**Far East Journal of Mathematical Sciences (FJMS)**
Volume 68, Number 1, 2012, Pages 127-135
Published Online: July 2012
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

# INTERACTION THE BURGERS' EQUATION WITH FINITE DIFFERENCE SCHEME

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## Abstract

Applying the finite difference scheme for the Burgers' equation is evaluated numerically. Analyzing the unconditionally stable of the finite difference form by Von Neumann stability analysis and examining the truncation error are shown. Two cases of this problem are numerated by invariants of motion and the results compare with the test problem that the results are implemented and effective.

## 1. Introduction

The Burgers' equation is used in fluid dynamics teaching and in engineering as a simplified model for turbulence, boundary layer behavior, shock wave formation, and mass transport. This equation has been studied and applied for many decades. Many different closed-form, series



2010 Mathematics Subject Classification: 65M06.

Keywords and phrases: finite difference, shock wave, Burgers' equation, marginally stable.

This work is supported by the grants from Science Faculty, Srinakarinwirot University.

Received January 22, 2012; Revised February 27, 2012

approximation, and numerical solutions are known for particular sets of boundary conditions. Many problems can be modeled by this equation as follows: the Burgers' equation is nonlinear model of the incompressible momentum equations, a one-dimensional analogue of the Navier-Stokes equations. The Burgers' equation is evaluated exactly for an arbitrary initial and boundary conditions in [1-5]. These analytic solutions are impractical for the small values of viscosity constant due to slow convergence of serious solutions, which was illustrated in the study of Miller [6]. Thus many numerical methods are constructed to get solutions of the Burgers' equation for small values of viscosity constant which corresponds to steep front in the propagation of dynamic wave forms. Recently, the mathematicians used many methods to solve this equation in the following:

In 2004, Aksan and Ozdes [7] solved a variational method constructed on the method of discretization in time. The numerical results obtained by these ways for various values of viscosity have been compared with the exact solution. Dogan [8] applied Galerkin finite element method for the numerical solution of Burgers' equation. A linear recurrence relationship for the numerical solution of the resulting system of ordinary differential equations is found via a Crank-Nicolson approach involving a product approximation. In 2005, Dag et al. [9] used cubic B-spline collocation method and applied to the time-splitted Burgers' equation. In 2006, Javidi [10] had a new method for solving of the Burger's equation by combination of method of lines (MOL) and matrix free modified extended backward differential formula (MF-MEBDF). The method of lines semi-discretization approach is used to transform the model partial differential equations (PDEs) in a system of first order ordinary differential equations (ODEs). Chen and Wu [11] developed a kind of univariate multiquadric (MQ) quasi-interpolation and used it to solve Burgers' equation (with viscosity). They constructed the MQ quasi-interpolation, which possesses the properties of linear reproducing and preserving monotonicity. In 2009, Dhawan [12] solved numerically using a finite element method, where a combination of cubic B-splines is used as an approximating function. Different comparisons for the test problem in hand are made to validate the proposed numerical technique. Ohwada [13], solving the Burgers' equation via the diffusion equation is proposed. The time variation of numerical solution is given by a rational function. The coefficients of the polynomials in the denominator and numerator are determined by simple algebra. In the case of vanishingly small viscosity, the numerical method becomes shock-capturing only by increasing the viscosity to the order of mesh spacing locally around shocks. Zhu and Wang [14] used the derivative of the quasi-interpolation to approximate the spatial derivative of the dependent variable and a low order forward difference to approximate the time derivative of the dependent variable. In 2010, Jiang and Wang [15] used the cubic B-spline quasi-interpolation and the compact finite difference method to find the solution of the Burgers' equation. They used the derivative of the quasi-interpolation to approximate the spatial derivative and a two-order compact scheme to approximate the time derivative.

In this paper, we consider the Burgers' equation

$$u_t + uu_x - \nu u_{xx} = 0, \tag{1}$$

where $\nu$ is the kinematics viscosity of the fluid, $x$ and $t$ are differentiation. With the initial and boundary conditions as follows:

$$u(x, 0) = f(x), \quad a \leq x \leq b, \tag{2}$$

$$u(a, t) = \beta_1, \ u(b, t) = \beta_2, \ t \in [0, T]. \tag{3}$$

We applied the finite difference scheme for this equation, check stability with linearized Fourier method and also show the truncation error. Finally, we choose two problems for the test and compare with the analytic solution.

## 2. Finite Difference Method and its Stability

Letting $g(u) = \frac{1}{2}u^2$ and substituting in equation (1) yield

$$u_t + g_x - \nu u_{xx} = 0. \tag{4}$$

Applying the finite difference scheme into equation (4), we get

$$(u_t)_m^n + \frac{1}{2}[(g_x)_{m+1}^n + (g_x)_m^n] - \frac{\nu}{2}[(u_{xx})_{m+1}^n + (u_{xx})_m^n] = 0, \tag{5}$$

with

$$(u_t)_m^n = \frac{u_m^{n+1} - u_m^n}{k},$$

$$(g_x)_m^n = \frac{g_m^{n+1} - g_m^{n-1}}{2h},$$

$$(u_{xx})_m^n = \frac{u_m^{n+1} - 2u_m^n + u_m^{n-1}}{h^2}, \tag{6}$$

where $h$ and $k$ are the mesh sizes in space and time. Substituting equation (6) into equation (5), we get the new finite difference form as

$$Au_{m+1}^{n-1} + Bu_{m+1}^n + Cu_{m+1}^{n+1} = (B-1)u_m^{n-1} + (2-B)u_m^n + (B-1)u_m^{n+1}, \tag{7}$$

where

$$A = -\frac{k}{4h}u_{m+1}^{n+1} - \frac{\nu k}{2h^2},$$

$$B = 1 + \frac{\nu k}{2h^2},$$

$$C = \frac{k}{4h}u_{m+1}^{n-1} - \frac{\nu k}{2h^2}.$$

**Lemma.** *The local truncation error of the finite difference scheme equation* (7) *is* $O(h^2 + k^2)$, *if* $u(x, t)$ *is smooth enough and this method in equation* (4) *is a marginally stable*.

**Proof.** Using the Von Neumann analysis to verified the difference scheme equation (7) and applied Fourier method in the form

$$u_m^n = p^n e^{im\omega}, \tag{8}$$

where $i$ is an imaginary unit, $\omega$ is an arbitrary real number, and $p = p(\omega)$ is a complex number whose value must be found.

Then substituting equation (8) into equation (7), we get a new form

$$p^{n+1} = q^2 p^{n-1}, \tag{9}$$

where $q$ is the growth factor which in the relation as follows:

$$q^2 - 2i \sin \psi + 1 = 0, \tag{10}$$

where $\sin \psi = \dfrac{D \sin m\omega}{(B-1)\cos m\omega - 2 + B}$, $D = \dfrac{C}{A}$ and $i = \sqrt{-1}$.

From equation (10), we get $|q_1| = |q_2| = 1$ implied that the finite difference method is marginally stable.

Let $v_m^n = u(x_m, t_n)$ be the analytical solution of equation (1) and $x$, $t$ be independent variables. Substitute $v_m^n$ into finite difference form of equation (1), then we get

$$t_m^n = (v_m^n)_{\hat{t}} + (v_m^n)(v_m^n)_{\hat{x}} - \nu(v_m^n)_{\hat{x}\hat{x}}. \tag{11}$$

Applying Taylor's expansion at $(x_m, t_n)$ becomes

$$T_m^n = (v_t + vv_x - vv_{xx})\big|_{(x_m, t_n)} + \frac{k^2}{6} v_{ttt}\Big|_{(x_m, t_n)} + \frac{h^2}{6} vv_{xxx}\Big|_{(x_m, t_n)} + \cdots. \tag{12}$$

Thus the local truncation error is $T_m^n = O(h^2 + k^2)$.

## 3. The Error Norms and Numerical Approximation

The test problems are studied in order to show the strongest and numerical accuracy of the proposed methods. Using the error norms $L_2$ and $L_\infty$ measure the accuracy that are shown in the following:

$$L_2 = |U - U_N|^2 = h \sum_{m=0}^{N} |U_m - (U_N)_m^n|^2,$$

$$L_\infty = |U - U_N|^\infty = \max_m |U_m - (U_N)_m^n|.$$

**Problem 1.** The exact solution (Shock-like solution) of the Burgers' equation (1) is

$$u(x, t) = \frac{x/t}{1 + \sqrt{(t/t_0)}\, e^{(x^2/4\nu t)}}, \quad 0 \le x \le 1, \quad t \ge 1, \tag{13}$$

where $e^{(1/8\nu)}$, with the initial condition at time $t = 1$ and boundary conditions are $u(0, t) = u(1, t) = 0$. We used $h = 0.01$, $\nu = 0.005$ to solve this problem. The time runs from 1.7 to 3.25. These values still in the interval $[0, 1]$ and the results of exact solution and approximate solution are shown in Table 3.1. Results of this problem compared with analytic solutions and compared with two norms of CBCM. The exact solutions for such $t$ compared with numerical solutions are smallest error. In time $t = 1.7$, $x =$ 0.3, 0.5, 0.9, $t = 2.5$, $x = 0.1, 0.2, 0.4, 0.5, 0.6$ and $t = 3.25$, $x = 0.1, 0.2,$ 0.3, 0.5, 0.6, the results of analytic solutions equal to numerical solutions. Error norms of the presented solutions are better than CBCM with [9].

**Table 3.1.** Comparison of solutions in different time when $\nu = 0.005$ with $h = 0.005$ and $\Delta t = 0.01$

| $x$ | Exact | Numerical | Exact | Numerical | Exact | Numerical |
|---|---|---|---|---|---|---|
| | $t = 1.7$ | $t = 1.7$ | $t = 2.5$ | $t = 2.5$ | $t = 3.25$ | $t = 3.25$ |
| 0.1 | 0.05882 | 0.05883 | 0.04000 | 0.04000 | 0.03077 | 0.03077 |
| 0.2 | 0.11765 | 0.11765 | 0.08000 | 0.08000 | 0.06154 | 0.06154 |
| 0.3 | 0.17646 | 0.17646 | 0.12000 | 0.12001 | 0.09231 | 0.09231 |
| 0.4 | 0.23517 | 0.23516 | 0.15998 | 0.15998 | 0.12307 | 0.12306 |
| 0.5 | 0.29192 | 0.29192 | 0.19983 | 0.19983 | 0.15380 | 0.15380 |
| 0.6 | 0.29591 | 0.29593 | 0.23812 | 0.23812 | 0.18430 | 0.18430 |
| 0.7 | 0.04193 | 0.04211 | 0.25310 | 0.25234 | 0.21270 | 0.21271 |
| 0.8 | 0.00065 | 0.00064 | 0.10210 | 0.10225 | 0.21844 | 0.21838 |
| 0.9 | 0.00000 | 0.00000 | 0.00554 | 0.00562 | 0.10126 | 0.10125 |
| $L_2 \times 10^3$ | | 2.46636 | | 2.11154 | | 1.86254 |
| $L_\infty \times 10^3$ | | 26.5260 | | 23.9634 | | 20.5387 |
| $L_2 \times 10^3$ CBCM | | 2.46642 | | 2.11870 | | 1.92482 |
| $L_\infty \times 10^3$ CBCM | | 27.5770 | | 25.1517 | | 21.0489 |

**Problem 2.** Consider the Burgers' equation (1) with initial condition

$$u(x, 0) = \sin \pi x, \quad 0 < x < 1, \tag{14}$$

and boundary conditions

$$u(0, t) = u(1, t) = 0, \quad t > 0. \tag{15}$$

The exact solution with boundary conditions from equations (2) and (3) are obtained as

$$u(x, 0) = 2\pi v \frac{\sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 vt} n \sin n\pi x}{a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 vt} n \cos n\pi x}, \quad n = 1, 2, \ldots, \tag{16}$$

where $a_0$ and $a_n$ are the Fourier coefficients and defined as the following [3]:

$$a_0 = \int_0^1 e^{-(2\pi v)^{-1}[1-\cos \pi x]} dx,$$

$$a_n = 2\int_0^1 e^{-(2\pi v)^{-1}[1-\cos \pi x]} \cos n\pi x dx.$$

In this problem, we used $v = 0.01$, $t = 1$ and $h = 0.01$. Table 3.2 shows the comparison of the numerical method with exact solution and other methods such as MQQI [11] and BSQI [14]. The MQQI methods used the slope parameter $c = 2.9 \times 10^{-3}$. We observed that the numerical solutions are better than MQQI and BSQI. The numerical solutions are closed to the analytic solutions.

## 4. Conclusions

In this paper, we get the efficient numerical method for the Burgers' equation which compared with exact solutions and other methods are acceptable. The comparison of two problems with analytic solutions is getting the small error. Therefore, we can conclude that the difference scheme is reliable and efficient to find the numerical results of the Burgers' equation.

**Table 3.2.** Comparison of solutions in different time when $v = 0.01,\ t = 1$ and $h = 0.01$

| $x$ | Exact | Numerical | MQQI | BSQI |
|---|---|---|---|---|
| 0.1 | 0.0754 | 0.07548 | 0.07868 | 0.07530 |
| 0.2 | 0.1506 | 0.15035 | 0.15202 | 0.15049 |
| 0.3 | 0.2257 | 0.22563 | 0.22554 | 0.22554 |
| 0.4 | 0.3003 | 0.29973 | 0.29904 | 0.30002 |
| 0.5 | 0.3744 | 0.37452 | 0.37226 | 0.37407 |
| 0.6 | 0.4478 | 0.44746 | 0.44484 | 0.44742 |
| 0.7 | 0.5203 | 0.51992 | 0.51643 | 0.51985 |
| 0.8 | 0.5915 | 0.59108 | 0.58622 | 0.59106 |
| 0.9 | 0.6600 | 0.65552 | 0.62956 | 0.65964 |

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