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SOME RESULTS ON HESSENBERG MATRICES

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Abstract

We obtain some new Hessenberg matrices and their corresponding determinants by adding two well-known Hessenberg matrices.

1. Introduction

A matrix is said to be a *Hessenberg matrix* [1] if all entries above the superdiagonal are zero. For instance, the matrix [1],

$$E_n = \begin{bmatrix} 3 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 3 & 1 & \cdots & \cdots & \vdots \\ 0 & 1 & 3 & 1 & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 1 & 3 & 1 \\ 0 & \cdots & \cdots & \cdots & 1 & 3 \end{bmatrix}$$

is a Hessenberg matrix and its determinant is F_{2n+2}. In [1], the author introduced several types of Hessenberg matrices whose determinants are

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Fibonacci numbers were calculated by using the basic definition of the determinant as a signed sum over the symmetric group. In [2], Li et al. proved the above results by investigating first the feasibility of LU factorization, i.e., a lower triangular matrix with unit main diagonal and an upper triangular matrix. Furthermore, they found the determinant of a new class of Hessenberg matrices.

In this paper, we try to calculate the determinants of some new Hessenberg matrices obtained by adding two well-known Hessenberg matrices.

2. Main Results

Let $A_{n,t}$ be the $n \times n$ Hessenberg matrix in which the superdiagonal entries are 1, all main diagonal entries are 1 except the last one, which is $t + 1$, and the entries of each column below the main diagonal alternate 0's and 1's, starting with 0 and t is an indeterminate. Let $C_{n,t}$ be the matrix in which the superdiagonal entries are –1's, all main diagonal are 2's except the last one, which is $t + 1$, and all entries below the diagonal are 1's. Let G_n be the Hessenberg matrix in which the superdiagonal entries are 1's, the main diagonal entries are 2's, and the entries of each column below the main diagonal alternate –1's and 1's, starting with –1. Let $G_{n,t}$ be the matrix obtained from G_n by replacing the lowest diagonal 2 with $t + 1$. Let H_n be the matrix obtained by changing the superdiagonal entries of G_n to –1's. Let $H_{n,t}$ be the matrix obtained from H_n by replacing the lowest diagonal 2 with $t + 1$.

Now we are in a position to state and prove the main theorems.

Theorem 2.1. $\det(A_{n,t} + C_{n,t}) = 2 \times 3^{n-1}(t + 1)$.

Proof. Because

$$A_{n,t} + C_{n,t} = \begin{bmatrix} 3 & 0 & & & & \\ 1 & 3 & 0 & & 0 & \\ 2 & 1 & 3 & \ddots & & \\ 1 & 2 & \ddots & \ddots & \ddots & \\ 2 & 1 & \ddots & \ddots & \ddots & 0 \\ \cdots & \cdots & \cdots & 2 & 1 & 2t+2 \end{bmatrix},$$

so $A_{n,t} + C_{n,t}$ is a lower triangular matrix and also a Hessenberg matrix. As we know, the determinant of a triangular matrix is the product of the main diagonals, hence

$$\det(A_{n,t} + C_{n,t}) = 3 \times 3 \times \cdots \times 3 \times (2t+2) = 2 \times 3^{n-1}(t+1).$$

This completes the proof.

Example 2.1. Let $n = 5$. Then

$$A_{5,t} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & t+1 \end{bmatrix};$$

$$C_{5,t} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 \\ 1 & 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 1 & t+1 \end{bmatrix};$$

$$A_{5,t} + C_{5,t} = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 3 & 0 \\ 2 & 1 & 2 & 1 & 2t+2 \end{bmatrix}.$$

And

$$\det(A_{5,t} + C_{5,t}) = 3 \times 3 \times 3 \times 3 \times (2t + 2) = 2 \times 3^4(t + 1).$$

Theorem 2.2. $\det(C_{n,t} + G_{n,t}) = 2^{2n-1}(t + 1).$

Proof. Because

$$C_{n,t} + G_{n,t} = \begin{bmatrix} 4 & 0 & & & & \\ 0 & 4 & 0 & & 0 & \\ 2 & 0 & 4 & \ddots & & \\ 0 & 2 & \ddots & \ddots & \ddots & \\ 2 & 0 & \ddots & \ddots & \ddots & 0 \\ \cdots & \cdots & \cdots & 2 & 0 & 2t+2 \end{bmatrix},$$

so it is obvious that $C_{n,t} + G_{n,t}$ is a lower triangular matrix and also a Hessenberg matrix. As we know, the determinant of a triangular matrix is the product of the main diagonals, hence

$$\det(C_{n,t} + G_{n,t}) = 4 \times 4 \times \cdots \times 4 \times (2t + 2) = 2^{2n-1}(t + 1).$$

This completes the proof.

Example 2.2. Let $n = 5$. Then

$$C_{5,t} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 \\ 1 & 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 1 & t+1 \end{bmatrix};$$

$$G_{5,t} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ -1 & 1 & -1 & 2 & 1 \\ 1 & -1 & 1 & -1 & t+1 \end{bmatrix};$$

$$C_{5,t} + G_{5,t} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 4 & 0 \\ 2 & 0 & 2 & 0 & 2t+2 \end{bmatrix}.$$

And

$$\det(C_{5,t} + G_{5,t}) = 4 \times 4 \times 4 \times 4 \times (2t+2) = 2^9(t+1).$$

Theorem 2.3. $\det(G_{n,t} + H_{n,t}) = 2^{2n-1}(t+1)$.

Proof. Because

$$G_{n,t} + H_{n,t} = \begin{bmatrix} 4 & 0 & & & & \\ -2 & 4 & 0 & & 0 & \\ 2 & -2 & 4 & \ddots & & \\ -2 & 2 & \ddots & \ddots & \ddots & \\ 2 & -2 & \ddots & \ddots & \ddots & 0 \\ \cdots & \cdots & \cdots & 2 & -2 & 2t+2 \end{bmatrix},$$

so it is obvious that $G_{n,t} + H_{n,t}$ is a lower triangular matrix and also a Hessenberg matrix. As we know, the determinant of a triangular matrix is the product of the main diagonals, hence

$$\det(G_{n,t} + H_{n,t}) = 4 \times 4 \times \cdots \times 4 \times (2t+2) = 2^{2n-1}(t+1).$$

This completes the proof.

Example 2.3. Let $n = 5$. Then

$$G_{5,t} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ -1 & 1 & -1 & 2 & 1 \\ 1 & -1 & 1 & -1 & t+1 \end{bmatrix};$$

$$H_{5,t} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ -1 & 1 & -1 & 2 & -1 \\ 1 & -1 & 1 & -1 & t+1 \end{bmatrix};$$

$$G_{5,t} + H_{5,t} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 \\ 2 & -2 & 4 & 0 & 0 \\ -2 & 2 & -2 & 4 & 0 \\ 2 & -2 & 2 & -2 & 2t+2 \end{bmatrix}.$$

And

$$\det(C_{5,t} + G_{5,t}) = 4 \times 4 \times 4 \times 4 \times (2t+2) = 2^9(t+1).$$

3. Conclusions

In the near future, we will discuss the sums of another two Hessenberg matrices and try to calculate the determinants of the referred Hessenberg matrices.

References

[1] Morteza Esmaeili, More on the Fibonacci sequence and Hessenberg matrices, Integers 6 (2006), #A32.

[2] Hsuan-Chu Li, Young-Ming Chen and Eng-Tjioe Tan, Counting determinants of Fibonacci-Hessenberg matrices using LU factorizations, Integers 9 (2009), #A37.