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ASYMPTOTIC DENSITIES IN NUMBER THEORY. PART I: A SURVEY

(*Dedicated to A. Fuchs and G. Letta*)

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Abstract

In this paper, we present a detailed study of the asymptotic and conditional asymptotic densities and give some applications. Some new existence criteria are established. We give some new results that prove and simplify those obtained by Diaconis [Weak and strong averages in probability and the theory of numbers, Ph.D. Dissertation, Harvard University, Cambridge, Mass., 1974].

1. Prelude

We consider in this approach a family $\mathfrak{R} = \{\mu_\alpha, \alpha \in T\}$ of σ-finitely additive probability measures on the set $\wp(\mathbb{N}^*)$ of subsets of $\mathbb{N}^*$ where for a subset E of $\mathbb{N}^*$,

$$\mu_\alpha(E) := \nu_n(E) := \frac{1}{n}\sum_{k=1}^{n} I_E(k)$$

and I_E is the indicator function of the subset E.

This family $\nu = \{\nu_n, n \geq 1\}$, where ν_n is the uniform probability

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measure on $\{1, ..., n\}$, is well known and studied. It is called a family of frequencies.

By taking the limit when n tends to infinity, we diffuse the considered measure, and we obtain that we call an asymptotic density. In this paper, we present a detailed study of asymptotic and conditional asymptotic densities. We give some new existence criteria and some applications.

2. Main Results

2.1. Asymptotic density of a subset E of $\mathbb{N}^*$

Definition 2.1. Let E be a subset of $\mathbb{N}^*$. We consider, for an integer $n \geq 1$, the expression defined by

$$\nu_n(E) := \frac{1}{n}\sum_{k=1}^{n} I_E(k),$$

where $I_E(k)$ is the indicator function of the set E.

(a) We say that E has the number ℓ as a lower asymptotic density, if

$$\ell = \liminf_n \nu_n(E),$$

when n tends to $+\infty$. We denote this limit by $\underline{d}(E)$; notice that this limit belongs to [0, 1].

(b) We say that E has the number ℓ as an upper asymptotic density, if

$$\ell = \limsup_n \nu_n(E),$$

when n tends to $+\infty$. We denote this limit by $\overline{d}(E)$; notice that this limit belongs to [0, 1].

Proposition 2.1. *Let E be a subset of $\mathbb{N}^*$. Then*

$$0 \leq \underline{d}(E) \leq \overline{d}(E) \leq 1.$$

We give a theorem which characterizes the set of upper and lower densities; its proof requires several constructions.

Theorem 2.2 ([1, pp. 38-39], [4, p. 179]). (1) *Let E be an infinite subset of* $\mathbb{N}^*$, *and let F be a subset of E. We put*

$$\overline{d}(E) = s \quad \text{and} \quad \underline{d}(E) = i.$$

Then, the set of couples

$$S(E) := \{(\overline{d}(F), \underline{d}(F)) \in [0, 1] \times [0, 1] \subset \mathbb{R}^2 : F \subset E\}$$

possesses the following properties:

(a) $S(E)$ *is enclosing in the closed trapezium of vertices*:

$$(0; 0), \quad (s; 0), \quad (s; i), \quad (i; i);$$

(b) $S(E)$ *contains the closed triangle of vertices*:

$$(0; 0), \quad (s; 0), \quad (s; i);$$

(c) $S(E)$ *is a convex set*;

(d) $S(E)$ *is a closed set.*

(2) *Conversely, consider two real numbers s and i such that*

$$0 \leq i \leq s \leq 1,$$

and a subset S of $\mathbb{R}^2$ *satisfies* (a)-(d) *as above, then there exists a subset E with*

$$\overline{d}(E) = s, \quad \underline{d}(E) = i \quad \text{and} \quad S(E) = S.$$

We are interested in the asymptotic behaviour of the sequence $(\nu_n(E))_{n \geq 1}$.

Proposition 2.3. *The set of limit points of the sequence* $(\nu_n(E))_{n \geq 1}$ *is the interval* $[\underline{d}(E), \overline{d}(E)]$.

Definition 2.2. We say that a subset E of $\mathbb{N}^*$ admits an asymptotic density ℓ, if the numerical sequence $(\nu_n(E))_{n \geq 1}$ admits a limit equal to ℓ, (necessarily this limit belongs to [0, 1]), when n tends to $+\infty$. If this is the case, we shall denote this limit by $d(E)$, and we shall say that $d(E)$ is an asymptotic density of the set E.

We write $\mathfrak{D}$ for a class of subsets of $\mathbb{N}^*$ which has an asymptotic density.

2.2. Some applications

Proposition 2.4. *The set E of even natural integers admits an asymptotic density $d(E)$, and $d(E) = \frac{1}{2}$.*

Proposition 2.5. *Generally, for all $m \in \mathbb{N}^*$, the class $m\mathbb{N}^*$ of integral multiples of m (or divisible by m) belongs to $\mathfrak{D}$, and $d(m\mathbb{N}^*) = \frac{1}{m}$.*

Theorem 2.6. *For an integer $b \geq 2$, the set*

$$E_b = \bigcup_{k \geq 0} E_k,$$

where $E_k = \{n : b^{2k} \leq n < b^{2k+1}\}$, $k \geq 0$ does not admit an asymptotic density.

Proof. It is enough to construct two subsequences of the sequence $(\nu_n(E))_{n \geq 1}$ which converge to two distinct limits. We choose as subsequences those whose indices coincide with the antecedents and the consequents of connected components $[b^{2r}, b^{2r+1}[$ of $\mathbb{N}^*$, i.e., with $b^{2r} - 1$ and b^{2r+1}.

– Subsequence indexed by $n = b^{2r} - 1$, $r \geq 1$.

$$N_n(E) = card(E \cap \{1, \ldots, b^{2r-1}\}) = card\left(\bigcup_{k=0}^{r-1} [b^{2k}, b^{2k+1}[\right)$$

$$= \sum_{k=0}^{r-1} (b^{2k+1} - b^{2k}) = (b-1) \sum_{k=0}^{r-1} (b^2)^k = \frac{b^{2r} - 1}{b+1}$$

$$\nu_n(E) = \frac{N_n(E)}{n} = \frac{b^{2r} - 1}{b+1} \cdot \frac{1}{b^{2r} - 1} = \frac{1}{b+1}$$

and

$$\underline{d}(E) = \frac{1}{b+1}.$$

– Subsequence indexed by $n = b^{2r+1}$, $r \geq 0$.

$$N_n(E) = (b-1)\sum_{k=0}^{r} b^{2k} = (b-1)\sum_{k=0}^{r} (b^2)^k = (b-1)\frac{b^{2(r+1)}-1}{(b^2-1)}.$$

$$\nu_n(E) = \frac{N_n(E)}{n} = \frac{1}{b^{2r+1}}\frac{b^{2(r+1)}-1}{b+1} = \frac{1}{b+1}\frac{b^{2r+2}-1}{b^{2r+1}} = \frac{b}{b+1}(1-b^{-2r-2})$$

and

$$\overline{d}(E) = \frac{b}{b+1}.$$

So, the sequence $(\nu_n(E))_{n\geq 1}$ does not converge, and hence $E \notin \mathfrak{D}$.

Below are examples of subsets of $\mathbb{N}^*$ which do not possess asymptotic density.

Corollary 2.7. *Let E be the set of natural integers whose decimal development contains an odd number of digits. Then, E does not have an asymptotic density.*

Proof. We write E as disjoint union of its connected components. Let m be an element of E with $(2k+1)$ digits. Then

$$E = \bigcup_{k\geq 0} E_k, \quad E_k = [10^{2k}, 10^{2k+1}[\text{ with } E_i \cap E_j = \varnothing \text{ for all } i \neq j.$$

We prove that E does not admit an asymptotic density. It is enough to construct two subsequences namely those whose indices coincide with the antecedents and the consequents of connected components $[10^{2k}, 10^{2k+1}[$, i.e., with $10^{2k} - 1$ and 10^{2k+1}. The results follow from Theorem 2.6:

• Subsequence indexed by $n = 10^{2r} - 1$, $r \geq 1$. Then $\underline{d}(E) = \frac{1}{11}$.

• Subsequence indexed by $n = 10^{2r+1}$, $r \geq 0$. Then $\overline{d}(E) = \frac{10}{11}$. It follows that the sequence $(\nu_n(E))_{n \geq 1}$ does not converge, and hence E does not admit an asymptotic density.

Corollary 2.8. *Let E be the set of natural integers whose binary development contains an odd number of digits. Then, E does not admit an asymptotic density.* In other words, *the set*

$$E = \bigcup_{k \geq 0} E_k,$$

where $E_k = \{n : 2^{2k} \leq n < 2^{2k+1}\}$, $k \geq 0$ *does not admit an asymptotic density.*

Proof. It is enough to construct two subsequences of the sequence $(\nu_n(E))_{n \geq 1}$ which converge to two distinct limits.

We choose as subsequences those whose indices coincide with the antecedents and the consequents of connected components $[2^{2r}, 2^{2r+1}[$ of $\mathbb{N}^*$, i.e., with $2^{2r} - 1$ and 2^{2r+1}. The results follow from Theorem 2.6:

• Subsequence indexed by $n = 2^{2r} - 1$, $r \geq 1$. Then $\underline{d}(E) = \frac{1}{3}$.

• Subsequence indexed by $n = 2^{2r+1}$, $r \geq 0$. Then $\overline{d}(E) = \frac{2}{3}$. So, the sequence $(\nu_n(E))_{n \geq 1}$ does not converge, and hence $E \notin \mathfrak{D}$.

3. Properties of Asymptotic Density of a Subset E of $\mathbb{N}^*$

Proposition 3.1. *The set $\mathfrak{D}$ contains the algebra $\mathfrak{A}$ of finite and cofinite subsets of $\mathbb{N}^*$.*

Proposition 3.2. *Let $E \in \mathfrak{D}$ and $k \in \mathbb{N}$ such that $E + k \in \wp(\mathbb{N}^*)$. Then, $E + k \in \mathfrak{D}$ and $d(E + k) = d(E)$.* In other words, *the asymptotic density d is invariant under translation.*

Proposition 3.3. *Let* $E \in \mathfrak{D}$ *and* $k \in \mathbb{N}^*$. *Then,* $kE \in \mathfrak{D}$ *and* $d(kE) = \frac{1}{k} d(E)$.

Proposition 3.4. (a) *Let* $E \subset \mathbb{N}^*$. *If E is finite, then* $d(E) = 0$; *if E is cofinite, then* $d(E) = 1$.

(b) $\mathbb{N}^* \in \mathfrak{D}$, *and* $d(\mathbb{N}^*) = 1$.

Proposition 3.5. (a) $\mathfrak{D}$ *is a weak Dynkin class,*

(b) *d is additive.*

Proposition 3.6. *Let* E_1 *and* E_2 *be two elements of* $\mathfrak{D}$. *Then, the following four properties are equivalent*:

(p_1) $E_1 \cup E_2 \in \mathfrak{D}$;

(p_2) $E_1 \cap E_2 \in \mathfrak{D}$;

(p_3) $E_1 \setminus E_2 \in \mathfrak{D}$;

(p_4) $E_2 \setminus E_1 \in \mathfrak{D}$.

Moreover, if any one of these four properties is satisfied, then

$$d(E_1 \cup E_2) = d(E_1) + d(E_2) - d(E_1 \cap E_2).$$

Proof. • $(\mathrm{p}_1) \Rightarrow (\mathrm{p}_2)$ For this purpose, let E_1 and E_2 be two elements of $\mathfrak{D}$. We suppose that $E_1 \cup E_2 \in \mathfrak{D}$, and prove that $E_1 \cap E_2$ is an element of $\mathfrak{D}$. $\mathfrak{D}$ is stable under proper difference. Let $E_1 \cup E_2 \in \mathfrak{D}$. Then

$$E = (E_1 \cup E_2) \setminus E_1 \in \mathfrak{D}, \quad F = (E_1 \cup E_2) \setminus E_2 \in \mathfrak{D}, \quad \text{and} \quad E \cap F = \varnothing.$$

So

$$(E_1 \cap E_2) = (E_1 \cup E_2) \setminus (E \cup F) \in \mathfrak{D}.$$

• $(\mathrm{p}_2) \Rightarrow (\mathrm{p}_3)$ For this purpose, let E_1 and E_2 be two elements of $\mathfrak{D}$. We suppose that $E_1 \cap E_2 \in \mathfrak{D}$, and that $E_1 \setminus E_2$ is an element of $\mathfrak{D}$. Then

$$E_1 \setminus E_2 = E_1 \setminus (E_1 \cap E_2),$$

whence the result, since $\mathfrak{D}$ is stable under proper difference.

• In an analogous way, we prove that

$$(\mathrm{p}_3) \Rightarrow (\mathrm{p}_4) \text{ and } (\mathrm{p}_4) \Rightarrow (\mathrm{p}_1),$$

and that

$$d(E_1 \cup E_2) = d(E_1) + d(E_2) - d(E_1 \cap E_2).$$

Theorem 3.7. *The class $\mathfrak{D}$ is not a σ-algebra.*

Proof. Since $\mathfrak{D}$ contains the finite and cofinite subsets of $\mathbb{N}^*$, the σ-algebra $\sigma(\mathfrak{D})$ generated by $\mathfrak{D}$ coincide with the set of subsets of $\mathbb{N}^*$, i.e.,

$$\sigma(\mathfrak{D}) = \wp(\mathbb{N}^*).$$

Thus $\mathfrak{D}$ being a σ-field, it coincides with $\wp(\mathbb{N}^*)$.

To prove that $\mathfrak{D}$ is not a σ-field, it is enough to construct a subset $E \in \wp(\mathbb{N}^*)$ which does not belong to $\mathfrak{D}$. It is precisely what we did in Corollaries 2.7 and 2.8.

Theorem 3.8. *The class $\mathfrak{D}$ is not stable under finite intersections, so it is not an algebra.*

Proof. We start from the set E of Corollary 2.8, which does not admit an asymptotic density.

$$E = \bigcup_{k \geq 0} E_k,$$

where

$$E_k = \{n : 2^{2k} \leq n < 2^{2k+1}\}, \quad k \geq 0.$$

We note that

$$E^c = \bigcup_{k \geq 0} F_k,$$

where

$$F_k = \{n : 2^{2k+1} \le n < 2^{2k+2}\}, \quad k \ge 0.$$

We introduce the sets

$$A = 2\mathbb{N}^* \text{ the set of even numbers } > 0;$$

$$B = 2\mathbb{N} + 1 \text{ the set of odd numbers } > 0.$$

By Corollary 2.7 and Proposition 3.2, it follows that $A, B \in \mathfrak{D}$ and that

$$d(A) = d(B) = \frac{1}{2}.$$

We consider at present the set

$$F = (A \cap E) \cup (B \cap E^c).$$

It is the disjoint union of even numbers of E, and odd numbers of E^c.

- We show that $F \in \mathfrak{D}$, and that $d(F) = \frac{1}{2}$. For this purpose, the union

$$(A \cap E) \cup (B \cap E^c)$$

is disjoint. For all $n \ge 1$,

$$\nu_n(F) = \nu_n(A \cap E) + \nu_n(B \cap E^c), \tag{1}$$

and the same structure of the set

$$E^c = \bigcup_{k \ge 0} F_k$$

shows that

$$N_n(B \cap E^c) = N_n(A \cap E^c) - I_{A \cap E^c}(n).$$

So, for all $n \ge 1$,

$$\nu_n(B \cap E^c) = \nu_n(A \cap E^c) - \frac{1}{n} I_{A \cap E^c}(n). \tag{2}$$

It follows from (1) and (2) that

$$\nu_n(F) = \nu_n(A \cap E) + \nu_n(A \cap E^c) - \frac{1}{n} I_{A \cap E^c}(n) = \nu_n(A) - \frac{1}{n} I_{A \cap E^c}(n).$$

So, by taking limit when n tends to $+\infty$, $\nu_n(F)$ tends to $\frac{1}{2}$. It follows that $F \in \mathfrak{D}$, and $d(F) = \frac{1}{2}$.

• For sets A and F in $\mathfrak{D}$; we show that the intersection $A \cap F$ does not belong to $\mathfrak{D}$, and thus $\mathfrak{D}$ is not an algebra. For this purpose, we have

$$A \cap F = A \cap E.$$

So, for all $n \geq 1$

$$\nu_n(A \cap E) = \frac{1}{2} \nu_n(E) + \frac{1}{n} \varepsilon_n,$$

where $|\varepsilon_n| \leq 1$.

But then, since E does not belong to $\mathfrak{D}$, the sequence $(\nu_n(E))_{n \geq 1}$ does not converge, so the sequence $(\nu_n(A \cap E))_{n \geq 1}$ does not converge. Again, then

$$A \cap F = A \cap E \notin \mathfrak{D}.$$

Theorem 3.9. *Consider $d : \mathfrak{D} \to [0, 1]$ which assigns an element $E \in \mathfrak{D}$ to its asymptotic density $d(E)$. Then d is not σ-additive on $\mathfrak{D}$.* In other words*, if $(E_n)_{n \geq 1}$ is a sequence of disjoint elements of $\mathfrak{D}$ such that*

$$E = \bigcup_{n \geq 1} E_n \in \mathfrak{D}, \text{ with } E_i \cap E_j = \varnothing, \text{ for all } i \neq j, \tag{3}$$

then the following is not necessarily true:

$$d(E) = \sum_{n \geq 1} d(E_n).$$

Proof. We take

$$E_n = \{n\}, \quad n \geq 1.$$

Then, for all $n \geq 1$,

$$E_n \in \mathfrak{D} \quad \text{and} \quad d(E_n) = 0.$$

The set of natural integers can be represented in the form

$$\mathbb{N}^* = \bigcup_{n \geq 1} E_n \in \mathfrak{D},$$

with $E_i \cap E_j = \varnothing$, for all $i \neq j$. Since

$$\mathbb{N}^* \in \mathfrak{D} \text{ and } d(\mathbb{N}^*) = 1,$$

we have

$$1 = d(\mathbb{N}^*) = d\left(\bigcup_{n \geq 1} E_n\right) \neq \sum_{n \geq 1} d(E_n) = 0.$$

So, d is not σ-additive on $\mathfrak{D}$.

We have the following theorem which gives the structure of d.

Theorem 3.10. *The assignment* $d : \mathfrak{D} \to [0, 1]$ *sending an element* $E \in \mathfrak{D}$ *to its asymptotic density* $d(E)$ *is a finitely additive probability measure, i.e., it satisfies*

(p_1) $d(\mathbb{N}^*) = 1$;

(p_2) if $(E_k)_{1 \leq k \leq n}$ *is a finite sequence of disjoint elements of* $\mathfrak{D}$, *then*

$$d\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} d(E_k);$$

(p_3) $(p_3)_1$ *d is invariant under translation, i.e., for all* $E \in \mathfrak{D}$ *and for all* $k \in \mathbb{N}$,

$$d(E + k) = d(E);$$

$(p_3)_2$ *d satisfies, for all* $E \in \mathfrak{D}$ *and for all* $k \in \mathbb{N}^*$,

$$d(kE) = \frac{1}{k} d(E).$$

Proof. The proof follows on the lines of the previous results.

Theorem 3.11. *The class* $\mathfrak{D}$ *of subsets of* $\mathbb{N}^*$ *possessing asymptotic densities is an* α*-class*. In other words, *it satisfies the following properties*:

(p_1) $\mathbb{N}^* \in \mathfrak{D}$;

(p_2) $\mathfrak{D}$ *is stable under complementation*;

(p_3) $\mathfrak{D}$ *is stable under finite disjoint unions*;

(p_4) $(\mathrm{p}_4)_1$ $\mathfrak{D}$ *is stable under translation*, *i.e.*, *for all* $E \in \mathfrak{D}$ *and for all* $k \in \mathbb{N}$,

$$E + k \in \mathfrak{D};$$

$(\mathrm{p}_4)_2$ $\mathfrak{D}$ *is stable under the change of scale*, *i.e.*, *for all* $E \in \mathfrak{D}$ *and for all* $k \in \mathbb{N}^*$,

$$kE \in \mathfrak{D}.$$

Proof. • Properties (p_1), (p_2), (p_3) and $(\mathrm{p}_4)_1$ can be verified easily.

• For $(\mathrm{p}_4)_2$ we observe that for all $E \in \mathfrak{D}$ for all $k \in \mathbb{N}^*$,

$$\lim_{(n \to +\infty)} \nu_n(kE) = \frac{1}{k} d(E).$$

Theorem 3.12. *Let* E *be an element of* $\mathfrak{D}$. *Then for all* ℓ, *with* $0 \leq \ell \leq d(E)$, *there exists a subset* E^* *of* E *such that* $d(E^*) = \ell$.

Proof. We look at the following three cases:

• $\ell = 0$, in which case the result is obvious;

• $\ell = d(E)$, in which case the result is obvious;

• We give the proof for $E \subset \mathbb{N}^*$; for all $\ell \in]0, 1[$, there exists $E \in \mathfrak{D}$ such that $d(E) = \ell$. We construct

$$E = \bigcup_{k \geq 1} [p_k, q_k[$$

as follows: p_1 : arbitrary, q_1 : the first integer $> p_1$ such that $\nu_{q_1}(E) > \ell$

$$\begin{cases} p_2 : \text{the first integer} > q_1 \text{ such that } \nu_{p_2}(E) < \ell, \\ q_2 : \text{the first integer} > p_2 \text{ such that } \nu_{q_2}(E) > \ell \end{cases}$$

and, by induction

$$\begin{cases} p_k : \text{the first integer} > q_{k-1} \text{ such that } \nu_{p_k}(E) < \ell, \\ q_k : \text{the first integer} > p_k \text{ such that } \nu_{q_k}(E) > \ell. \end{cases}$$

Then, we have for all $k > 1$

$$\begin{cases} |\nu_{p_k}(E) - \ell| \leq |\nu_{p_k}(E) - \nu_{p_{k-1}}(E)| \leq \dfrac{1}{p_k}, \\ |\nu_{q_k}(E) - \ell| \leq |\nu_{q_k}(E) - \nu_{q_{k-1}}(E)| \leq \dfrac{1}{q_k}, \end{cases}$$

so

$$\nu_{p_k}(E) \to \ell \text{ and } \nu_{q_k}(E) \to \ell, \text{ as } (k \to +\infty).$$

The proof, for $E^* \subset E$, holds immediately from the above argument. For this purpose, p_1 : arbitrary, q_1 : the first integer $> p_1$ such that $\nu_{q_1}(E \cap E^*) > \ell$

$$\begin{cases} p_2 : \text{the first integer} > q_1 \text{ such that } \nu_{p_2}(E \cap E^*) < \ell, \\ q_2 : \text{the first integer} > p_2 \text{ such that } \nu_{q_2}(E \cap E^*) > \ell \end{cases}$$

and, by induction

$$\begin{cases} p_k : \text{the first integer} > q_{k-1} \text{ such that } \nu_{p_k}(E \cap E^*) < \ell, \\ q_k : \text{the first integer} > p_k \text{ such that } \nu_{q_k}(E \cap E^*) > \ell. \end{cases}$$

Then, we have for all $k > 1$

$$\begin{cases} |\nu_{p_k}(E \cap E^*) - \ell| \leq |\nu_{p_k}(E \cap E^*) - \nu_{p_{k-1}}(E \cap E^*)| \leq \dfrac{1}{p_k}, \\ |\nu_{q_k}(E \cap E^*) - \ell| \leq |\nu_{q_k}(E \cap E^*) - \nu_{q_{k-1}}(E \cap E^*)| \leq \dfrac{1}{q_k}, \end{cases}$$

so

$$\nu_{p_k}(E \cap E^*) \to \ell \text{ and where } \nu_{q_k}(E \cap E^*) \to \ell, \text{ as } (k \to +\infty).$$

4. Existence Criteria of an Asymptotic Density of a Subset E of $\mathbb{N}^*$

Let E be a subset of $\mathbb{N}^*$, which is neither finite nor cofinite. It can be represented by the disjoint union of its connected components, i.e., by

$$E = \bigcup_{n \geq 1} [p_n, q_n [,$$

where $(p_n)_{n\geq 1}$, $(q_n)_{n\geq 1}$ denote two sequences of strictly positive integers satisfying

$$\forall n \geq 1,\ p_n < q_n < p_{n+1},$$

where

$$[p_n, q_n [= \{k \in \mathbb{N}^* : p_n \leq k < q_n\}$$

denotes the nth connected component of E.

Criterion $\triangleright$ ([1, p.47], [3]). Let

$$E = \bigcup_{n \geq 1} [p_n, q_n [,$$

be a subset of $\mathbb{N}^*$, neither finite nor cofinite and

$$\rho_n = q_n - p_n,\ \sigma_n = q_n - q_{n-1} \quad (n \geq 1) \quad (q_0 = 1).$$

Let ℓ be a real number such that $0 < \ell \leq 1$ and the following properties hold:

(p_1) $p_n \sim q_{n-1}$, as $(n \to +\infty)$;

(p_2) $\dfrac{\rho_n}{\sigma_n} \to \ell$, as $(n \to +\infty)$.

Then, E admits an asymptotic density $d(E)$ and $d(E) = \ell$. If $\ell = 0$, then the only condition (p_2) implies that $d(E) = 0$.

Proposition 4.1. *Let*

$$E = \bigcup_{k \geq 1} [P(k), Q(k)[,$$

where $P(k)$ *and* $Q(k)$ *are polynomials with integral coefficients* > 0 *in* k, *defined by*

$$\begin{cases} P(k) = ak^n + bk^{n-1} + O(k^{n-1}), \\ Q(k) = ak^n + ck^{n-1} + O(k^{n-1}), \end{cases}$$

where $n \in \mathbb{N}^*$ *is the common degree of* P *and of* Q.

We suppose that

$$0 < \frac{c-b}{na} < 1.$$

Then, E admits an asymptotic density $d(E)$ and $d(E) = \dfrac{c-b}{na}$.

Remark 4.1.

• If $\deg(P) > \deg(Q)$, then from some rank each of the intervals $[P(k), Q(k)[$ is empty, E is finite and $d(E) = 0$.

• If $\deg(P) < \deg(Q)$, then each of intervals $[P(k), Q(k)[$ overlaps the next, E is cofinite and $d(E) = 1$.

The condition $\deg(P) = \deg(Q)$ is thus a necessary condition in order that from some rank all intervals $[P(k), Q(k)[$ are nonempty and disjoint. In other words, there exists a $k_0 \in \mathbb{N}^*$ such that

$$\forall k \geq k_0, \quad P(k) < Q(k) < P(k+1). \tag{4}$$

We say this case to be the regular case, E is then neither finite nor cofinite.

Remark 4.2. We put

$$\begin{cases} P(k) = ak^n + O(k^n), \\ Q(k) = a^* k^n + O(k^n), \end{cases}$$

where $n \in \mathbb{N}^*$ is the common degree of P and Q.

The condition (4) implies that

$$ak^n + O(k^n) < a^* k^n + O(k^n) < a(k+1)^n + O(k^n)$$

so it holds that

$$a \le a^*, \quad a^* \le a, \quad \text{i.e.,} \quad a = a^*.$$

By taking this condition into account, we put

$$\begin{cases} P(k) = ak^n + bk^{n-1} + O(k^{n-1}), \\ Q(k) = ak^n + ck^{n-1} + O(k^{n-1}), \end{cases}$$

where $n \in \mathbb{N}^*$ is the common degree of P and Q.

The condition (4) implies the following two conditions:

$$\begin{cases} bk^{n-1} + O(k^{n-1}) < ck^{n-1} + O(k^{n-1}), \\ (c - na)k^{n-1} + O(k^{n-1}) < bk^{n-1} + O(k^{n-1}), \end{cases}$$

so it holds that

$$b \le c, \quad c - na \le b.$$

In other words,

$$0 \le \frac{c-b}{na} \le 1. \tag{5}$$

This condition is a necessary condition in order to have (4), but it is not sufficient.

We obtain a sufficient condition to fill in (5) the sign of weak inequalities by the sign of strict inequalities. In other words to fill (5) by

$$0 < \frac{c-b}{na} < 1. \tag{6}$$

In this case, we put

$$\begin{cases} p_k = P(k) = ak^n + bk^{n-1} + O(k^{n-1}), \\ q_k = Q(k) = ak^n + ck^{n-1} + O(k^{n-1}), \end{cases} \quad 0 < \frac{c-b}{na} < 1$$

$$\begin{cases} \rho_k = q_k - p_k = (c-b)k^{n-1} + O(k^{n-1}), \\ \sigma_k = q_k - q_{k-1} = nak^{n-1} + O(k^{n-1}). \end{cases}$$

We verify that

(p_1) $p_k \sim q_{k-1}$, as $(k \to +\infty)$;

(p_2) $\dfrac{\rho_k}{\sigma_k} \to \dfrac{c-b}{na}$, as $(k \to +\infty)$.

It follows that E admits an asymptotic density $d(E)$ and $d(E) = \dfrac{c-b}{na}$.

We have the following particular case of the previous proposition.

Proposition 4.2. *Let*

$$E = \bigcup_{n \geq 1} [2n-1, 2n[$$

be the set of odd integers > 0. *Then*, E *admits an asymptotic density* $d(E)$ *and* $d(E) = \dfrac{1}{2}$.

Proof. For this purpose, we put

$$\begin{cases} p_n = 2n-1, & q_n = 2n, \\ \rho_n = q_n - p_n = 1, \\ \sigma_n = q_n - q_{n-1} = 2. \end{cases}$$

We verify that

(p_1) $p_n \sim q_{n-1}$, as $(n \to +\infty)$;

(p_2) $\dfrac{\rho_n}{\sigma_n} = \dfrac{1}{2}$.

Then E admits an asymptotic density $d(E)$ and $d(E) = \dfrac{1}{2}$.

Proposition 4.3. *Let* E *be a subset of* $\mathbb{N}^*$, *which we write in the form*

$$E = \bigcup_{k \geq 1} [p_k, q_k[,$$

$$\begin{cases} p_k = 10^{P(k)}, & P(k) = ak + b, \\ q_k = 10^{Q(k)}, & Q(k) = ak + c. \end{cases}$$

We suppose that a, b, c *are real numbers satisfying*:

(1) $a > 0$;

(2) *for all* $k \geq 1$, p_k *and* q_k *are integers* ≥ 1;

(3) $0 < \dfrac{c-b}{a} < 1$.

Then, E *does not admit an asymptotic density*.

Proof. For this purpose, we put

$$\begin{cases} \rho_k = q_k - p_k = 10^{ak+c} - 10^{ak+b} \\ \sigma_k = q_k - q_{k-1} = 10^{ak+c} - 10^{a(k-1)+c}. \end{cases}$$

We verify that

$$(\mathrm{p}_1) \quad \frac{q_{k-1}}{p_k} = 10^{-a\left(1-\frac{c-b}{a}\right)} < 1;$$

$$(\mathrm{p}_2) \quad \frac{\rho_k}{\sigma_k} = \frac{1-10^{-(c-b)}}{1-10^{-a}}.$$

It follows from Theorem VII.9 in [1, Chapter VII] that E does not admit an asymptotic density but

$$\underline{d}(E) = 10^{-a\left(1-\frac{c-b}{a}\right)} \frac{1-10^{-(c-b)}}{1-10^{-a}}$$

and

$$\overline{d}(E) = \frac{1-10^{-(c-b)}}{1-10^{-a}}.$$

5. Conditional Asymptotic Density

A good amount of work in analytic number theory is concerned with the distribution of prime numbers. To answer some questions in this area, it is natural to consider conditional asymptotic density on the prime numbers.

Definition 5.1. Let E be a subset of $\mathbb{N}^*$, $\prod(n)$ be the number of prime numbers $\leq n$ and $\prod_E(n)$ be the number of prime numbers $\leq n$ in E. Then, we put

$$\nu_n(E \mid \mathbb{P}) := \frac{\prod_E(n)}{\prod(n)}.$$

We say that E admits the number ℓ as a conditional asymptotic density, related to $\mathbb{P}$, if the limit, $\lim\limits_{(n \to +\infty)} \nu_n(E \mid \mathbb{P})$ exists and equals to ℓ, (necessarily this limit belongs to [0, 1]). If this is the case, then we shall denote the conditional density by $d(E \mid \mathbb{P})$, or plainly by $d_c(E)$.

Definition 5.2 [5]. Let $f : \mathbb{N}^* \to \mathbb{R}^+$ be a positive arithmetic function such that

$$\sum_{n=1}^{+\infty} f(n) = +\infty$$

and

$$\lim_{(n \to +\infty)} \frac{f(n) + \sum\limits_{k=1}^{n-1} |f(k) - f(k+1)|}{\sum\limits_{k=1}^{n} f(k)} = 0.$$

Let E be a subset of $\mathbb{N}^*$ and

$$\nu_n(E \mid f) := \frac{\sum\limits_{1 \leq k \leq n,\, k \in E} f(k)}{\sum\limits_{k=1}^{n} f(k)}.$$

Then we say that E admits the number ℓ as a conditional asymptotic density, conditionally to f, if $\nu_n(E \mid f)$ tends to ℓ, when n tends to $+\infty$ (necessarily this limit value belongs to [0, 1]). If this is the case, we denote this conditional density by $d_c(E)$.

We have the following theorem:

Theorem 5.1 [1]**.** (1) *Let E be an infinite subset of $\mathbb{N}^*$ and let F be a subset of E. We denote by $\underline{d}_c(E)$ the lower conditional asymptotic density and by $\overline{d}_c(E)$ the upper conditional asymptotic density of the sequence $\nu_n(E \mid f)$. Then the set of couples*

$$S_c(E) := \{(\overline{d}_c(F); \underline{d}_c(F)) \in [0, 1] \times [0, 1] \subset \mathbb{R}^2 : F \subset E\}$$

has the following properties:

(a) $S_c(E)$ *encloses the closed trapezium having vertices*:

$$(0; 0), \quad (s; 0), \quad (s; i), \quad (i; i);$$

(b) $S_c(E)$ *contains the closed triangle of vertices*:

$$(0; 0), \quad (s; 0), \quad (s; i);$$

(c) $S_c(E)$ *is a convex set*;

(d) $S_c(E)$ *is a closed set.*

(2) *Conversely*, *consider two real numbers s and i*, *such that*

$$0 \le i \le s \le 1.$$

If a subset S of $\mathbb{R}^2$ satisfies (a)-(d) *as above*, *then there exists a subset E with*

$$\overline{d}_c(E) = s, \quad \underline{d}_c(E) = i \quad \text{and} \quad S_c(E) = S.$$

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