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NEW MULTIPLE SOLITON-LIKE SOLUTIONS FOR COUPLED NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS USING EXTENDED COUPLED SUB-ODEs EXPANSION METHOD

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Abstract

In this paper, the extended coupled sub-ODEs expansion method has been used to construct a series of double soliton-like solutions, double triangular function solitons and complexiton solitons for the $(2+1)$-dimensional Painlevé integrable Burgers equations with variable coefficients. This method can also apply to other NLPDEs.

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1. Introduction

In recent years, the exact solitons of nonlinear PDEs have been investigated by many authors (see, for example, [11, 12]) who are investigated in nonlinear physical phenomena. Many powerful methods have been presented by those authors such as the inverse scattering transform [1], the Bäcklund transform [2], the generalized Riccati equation [3], the Jacobi elliptic expansion [4], the extended tanh-function method [5], the *F*-expansion method [6], the exp-function expansion method [7], the sub-ODE method [8], the homogeneous balance method [9], the extended sine-cosine methods [10], the complex hyperbolic function method [11], the $\left(\frac{G'}{G}\right)$-expansion method [12] and so on. Recently, the coupled sub-equations expansion method as the extension of multiple Riccati equations expansion method is efficiently applied by many researchers to a great variety of NLPDEs [13]. In these papers, the solutions of different Riccati equations with different parameters are used as different variables in the components of rational expansion.

The present paper is motivated by the desire to extend the multiple Riccati equation expansion method and use two ODEs expansion method to construct a series of some types of traveling wave solutions, namely, the doubled soliton-like solutions, double triangular function solutions and complexiton soliton solutions for the $(2+1)$-dimensional Painlevé integrable Burgers equations with variable coefficients.

2. Summary of the Extended Coupled Sub-equations Expansion Method

In this section, we would like to outline the main steps of this method as follows:

Step 1. We consider the following nonlinear partial differential equation with some physical field $u(x, y, t)$:

$$U(u, u_t, u_x, u_y, u_{xy}, u_{xt}, u_{ty}, u_{tt}, u_{xx}, u_{yy}, \ldots) = 0. \tag{2.1}$$

Step 2. We introduce a more generalized ansatz in terms of a finite form expansion in the following forms:

$$u(x,\ y,\ t) = a_0(X) + \sum_{k=1}^{n} \sum_{i+j=k} a_i^j(X)\left(\frac{F'(\xi)}{F(\xi)}\right)^i \left(\frac{G'(\eta)}{G(\eta)}\right)^j, \tag{2.2}$$

where $a_0(X),\ a_i^j(X)\ (i,\ j = 0, 1, 2, ..., n)$ are functions of $X = (x,\ y,\ t)$ to be determined later, while the new variables $F(\xi)$ and $G(\eta)$ satisfy the following two equations:

$$F''(\xi) + \varepsilon F'(\xi) + \theta F(\xi) = 0, \tag{2.3}$$

$$G''(\eta) + \lambda G'(\eta) + \mu G(\eta) = 0, \tag{2.4}$$

where ε, θ, λ, μ are arbitrary constants, $F''(\xi) = \dfrac{d^2F(\xi)}{d^2\xi}$, $F'(\xi) = \dfrac{dF(\xi)}{d\xi}$, $G''(\eta) = \dfrac{d^2G(\eta)}{d^2\eta}$, $G'(\eta) = \dfrac{dG(\eta)}{d\eta}$. The parameters ξ, η are given by $\xi = k_1x + l_1y + \lambda_1(t)$ and $\eta = k_2x + l_2y + \lambda_2(t)$, where k_1, l_1, k_2, l_2 are arbitrary constants, $\lambda_1(t)$, $\lambda_2(t)$ are functions of t.

Step 3. Determine the positive integer n of the formal polynomial solution (2.2) by balancing the highest nonlinear terms and the highest partial derivative terms in the given system equations, and then give the formal solution.

Step 4. Substitute (2.2) into (2.1), along with (2.3) and (2.4) and then set all the coefficients of $\left[\dfrac{F'(\xi)}{F(\xi)}\right]^i \left[\dfrac{G'(\eta)}{G(\eta)}\right]^j$ $(i,\ j = 0, 1, 2, ...)$ of the resulting system's numerator to be zero. We get an over-determined system of differential equations with respect to k_1, k_2, l_1, l_2, $\lambda_1(t)$, $\lambda_2(t)$, ε, θ, λ, μ, $a_0(X)$ and $a_i^j(X)\ (i,\ j = 0, 1, 2, ..., n)$.

Step 5. Solving the over-determined system of differential equations by using the symbolic computation as Maple, we end up with explicit expressions for $k_1,\ k_2,\ l_1,\ l_2,\ \lambda_1(t),\ \lambda_2(t),\ \varepsilon, \theta, \lambda, \mu,\ a_0(X)$ and $a_i^j(X)$ $(i,\ j = 0, 1, 2, \ldots, n)$.

Step 6. It is well known that the general solutions of the differential equations (2.3) and (2.4) are listed as follows [14]:

(i) when $\varepsilon^2 > 4\theta$ and $\lambda^2 > 4\mu$, then

$$\frac{F'(\xi)}{F(\xi)} = -\frac{\varepsilon}{2} + \frac{\sqrt{\varepsilon^2 - 4\theta}}{2}\left\{\frac{H_1 \sinh \frac{\sqrt{\varepsilon^2 - 4\theta}}{2}\xi + H_2 \cosh \frac{\sqrt{\varepsilon^2 - 4\theta}}{2}\xi}{H_1 \cosh \frac{\sqrt{\varepsilon^2 - 4\theta}}{2}\xi + H_2 \sinh \frac{\sqrt{\varepsilon^2 - 4\theta}}{2}\xi}\right\},$$

$$\frac{G'(\eta)}{G(\eta)} = -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2}\left\{\frac{C_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2}\eta + C_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2}\eta}{C_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2}\eta + C_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2}\eta}\right\}. \tag{2.5}$$

(ii) When $\varepsilon^2 < 4\theta$ and $\lambda^2 < 4\mu$, then

$$\frac{F'(\xi)}{F(\xi)} = -\frac{\varepsilon}{2} + \frac{\sqrt{4\theta - \varepsilon^2}}{2}\left\{\frac{-H_1 \sin \frac{\sqrt{4\theta - \varepsilon^2}}{2}\xi + H_2 \cos \frac{\sqrt{4\theta - \varepsilon^2}}{2}\xi}{H_1 \cos \frac{\sqrt{4\theta - \varepsilon^2}}{2}\xi + H_2 \sin \frac{\sqrt{4\theta - \varepsilon^2}}{2}\xi}\right\},$$

$$\frac{G'(\eta)}{G(\eta)} = -\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2}\left\{\frac{-C_1 \sin \frac{\sqrt{4\mu - \lambda^2}}{2}\eta + C_2 \cos \frac{\sqrt{4\mu - \lambda^2}}{2}\eta}{C_1 \cos \frac{\sqrt{4\mu - \lambda^2}}{2}\eta + C_2 \sin \frac{\sqrt{4\mu - \lambda^2}}{2}\eta}\right\}. \tag{2.6}$$

(iii) When $\varepsilon^2 < 4\theta$ and $\lambda^2 > 4\mu$, then

$$\frac{F'(\xi)}{F(\xi)} = -\frac{\varepsilon}{2} + \frac{\sqrt{4\theta - \varepsilon^2}}{2}\left\{\frac{-H_1 \sin \frac{\sqrt{4\theta - \varepsilon^2}}{2}\xi + H_2 \cos \frac{\sqrt{4\theta - \varepsilon^2}}{2}\xi}{H_1 \cos \frac{\sqrt{4\theta - \varepsilon^2}}{2}\xi + H_2 \sin \frac{\sqrt{4\theta - \varepsilon^2}}{2}\xi}\right\},$$

$$\frac{G'(\eta)}{G(\eta)} = -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2}\left\{\frac{C_1 \sinh\frac{\sqrt{\lambda^2 - 4\mu}}{2}\eta + C_2 \cosh\frac{\sqrt{\lambda^2 - 4\mu}}{2}\eta}{C_1 \cosh\frac{\sqrt{\lambda^2 - 4\mu}}{2}\eta + C_2 \sinh\frac{\sqrt{\lambda^2 - 4\mu}}{2}\eta}\right\}. \quad (2.7)$$

(iv) When $\varepsilon^2 = 4\theta$ and $\lambda^2 > 4\mu$, then

$$\frac{F'(\xi)}{F(\xi)} = -\frac{\varepsilon}{2} + \frac{H_2}{H_1 + H_2\xi},$$

$$\frac{G'(\eta)}{G(\eta)} = -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2}\left\{\frac{C_1 \sinh\frac{\sqrt{\lambda^2 - 4\mu}}{2}\eta + C_2 \cosh\frac{\sqrt{\lambda^2 - 4\mu}}{2}\eta}{C_1 \cosh\frac{\sqrt{\lambda^2 - 4\mu}}{2}\eta + C_2 \sinh\frac{\sqrt{\lambda^2 - 4\mu}}{2}\eta}\right\}, \quad (2.8)$$

where H_1, H_2, C_1, C_2 are constants and $H_2C_2 \neq 0$.

3. New Multiple Soliton-like Solutions for the $(2+1)$-dimensional Painlevé Integrable Burgers Equations with Variable Coefficients

In this section, we will apply the extended coupled sub-ODEs expansion method to construct travelling wave solutions for the $(2+1)$-dimensional Painlevé integrable Burgers equations with variable coefficients [14]:

$$-u_t + uu_y + \alpha(t)vu_x + \beta(t)u_{yy} + \alpha(t)\beta(t)u_{xx} = 0,$$
$$u_x - v_y = 0. \quad (3.1)$$

Let us now solve the system (3.1) by the sub-ODEs expansion method. By balancing the nonlinear terms and the highest order linear partial derivative terms of (3.1), we get $m_1 = 1$ and $m_2 = 1$. Thus, the $(2+1)$-dimensional Painlevé integrable Burgers equations with variable coefficients (3.1) have the following exact solutions:

$$u(x, y, t) = a_0(t) + a_1(t)\frac{F'(\xi)}{F(\xi)} + a_2(t)\frac{G'(\eta)}{G(\eta)},$$
$$v(x, y, t) = b_0(t) + b_1(t)\frac{F'(\xi)}{F(\xi)} + b_2(t)\frac{G'(\eta)}{G(\eta)}, \quad (3.2)$$

where $\xi = k_1 x + l_1 y + \lambda_1(t)$, $\eta = k_2 x + l_2 y + \lambda_2(t)$ and $a_0(t)$, $a_1(t)$, $a_2(t)$, $b_0(t)$, $b_1(t)$, $b_2(t)$, $\lambda_1(t)$, $\lambda_2(t)$ are functions of t to be determined later. With the aid of Maple, we substitute (3.2) along with (2.3) and (2.4) into (3.1) and set the coefficients of $\left[\frac{F'(\xi)}{F(\xi)}\right]^m \left[\frac{G'(\eta)}{G(\eta)}\right]^n (m, n = 0, 1, 2, \ldots)$ to be zero, then yield a set of over-determined differential equations with respect to k_1, k_2, l_1, l_2, $a_i(t)\ (i = 0, 1, 2)$ and $b_j(t)\ (j = 0, 1, 2)$. On using the Maple software package, we solve the over-determined differential equations. Consequently, we get the following results:

Case 1.

$$l_1 = \frac{a_1(t)k_1}{b_1(t)},\ l_2 = -\frac{k_2 a_1(t)}{b_1(t)},$$

$$a_0(t) = C_3,\ a_1(t) = C_4,\ a_2(t) = C_5,$$

$$b_1(t) = C_6,\ b_2(t) = -\frac{a_2(t)b_1(t)}{a_1(t)},$$

$$a_1^2(t) + \alpha(t)b_1^2(t) = 0,$$

$$\lambda_1(t) = \int \frac{-(a_0(t)b_1(t) + a_1(t)b_0(t))k_1 a_1(t)}{b_1^2(t)}\,dt + C_7,$$

$$\lambda_2(t) = \int \frac{-(a_0(t)b_1(t) + a_1(t)b_0(t))k_2 a_1(t)}{b_1^2(t)}\,dt + C_8. \tag{3.3}$$

Case 2.

$$k_1 = -\frac{b_1(t)k_2}{b_2(t)},\ l_1 = -\frac{a_1(t)k_2}{b_2(t)},\ l_2 = \frac{a_2(t)k_2}{b_2(t)},\ \beta(t) = \frac{1}{2}\frac{b_2(t)}{k_2},$$

$$a_0(t) = C_3,\ a_1(t) = C_4,\ a_2(t) = C_5,$$

$$b_1(t) = C_6, \ b_2(t) = C_7,$$

$$\alpha(t) = -\frac{a_1^2(t)}{b_2^2(t)},$$

$$\lambda_1(t) = \int \frac{1}{2} \frac{k_2 a_1(t)(a_2(t)b_1(t)\varepsilon + 2a_0(t)b_2(t) - b_2(t)a_1(t)\varepsilon - 2a_2(t)b_0(t))}{b_2^2(t)} dt + C_8,$$

$$\lambda_2(t) = \int \frac{1}{2} \frac{a_2(t)k_2(-b_1(t)a_2(t)\tau + 2a_0(t)b_1(t) - 2a_1(t)b_0(t) + a_1(t)\tau b_0(t))}{b_1(t)b_2(t)} dt + C_9, \tag{3.4}$$

where $k_2 b_1(t) b_2(t) a_1(t) \neq 0.$ Note that, there are other cases, are omitted here. According to (2.3) and (2.4) and the general solutions (2.5)-(2.10) listed in Step 6, we obtain the following families of new multiple soliton-like solutions corresponding to Case 1 for equations (3.1):

Family 1. When $\varepsilon^2 > 4\theta,$ $\lambda^2 > 4\mu$ and $k_2 b_1(t) b_2(t) a_1(t) \neq 0,$ the double soliton-like solutions of equations (3.1) have the following forms:

$$u_1(x, y, t) = C_3 - C_4 \frac{\varepsilon}{2} + C_4 \frac{\sqrt{\varepsilon^2 - 4\theta}}{2} \left\{ \frac{H_1 \sinh \frac{\sqrt{\varepsilon^2 - 4\theta}}{2} \xi + H_2 \cosh \frac{\sqrt{\varepsilon^2 - 4\theta}}{2} \xi}{H_1 \cosh \frac{\sqrt{\varepsilon^2 - 4\theta}}{2} \xi + H_2 \sinh \frac{\sqrt{\varepsilon^2 - 4\theta}}{2} \xi} \right\}$$

$$- C_5 \frac{\lambda}{2} + C_5 \frac{\sqrt{\lambda^2 - 4\mu}}{2} \times \left\{ \frac{C_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta + C_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta}{C_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta + C_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta} \right\},$$

$$v_1(x, y, t)$$

$$= b_0(t) - C_6 \frac{\varepsilon}{2}$$

$$+ C_6 \frac{\sqrt{\varepsilon^2 - 4\theta}}{2} \left\{ \frac{H_1 \sinh \frac{\sqrt{\varepsilon^2 - 4\theta}}{2} \xi + H_2 \cosh \frac{\sqrt{\varepsilon^2 - 4\theta}}{2} \xi}{H_1 \cosh \frac{\sqrt{\varepsilon^2 - 4\theta}}{2} \xi + H_2 \sinh \frac{\sqrt{\varepsilon^2 - 4\theta}}{2} \xi} \right\} - \frac{C_5 C_6}{C_4} \frac{\lambda}{2}$$

$$+ \frac{C_5 C_6}{C_4} \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left\{ \frac{C_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta + C_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta}{C_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta + C_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta} \right\}. \quad (3.5)$$

Family 2. When $\varepsilon^2 < 4\theta$, $\lambda^2 < 4\mu$ and $k_2 b_1(t) b_2(t) a_1(t) \neq 0$, the double triangular function solitons of equations (3.1) have the following forms:

$$u_2(x, y, t)$$

$$= C_3 - C_4 \frac{\varepsilon}{2} + C_4 \frac{\sqrt{4\theta - \varepsilon^2}}{2} \left\{ \frac{-H_1 \sin \frac{\sqrt{4\theta - \varepsilon^2}}{2} \xi + H_2 \cos \frac{\sqrt{4\theta - \varepsilon^2}}{2} \xi}{H_1 \cos \frac{\sqrt{4\theta - \varepsilon^2}}{2} \xi + H_2 \sin \frac{\sqrt{4\theta - \varepsilon^2}}{2} \xi} \right\}$$

$$- C_5 \frac{\lambda}{2} + C_5 \frac{\sqrt{4\mu - \lambda^2}}{2}$$

$$\times \left\{ \frac{C_1 \sin \frac{\sqrt{4\mu - \lambda^2}}{2} \eta + C_2 \cos \frac{\sqrt{4\mu - \lambda^2}}{2} \eta}{C_1 \cos \frac{\sqrt{4\mu - \lambda^2}}{2} \eta + C_2 \sin \frac{\sqrt{4\mu - \lambda^2}}{2} \eta} \right\},$$

$$v_2(x, y, t)$$

$$= b_0(t) - C_6 \frac{\varepsilon}{2} + C_6 \frac{\sqrt{4\theta - \varepsilon^2}}{2} \left\{ \frac{-H_1 \sin \frac{\sqrt{4\theta - \varepsilon^2}}{2} \xi + H_2 \cos \frac{\sqrt{4\theta - \varepsilon^2}}{2} \xi}{H_1 \cos \frac{\sqrt{4\theta - \varepsilon^2}}{2} \xi + H_2 \sin \frac{\sqrt{4\theta - \varepsilon^2}}{2} \xi} \right\}$$

$$-\frac{C_5C_6}{C_4}\frac{\lambda}{2}$$

$$+\frac{C_5C_6}{C_4}\frac{\sqrt{4\mu-\lambda^2}}{2}\left\{\frac{C_1\sin\frac{\sqrt{\lambda^2-4\mu}}{2}\eta+C_2\cos\frac{\sqrt{\lambda^2-4\mu}}{2}\eta}{C_1\cos\frac{\sqrt{\lambda^2-4\mu}}{2}\eta+C_2\sin\frac{\sqrt{\lambda^2-4\mu}}{2}\eta}\right\}. \tag{3.6}$$

Family 3. When $\varepsilon^2 < 4\theta$, $\lambda^2 > 4\mu$ and $k_2b_1(t)b_2(t)a_1(t) \neq 0$, the complexiton soliton solutions of equations (3.1) have the following forms:

$$u_3(x, y, t)$$

$$= C_3 - C_4\frac{\varepsilon}{2} + C_4\frac{\sqrt{4\theta-\varepsilon^2}}{2}\left\{\frac{-H_1\sin\frac{\sqrt{4\theta-\varepsilon^2}}{2}\xi+H_2\cos\frac{\sqrt{4\theta-\varepsilon^2}}{2}\xi}{H_1\cos\frac{\sqrt{4\theta-\varepsilon^2}}{2}\xi+H_2\sin\frac{\sqrt{4\theta-\varepsilon^2}}{2}\xi}\right\}$$

$$-C_5\frac{\lambda}{2}+C_5\frac{\sqrt{\lambda^2-4\mu}}{2}\times\left\{\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\eta+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\eta}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\eta+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\eta}\right\},$$

$$v_3(x, y, t)$$

$$= b_0(t) - C_6\frac{\varepsilon}{2} + C_6\frac{\sqrt{4\theta-\varepsilon^2}}{2}\left\{\frac{-H_1\sin\frac{\sqrt{4\theta-\varepsilon^2}}{2}\xi+H_2\cos\frac{\sqrt{4\theta-\varepsilon^2}}{2}\xi}{H_1\cos\frac{\sqrt{4\theta-\varepsilon^2}}{2}\xi+H_2\sin\frac{\sqrt{4\theta-\varepsilon^2}}{2}\xi}\right\}$$

$$-\frac{C_5C_6}{C_4}\frac{\lambda}{2}+\frac{C_5C_6}{C_4}$$

$$\times\frac{\sqrt{\lambda^2-4\mu}}{2}\left\{\frac{C_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\eta+C_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\eta}{C_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\eta+C_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\eta}\right\}. \tag{3.7}$$

Family 4. When $\varepsilon^2 = 4\theta$, $\lambda^2 > 4\mu$ and $k_2 b_1(t) b_2(t) a_1(t) \neq 0$, the complexiton soliton solutions of equations (3.1) have the following forms:

$$u_4(x, y, t) = C_3 - C_4 \frac{\varepsilon}{2} + C_4 \frac{H_2}{H_1 + H_2\xi} - C_5 \frac{\lambda}{2}$$

$$+ C_5 \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left\{ \frac{C_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta + C_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta}{C_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta + C_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta} \right\},$$

$$v_4(x, y, t) = b_0(t) - C_6 \frac{\varepsilon}{2} + C_6 \frac{H_2}{H_1 + H_2\xi} - \frac{C_5 C_6}{C_4} \frac{\lambda}{2} + \frac{C_5 C_6}{C_4} \frac{\sqrt{\lambda^2 - 4\mu}}{2}$$

$$\times \left\{ \frac{C_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta + C_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta}{C_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta + C_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta} \right\}, \tag{3.8}$$

where

$$\xi = k_1 x + \frac{C_4 k_1}{C_6} y + \int \frac{-(C_3 C_6 + C_4 b_0(t)) k_1 C_4}{C_6^2} dt + C_7,$$

$$\eta = k_2 x + \frac{C_4 k_2}{C_6} y + \int \frac{-(-C_3 C_6 + C_4 b_0(t)) k_2 C_4}{C_6^2} dt + C_8.$$

We should point out that the solutions obtained in this paper are not only the (3.5)-(3.8). We list some new ones corresponding to Case 1, to show that our method is effective in constructing the new multiple soliton-like solutions of equations (3.1).

4. Summary and Conclusion

In summary, the extended coupled sub-ODEs method with symbolic computation is developed to deal with the nonlinear $(2+1)$-dimensional

Painlevé integrable Burgers equations with variable coefficients (3.1). Then when applying the proposed method to the above nonlinear equations, a rich variety of exact solutions which include: (a) double solitary-like wave solutions, (b) double trigonometric function solutions, (c) complexiton soliton solutions, is obtained. The extended sub-ODEs can also apply to other NLPDEs.

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