & I_{r_{i}-1} \\
0 & 0
\end{array}\right] \in M_{r_{i}}(\bar{F}), \quad i \in\{1, \ldots, t\} .
$$

Recall that the elementary divisors of $a$, over $\bar{F}$, are the polynomials

$$
\left(x-\sigma_{1}\right)^{r_{1}}, \ldots,\left(x-\sigma_{t}\right)^{r_{t}} .
$$

The equality (6) is satisfied if and only if, for every $\lambda \in \Lambda$, there exists a positive integer $k$ such that $\lambda a=a^{k}$. Note that $\lambda a=a^{k}$ is equivalent to

$$
\begin{equation*}
\lambda J_{r_{i}}\left(\sigma_{i}\right)=\left(J_{r_{i}}\left(\sigma_{i}\right)\right)^{k}, \quad i \in\{1, \ldots, t\} . \tag{7}
\end{equation*}
$$

Let $\lambda \in \Lambda$ and $\sigma \in \bar{F}$. Let $r$ and $k$ be positive integers. It is not hard to calculate that $\left(J_{r}(\sigma)\right)^{k}$ is an upper triangular matrix with its $(u, v)$
entry, where $1 \leq u \leq v \leq r$, equal to

$$
\begin{array}{ll}
0, & \text { if } k<u-v, \\
\frac{k!}{(u-v)!(k-u+v)!} \sigma^{k-u+v}, & \text { if } k \geq u-v
\end{array}
$$

In the following results we use the remarks above to study the equality (6), although it does not seem easy to write a simple and elegant solution.

Let $\Lambda_{a}$ be the semigroup of all $\lambda \in F$ such that there exists a nonnegative integer $l$ such that, for every nonzero eigenvalue $\sigma$ of $a$ in $\bar{F}, \lambda=\sigma^{l}$.

Theorem 4. Suppose that a is diagonalizable over $\bar{F}$. Then (6) holds if and only if $\Lambda \subseteq \Lambda_{a}$.

Proof. Suppose that (6) holds. Let $\lambda \in \Lambda$. As $\lambda a \in\langle a: G\rangle=\langle a\rangle$, there exists a positive integer $k$ such that $\lambda a=a^{k}$. Then (7) is satisfied. From (7) it follows that, for every $i \in\{1, \ldots, t\}, \lambda \sigma_{i}=\sigma_{i}^{k}$. Therefore, for every $i \in\{1, \ldots, t\}$ such that $\sigma_{i} \neq 0, \lambda=\sigma_{i}^{k-1}$.

Conversely, suppose that $\Lambda \subseteq \Lambda_{a}$. Let $\lambda \in \Lambda \subseteq \Lambda_{a}$ and let $l$ be a nonnegative integer such that, for every nonzero eigenvalue $\sigma$ of $a$ in $\bar{F}$, $\lambda=\sigma^{l}$. Let $k=l+1$. It is not hard to see that (7) is satisfied. Note that, as $a$ is diagonalizable, $t=n$ and $r_{1}=\cdots=r_{n}=1$. Then $\lambda a=a^{k}$, that is, $\lambda a \in\langle a\rangle=\langle a: G\rangle$. It follows that (6) holds.

Theorem 5. Suppose that $F$ has characteristic 0. Also suppose that $\Lambda \neq\{1\}$. Then (6) holds if and only if $a$ is diagonalizable over $\bar{F}$ and $\Lambda \subseteq \Lambda_{a}$.

Proof. Bearing in mind Theorem 4, it remains to prove that, if (6) holds, then $a$ is diagonalizable.

Suppose that (6) holds. Let $\lambda \in \Lambda \backslash\{1\}$. As $\lambda a \in\langle a: G\rangle=\langle a\rangle$, there exists a positive integer $k$ such that $\lambda a=a^{k}$. Then (7) is satisfied.

Suppose that there exists $i \in\{1, \ldots, t\}$ such that $r_{i} \geq 2$. Bearing in mind the form of the powers of the Jordan blocks, we deduce that $\lambda \sigma_{i}=\sigma_{i}^{k}$ and $\lambda=k \sigma_{i}^{k-1}$. If $\sigma_{i}=0$, we would have $\lambda=0$, a contradiction. Therefore $\quad \sigma_{i} \neq 0$. Then $\sigma_{i}^{k-1}=\lambda=k \sigma_{i}^{k-1} \quad$ and $1=k \quad$ so that $\lambda=k \sigma_{i}^{k-1}=1$, a contradiction. Therefore $r_{i}=1$, for every $i \in\{1, \ldots, t\}$, and $a$ is diagonalizable over $\bar{F}$.

Lemma 6. Suppose that $\Lambda \neq\{1\}$. If (6) holds, then every elementary divisor of a, that is, a power of $x$ has degree 1 .

Proof. Suppose that (6) holds and there exists $i \in\{1, \ldots, t\}$ such that $r_{i} \geq 2$ and $\sigma_{i}=0$. Let $\lambda \in \Lambda \backslash\{1\}$. As $\lambda a \in\langle a: G\rangle=\langle a\rangle$, there exists a positive integer $k$ such that $\lambda a=a^{k}$ and $\lambda J_{r_{i}}(0)=\left(J_{r_{i}}(0)\right)^{k}$. A simple calculation shows that this equality is impossible.

Corollary 7. Suppose that $\Lambda \neq\{1\}$ and that $a$ is nilpotent. Then (6) holds if and only if $a=0$.

Proof. It follows trivially from Lemma 6. Recall that $a$ is nilpotent if and only if all the eigenvalues of $a$ are equal to 0 .

Proposition 8. Suppose that $\Lambda \neq\{1\}$. Let $i \in\{1, \ldots, t\}$. If (6) holds, $r_{i} \geq 2$ and $\sigma_{i} \neq 0$, then $\sigma_{i}$ is a root of the unity.

Proof. Let $\lambda \in \Lambda \backslash\{1\}$. As $\lambda a \in\langle a: G\rangle=\langle a\rangle$, there exists a positive integer $k$ such that $\lambda a=a^{k}$ and $\lambda J_{r_{i}}\left(\sigma_{i}\right)=\left(J_{r_{i}}\left(\sigma_{i}\right)\right)^{k}$. Then $\lambda \sigma_{i}=\sigma_{i}^{k}$ and $\lambda=k \sigma_{i}^{k-1}$. As $\lambda \neq 1$, it follows that $k \geq 2$ and $\lambda=\sigma_{i}^{k-1}$, where $k-1$ is a positive integer. Analogously, as $\lambda^{-1} \in \Lambda \backslash\{1\}$, there exists a positive integer $l$ such that $\lambda^{-1}=\sigma_{i}^{l}$. Hence $1=\sigma_{i}^{l+k-1}$.

Define $r_{a}$ as follows:

$$
\begin{array}{ll}
r_{a}=0, & \text { if } a \text { is nilpotent, } \\
r_{a}=\max \left\{r_{i}: i \in\{1, \ldots, t\}, \sigma_{i} \neq 0\right\}, & \text { otherwise }
\end{array}
$$

Theorem 9. Suppose that $F$ has characteristic $p \neq 0$. Also suppose that $\Lambda \neq\{1\}$. Then (6) holds if and only if
(a) every elementary divisor of a, that is, a power of $x$ has degree 1, and
(b) for every $\lambda \in \Lambda$, there exists a nonnegative integer $l$ such that

- for every nonzero eigenvalue $\sigma$ of $a$ in $\bar{F}, \lambda=\sigma^{l}$,
$-p$ divides $l$, if $r_{a} \geq 2$,
- $p$ divides $\frac{(l+1)!}{s!(l-s+1)!}$, if $s \in\left\{2, \ldots, r_{a}-1\right\}$ and $r_{a} \geq 3$.

Proof. The proof follows easily from the remarks above and using arguments already applied previously.

Necessity. Condition (a) has been proved in Lemma 6. Let $\lambda \in \Lambda$. As $\lambda a \in\langle a: G\rangle=\langle a\rangle$, there exists a positive integer $k$ such that $\lambda a=a^{k}$ and (7) is satisfied. It is not hard to see that (b) is satisfied, with $l=k-1$.

Sufficiency. Let $\lambda \in \Lambda$ and choose a nonnegative integer $l$ satisfying the conditions of (b). Let $k=l+1$. It is not hard to see that (7) is satisfied. Then $\lambda a=a^{k}$, that is, $\lambda a \in\langle a\rangle=\langle a: G\rangle$. Therefore (6) holds.

Examples 10. If $|F|=2$, then there is a unique group of scalar matrices, $G=\left\{I_{n}\right\}$. Therefore, (6) holds.

Suppose that $|F|=3$ and $n=2$ and let

$$
G=\left\{I_{2},-I_{2}\right\}=\left\{\lambda I_{2}: \lambda \in \Lambda=\{1,-1\}\right\}
$$

the unique nontrivial group of scalar matrices. Let $a \in M_{n}(F) \backslash G L_{n}(F)$. The canonical Jordan form of $a$ has one of the following forms:

$$
0, \quad \operatorname{diag}(1,0), \quad \operatorname{diag}(-1,0), \quad J_{2}(0)
$$

- If $a=0$, then (6) holds.
- If $a$ is similar to $\operatorname{diag}(1,0)$, then $\Lambda \nsubseteq \Lambda_{a}=\{1\}$; therefore (6) does not hold.
- If $a$ is similar to $\operatorname{diag}(-1,0)$, then $\Lambda=\Lambda_{a}$ and (6) holds.
- If $a$ is similar to $J_{2}(0)$, then (6) does not hold, according to Lemma 6.


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