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THE PARTITION DIMENSION OF THE CORONA PRODUCT OF TWO GRAPHS

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Abstract

Let $G(V, E)$ be a connected graph. For a vertex $v \in V(G)$ and a subset S of $V(G)$, the distance $d(v, S)$ from v to S is $\min\{d(v, w) | w \in S\}$. For an ordered k-partition $\Pi = \{S_1, S_2, ..., S_k\}$

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of $V(G)$, the representation of v with respect to Π is $r(v|\Pi) = (d(v, S_1), d(v, S_2), ..., d(v, S_k))$. The k-partition Π is called a resolving partition of G if all $r(v|\Pi)$ for all $v \in V(G)$ are distinct. The partition dimension of a graph G is the smallest k such that G has a resolving k-partition. In this paper, we derive an upper bound of the partition dimension of the corona product $G \odot H$, where G, H are connected graphs and the diameter of H is at most 2. Furthermore, we also determine the exact value of the partition dimension of this corona product if G is either a path or a complete graph and H is a complete graph.

1. Introduction

One of the problems in graph theory with applications to chemistry deals with determining representations for the vertices of a graph such that distinct vertices have distinct representations. A representation defined in terms of distances and partitions was firstly studied by Chartrand et al. [4]. For any $u, v \in V(G)$, define the *distance* $d(u, v)$ from u to v as the length of the shortest path connecting these two vertices in G. For $v \in V(G)$ and $S \subset E(G)$, the *distance* $d(v, S)$ from u to S is defined as $\min\{d(v, x) | x \in S\}$. In particular, if $d(x, S) \neq d(y, S)$, then we shall say that x and y are *distinguished* by S or x and y are *distinguishable*. For an ordered k-partition $\Pi = \{S_1, S_2, ..., S_k\}$ of $V(G)$ and $v \in V(G)$, the *representation* of $v \in V(G)$ with respect to Π is the k-vector

$$r(v|\Pi) = (d(v, S_1), d(v, S_2), ..., d(v, S_k)).$$

We call Π a *resolving partition* if $r(u|\Pi) \neq r(v|\Pi)$ for every two distinct vertices $u, v \in G$. The *partition dimension* $pd(G)$ of graph G is the minimum cardinality of any resolving partition of $V(G)$.

Let $S_{m,n}$ be a double star, namely, a tree with two vertices of degree m and n and the remaining vertices of degree 1. In [4], Chartrand et al. showed that the partition dimension of $S_{m,n}$ is $\max\{m, n\} - 1$. Moreover, they also gave the sharp lower and upper bounds of the partition dimension of a caterpillar, namely, a tree having the property that the removal of its end-vertices results in a path. A construction of a tree T on n vertices with partition dimension k (for any k, $2 \leq k \leq n-1$, but $k \neq n-2$) is also given. Other result concerning caterpillar can be also seen in [5]. However, the partition dimension of any general tree is an open problem.

Finding the partition dimension of any graph in general is classified as an *NP*-hard problem [2]. The characterization studies for all graphs having certain partition dimension have been also conducted, for instance, see [2] and [10].

Some investigations have been also conducted to determine partition dimensions with some additional criteria for certain classes of graphs. For instance, Saenpholphat and Zhang [9] and Tomescu et al. [11] considered *connected resolving partition* in which the induced subgraph of each set in the partition is connected. Marinescu-Ghemeci and Tomescu [7] investigated *star partition dimension* of generalized gear graphs and Ruxandra [8] studied partition dimension of graph in which the induced subgraph of each set in the partition is a *path*.

Finding a relationship (in terms of partition dimension) between the original graphs and the resulting graph under some graph operation is also interesting to be considered. For instances, let us define the *corona product* $G \odot H$ between G and H as the graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and then joining by an edge each vertex of the ith-copy of H with the ith-vertex of G. In this paper, we are interested in determining the partition dimension of graph $G \odot H$. We derive an upper bound of the partition dimension of a corona product graph $G \odot H$ for any connected graphs G and H with the diameter of H is at most 2, namely, $pd(G \odot H) \leq pd(F) + pd(H)$. We also show that this upper bound is tight. Furthermore, we determine the partition dimension of $G \odot H$, if G is either a path or a complete graph and H is a complete graph.

The following lemma is useful in determining the partition dimension of a graph *G*.

Lemma 1 [3]**.** *Let G be a connected non trivial graph. Let* Π *be a resolving partition for G and* $u, v \in V(G)$. *If* $d(u, w) = d(v, w)$ *for all* $w \in V(G) - \{u, v\}$, *then u and v belong to different sets in* Π.

2. The Upper Bound of $pd(G \odot H)$

The *diameter* of a graph *G* is $\max\{d(x, y) \mid x, y \in V(G)\}$. In this section, we shall derive an upper bound of $pd(G \odot H)$ for any connected graphs *G* and *H* with diameter of *H* is at most 2.

Lemma 2. *Let G and H be connected graphs. Let* H_i *be ith-copy of H in* $G \odot H$. *Then any two vertices u and v of* H_i *can be only distinguished by some set in which has intersection not empty with the set of vertices of* H_i.

Proof. Since $d(u, w) = d(v, w)$ for all $w \in V(G \odot H) \backslash H_i$, vertices *u* and *v* can be only distinguished by some vertex in H_i. □

Theorem 1. *Let G and H be connected graphs. If the diameter of H is at most* 2, *then* $pd(G \odot H) \leq pd(G) + pd(H)$.

Proof. Let Π_G and Π_H be minimum resolving partitions of *G* and *H*, respectively. Let $|V(G)| = n$. *For* $i = 1, 2, \ldots, n$, partition the vertices of each H_i according to Π_H, say $\{H_i^1, H_i^2, \ldots, H_i^s\}$, where $s = pd(H)$. Now, consider the partition $\Pi = \Pi_1 \cup \Pi_2$, where $\Pi_1 = \{\bigcup_{i=1}^n H_i^1, \bigcup_{i=1}^n H_i^2, \ldots, \bigcup_{i=1}^n H_i^s\}$ and $\Pi_2 = \Pi_G$. Then we shall show that Π is a resolving partition of $G \odot H$. Note that since the diameter of *H* is at most 2, the distance of any two vertices $u, v \in V(H_i)$, for any *i*, under the corona graph $G \odot H$ is the same as its distance under the original graph *H*. Therefore, if the vertices $u, v \in V(H_i)$, for any *i*, are distinguishable by Π_H, then they are distinguishable too by Π_1. Let u and v be any two vertices of $G \odot H$. If $u, v \in V(H_i)$, then they will be clearly distinguished by $\bigcup_{i=1}^{n} H_i^t$ for some t. If $u, v \in V(G)$, then they will be distinguished by some set in Π_G. Now, assume that $u \in V(H_i)$ and $v \in V(G)$. If $u \in \bigcup_{i=1}^{n} H_i^t$ for some t, then the distances between u and v to $\bigcup_{i=1}^{n} H_i^t$ is 0 and 1, respectively. Therefore, u and v are distinguished. Now, the only case we have not considered is $u \in V(H_i)$ and $v \in V(H_j)$, for $i \neq j$. If $u, v \in \bigcup_{i=1}^{n} H_i^t$ for some t, then u, v are distinguished by some set in Π_G since Π_G is a resolving partition for G. □

In the following sections, we will determine the exact value of $pd(G \odot H)$ if $H \cong K_n$ and G is either a path or a complete. We also show that the bound in Theorem 1 is tight.

3. The Corona Product $P_m \odot K_n$

Now, we consider the corona product $G \cong P_m \odot K_n$, where P_m represents a path order m and K_n is the complete graph on n vertices. Let the vertex-set $V(G) = \{x_i \mid 1 \le i \le m\} \cup \{a_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$ and the edge-set

$$E(G) = \{x_{i-1}x_i \mid 2 \le i \le m\} \cup \{x_i a_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$$

$$\cup \{a_{ik}a_{il} \mid 1 \le i \le m, 1 \le k \le l \le n\}.$$

We will show that the upper bound of Theorem 1 is satisfied by $pd(P_m \odot K_n)$ provided $m > n + 2$.

Theorem 2. *For* $m \ge 2$ *and* $n \ge 4$, *the partition dimension of* $P_m \odot K_n$ *is as follows*:

$$pd(P_m \odot K_n) = \begin{cases} n+1, & if\ m \le n+2, \\ n+2, & if\ m \ge n+3. \end{cases}$$

Proof. Let $\Pi = \{S_1, S_2, ..., S_k\}$ be an ordered resolving partition of $G \cong P_m \odot K_n$. For $i = 1, 2, ..., m$, let $V(H_i) = \{a_{i1}, a_{i2}, ..., a_{in}\}$ be vertices of the ith-copy of K_n in G. Then each vertex in H_i must be in a different set in Π. Since $m \geq 2$, we need at least $n + 1$ sets in Π. Otherwise, the representations of a_{i1} and a_{j1} belonging to the same set in Π, for $i \neq j$, are the same. Therefore, $k \geq n + 1$.

Now, consider the case of $m \leq n + 2$. Define an ordered partition $\Pi = \{S_1, S_2, ..., S_{n+1}\}$ of G such that:

a. $x_1 \in S_1$, $\{x_2, x_3, x_4\} \subset S_5$, $\{x_5, x_6, ..., x_m\} \subset S_1$;

b. All vertices of H_1 are distributed equally into n partitions other than S_1;

c. All vertices of H_2 are distributed equally into n partitions other than S_2;

d. All vertices of H_3 are distributed equally into n partitions other than S_1;

e. For $t = 4, 5, ..., m$, all vertices of H_t are distributed equally into n partitions other than S_{t-1}. See Figure 1.

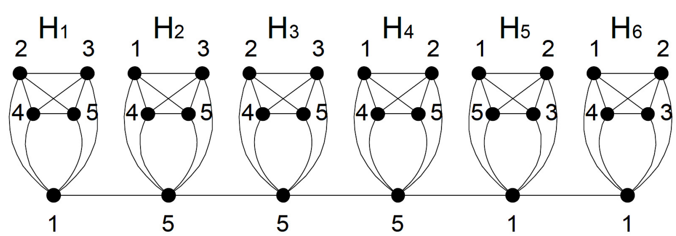

Figure 1. Resolving partition for corona product graph $P_6 \odot K_4$.

We claim that Π is a resolving partition of G. To prove it, let us consider two different vertices u, v of G in the same set in Π. If $u \in V(H_i)$,

$v \in V(H_j)$ for $i < j$, and $\{i, j\} \neq \{1, 3\}$, then $d(u, S_{j-1}) \neq d(v, S_{j-1})$ or $d(u, S_1) \neq d(v, S_1)$. Therefore, $r(u|\Pi) \neq r(v|\Pi)$. Now, if $u \in \{x_1, x_2, ..., x_m\}$ and $(v \in \{x_1, x_2, ..., x_m\}$ or $v \in S_t)$, then $d(u, S_b) \neq d(v, S_b)$, where S_b is the partition not containing any vertex of H_t. Therefore, again $r(u|\Pi) \neq r(v|\Pi)$. Thus, we obtain that Π is the revolving partition of G. This implies that $pd(G) = n + 1$ if $m \leq n + 2$.

Now, consider the case of $m \geq n + 3$. We will show that $pd(G) = n + 2$. To show the lower bound, for a contradiction assume there is an ordered resolving partition Π of G with $n + 1$ sets. Let $\Pi = \{S_1, S_2, ..., S_{n+1}\}$.

By Lemma 1, any two vertices in H_i, for each i, belong to different sets of Π. Therefore, for $i = 1, 2, ..., m$ we can define $c_i = b$ if no vertex of H_i is in S_b. Then since $m \geq n + 3$ and $1 \leq b \leq n + 1$, there exist i, j, l and $i < j < l$ such that $c_i = c_j = c_l = b$ for some b or there exist i, j, l, s and $i < j < l < s$ *such that* $c_i = c_j = b$ and $c_l = c_s = c$ for some b and c.

It is clear that the sets H_i and H_j which are the same cannot be adjacent, namely, $j \neq i + 1$. Since otherwise $d(x_j, S_b) = d(w, S_b)$ for some $w \in V(H_i)$ or $d(x_i, S_b) = d(w, S_b)$ for some $w \in V(H_j)$. Since $d(x_i, S_t) = d(x_j, S_t) = d(w, S_t) = 1$ for all $t \neq b$, $r(x_j|\Pi) = r(w|\Pi)$ or $r(x_i|\Pi) = r(w|\Pi)$, a contradiction. Therefore, $j - i > 1$, $l - j > 1$, and $s - l > 1$.

Now, consider the first case, namely, $c_i = c_j = c_l = b$. In order to have the representation of each vertex of G with respect to Π is distinct, then $\{d(H_i, S_b), d(H_j, S_b), d(H_l, S_b)\} = \{1, 2, 3\}$ (since $j - i > 1$ and $l - j > 1$), where $d(H_i, S_b)$ is the distance between the whole vertices of H_i to S_b. This implies that one of $\{x_i, x_j, x_l\}$ is in S_b, say $x_i \in S_b$, and one of them has distance 1 to S_b, say x_j. But, then we get $r(x_j|\Pi) = r(w|\Pi)$ for some $w \in V(H_i)$, a contradiction. Therefore, the first case is not possible.

Next, we consider the case of $c_i = c_j = b$ and $c_l = c_s = c$ for some b and c, $b \neq c$. Again, since $j - i > 1$, $l - j > 1$, and $s - l > 1$, $\{d(H_i, S_b), d(H_j, S_b)\} \subset \{1, 2, 3\}$ and $\{d(H_l, S_c), d(H_s, S_c)\} \subset \{1, 2, 3\}$. Clearly, at most one of $\{x_i, x_j\}$ is in S_b. If $x_i \in S_b$, then all vertices of H_i together with x_i are dominant, namely, all ordinates of its representation with respect to Π are 1's. Furthermore, if $x_i \in S_b$, then no one of $\{x_l, x_s\}$ is in S_c. Since otherwise, there are too many dominant vertices in G, namely, the number of dominant vertices greater than the partition dimension. Therefore, $\{d(H_l, S_c), d(H_s, S_c)\} = \{2, 3\}$. Thus, either one of $\{x_l, x_s\}$ has distance 1 to S_c, say x_l. This yields x_l as a dominant vertex; But now $r(x_l | \Pi) = r(w|\Pi)$, for some $w \in V(H_i) \cup \{x_i\}$. Therefore, as a conclusion, no one of $\{x_i, x_j\}$ is in S_b (similarly, no one of $\{x_l, x_s\}$ is in S_c). Hence, $\{d(H_i, S_b), d(H_j, S_b)\} = \{2, 3\}$ and $\{d(H_l, S_c), d(H_s, S_c)\} = \{2, 3\}$. In this case, we may assume $d(H_i, S_b) = 2$ and $d(H_j, S_b) = 3$. But, then $r(x_j | \Pi) = r(w|\Pi)$, for some $w \in V(H_i)$, a contradiction. Therefore, the second case is also not possible. This means that $pd(G) \geq n + 2$ if $m \geq n + 3$.

Now, to show the upper bound, for $m \geq n + 3$, define a resolving partition $\Pi = \{S_1, S_2, ..., S_{n+2}\}$ of G such that:

$$S_k = \begin{cases} \{a_{1k}, a_{2k}, ..., a_{mk}\}, & \text{if } 1 \leq k \leq n, \\ \{x_2, x_3, ..., x_m\}, & \text{if } k = n + 1, \\ \{x_1\}, & \text{if } k = n + 2. \end{cases}$$

Clearly, any two vertices in S_k, for $k \in \{1, 2, ..., n + 1\}$, have different distances to S_{n+2}. Therefore, their representations with respect to Π will be not the same. This means Π is the resolving partition of G; thus $pd(G) \leq n + 2$ for $m \geq n + 3$. □

Now, let us consider the graph $G \cong P_m \odot K_n$, with $m \geq 2$, and $n = 2, 3$. For $m \leq n + 2$, define a partition $\Pi = \{S_1, S_2, ..., S_{n+1}\}$ of G such that:

a.

$$S_1 = \{a_{21}, a_{41}, x_3\}, S_2 = \{a_{11}, a_{31}, a_{42}, x_4\},$$

$$S_3 = \{a_{12}, a_{22}, a_{32}, x_1, x_2\}, \text{ for } n = 2, m = 4;$$

b.

$$S_1 = \{a_{21}, a_{41}, a_{51}, x_3\}, S_2 = \{a_{11}, a_{31}, a_{42}, a_{52}, x_4, x_5\},$$

$$S_3 = \{a_{12}, a_{22}, a_{32}, a_{53}, x_1, x_2\}, S_4 = \{a_{13}, a_{23}, a_{33}, a_{43}\},$$

for $n = 3, m = 5$.

It is easy to see that Π is a resolving partition of G. Now, if $2 \leq m \leq n+1$, then by removing all elements a_{ij} and x_i with $i \geq m+1$ from all sets in the above Π, we will get the resolving partition of G for this case m. Next, consider $m \geq n+3$ and $n = 2, 3$. By using the same argument and the same partition like in the proof of the case $m \geq n+3$ and $n \geq 4$, we can show that $pd(G) = n + 2$. Therefore, we have the following theorem:

Theorem 3. *For $m \geq 2$ and $n = 2, 3$, the partition dimension of $P_m \odot K_n$ is as follows*:

$$pd(P_m \odot K_n) = \begin{cases} n+1, \text{ if } m \leq n+2, \\ n+2, \text{ if } m \geq n+3. \end{cases}$$

From Theorems 2 and 3, note that for $m \geq n+3$ the partition dimension $pd(P_m \odot K_n)$ is $n + 2$. This means that the upper bound of Theorem 1 is sharp.

4. The Corona Product $K_m \odot K_n$

In this section, we determine the partition dimension of $G \cong K_m \odot K_n$, the corona product of the complete graph K_m to K_n. Let the vertex-set $V(G) = \{x_i \mid 1 \le i \le m\} \cup \{a_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$ and the edge-set

$$E(G) = \{x_i x_j \mid 1 \le i < j \le m\} \cup \{x_i a_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$$
$$\cup \{a_{ik} a_{il} \mid 1 \le i \le m, 1 \le k < l \le n\}.$$

For simplicity, denote by $V(H_i) = \{a_{i1}, a_{i2}, ..., a_{in}\}$ the vertices of ith-copy of K_n with attach to vertex x_i in K_m.

Theorem 4. *Let* $G \cong K_m \odot K_n$, *with* $m \ge 2$ *and* $n \ge 3$. *Then*

a. $pd(G) = n + 1$ *iff* $2 \le m \le \binom{n+1}{n}$.

b. $pd(G) = n + 2$ *iff* $\binom{n+1}{n} + 1 \le m \le \binom{n+2}{n} + 1$.

c. $pd(G) \le n + k$, *if* $\binom{n+k-1}{n} + 1 \le m \le \binom{n+k}{n}$, *and* $k \ge 3$.

Proof. We shall divide the proof into three cases:

Case 1. $2 \le m \le \binom{n+1}{n}$.

Consider the vertices in H_i in G, for some i. By Lemma 1, any two of them must be in different partitions in a resolving partition Π of G. Therefore, we require n distinct partitions in Π for the vertices of H_i only. But, since $m \ge 2$, $|\Pi| \ge n + 1$. Now, if $m \le \binom{n+1}{n}$, then define an ordered partition $\Pi = \{S_1, S_2, ..., S_{n+1}\}$ of G such that:

a. All x_i's, for $i = 1, 2, ..., m$ belong to S_1;

b. For each i, distribute equally all n vertices of H_i into n distinct partitions other than S_i.

Then, by this definition, it is easy to verify that Π is a resolving partition of G. Now, let $m \geq \binom{n+1}{n} + 1$ and assume for a contradiction $|\Pi| = n + 1$. Then there are two distinct H_i and H_j such that their vertices are distributed to the same combination of n partitions of Π. Let $c_i = c_j = b$ if no vertex of $H_i(H_j)$ is in S_b. Then x_i and x_j must be in different partitions and one of $\{x_i, x_j\}$ is in S_b, say x_i. However, now $r(x_j | \Pi) = r(w|\Pi)$ for some $w \in V(H_i)$, a contradiction. Therefore, the first statement and the lower bound of the second statement have been proved.

Case 2. $\binom{n+1}{n} + 1 \leq m \leq \binom{n+2}{n} + 1.$

Let $T = \{\text{all } n\text{-combinations from } n + 2 \text{ distinct numbers}\}.$

Let $\Pi = \{S_1, S_2, ..., S_{n+2}\}$. Since all vertices of each H_i must be in n different partitions, each H_i can be associated with an n-combination in T. Now, we can define $c_i = \{a, b\}$ if S_a and S_b both do not contain any vertex of H_i. To show $pd(G) = n + 2$, define Π as follows:

a. Assign H_i, for $i = 1, 2, ..., m$ to all members in T such that

$$c_1 = \{1, 2\},\ c_2 = \{1, 2\},\ c_3 = \{1, 3\},\ c_4 = \{1, 4\}, ..., c_{m-1},\ c_m$$

are in a lexicographical order,

b. $x_1 \in S_1$, $x_2 \in S_2$, and

c. For $i = 3, 4, ..., m$, put x_i into S_1 if $2 \in c_i$; Otherwise x_i is put into S_2. See Figure 2.

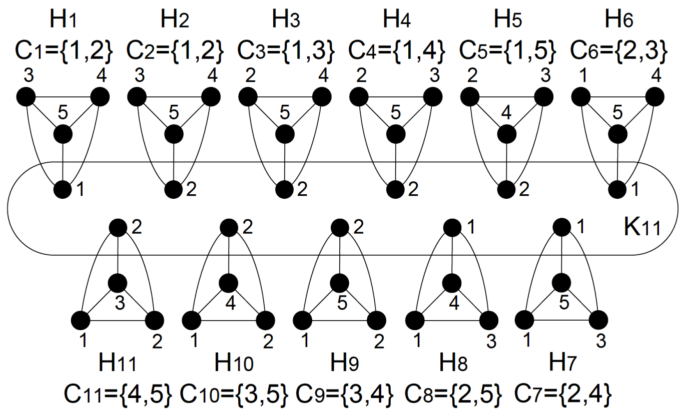

Figure 2. Resolving partition for corona product graph $K_{11} \odot K_3$.

We shall show that Π is a resolving partition of G. To do so, take any two vertices u, v in the same partition in Π. If $u \in V(H_i)$ and $v \in (H_j)$, for $i < j$, then $d(u, S_b) \neq d(v, S_b)$, where $b \in c_i - c_j$ and $b \neq 1, 2$ (provided $c_i - c_j \neq \varnothing$; otherwise set $b = 1$). Therefore, $r(u|\Pi) \neq r(v|\Pi)$. If $u \in V(H_i)$ and $v = x_j$ for some i and j, then $\{u, v\} \subset S_1$ or $\{u, v\} \subset S_2$. In both cases, we will get $d(u, S_b) \neq d(v, S_b)$, where $b \in c_i - c_j$ and $b \neq 1, 2$ (provided $i \neq j$; otherwise take any $b \in c_i$). Therefore, again, $r(u|\Pi) \neq r(v|\Pi)$. Now, let $u \in x_i$ and $v \in x_j$ for $i < j$. By a similar argument, we can show that $r(u|\Pi) \neq r(v|\Pi)$. Therefore, Π is a resolving partition of G provided $\binom{n+1}{n} + 1 \leq m \leq \binom{n+2}{n} + 1$.

Next, we shall show that if $pd(K_m \odot K_n) = n + 2$, then $\binom{n+1}{n} + 1 \leq m \leq \binom{n+2}{n} + 1$. To do so, for a contradiction assume that $pd(K_m \odot K_n) = n + 2$ for $m = \binom{n+2}{n} + 2$. Let Π be a resolving partition of $K_m \odot K_n$. Since $m = \binom{n+2}{n} + 2$, there exist i, j, l and $i < j < l$ such that $c_i =$

$c_j = c_l = \{a, b\}$ or there exist i, j, l, s and $i < j < l < s$ such that $c_i = c_j = \{a, b\}$ and $c_l = c_s = \{s, t\}$.

For the first case, we may assume, without loss of generality, $c_i = c_j = c_l = \{1, 2\}$. To distinguish all the vertices in H_i, H_j, and H_l, the vertices x_i, x_j, x_l must be in different partition of Π and two of them must be in S_1 and S_2, say $x_i \in S_1$, $x_j \in S_2$, $x_l \in S_3$. Then these three x_i, x_j, x_l are dominant vertices, namely, the ordinates of their representations are all 1's. Now, consider the vertex x_{r_1} adjacent to H_{r_1} with $c_{r_1} = \{1, 3\}$. Then x_{r_1} must be also dominant. Therefore, $x_{r_1} \notin S_1 \cup S_2 \cup S_3$. We may assume $x_{r_1} \in S_4$. Now, consider the vertex x_{r_2} adjacent to H_{r_2} with $c_{r_2} = \{1, 4\}$. Similarly, $x_{r_2} \in S_5$. We do this process for all x_h in K_m to obtain that all these vertices are dominant. Therefore, we have more than $n + 2$ dominant vertices, a contradiction. Thus, the first case is not possible.

For the second case, we assume, without loss of generality, $(c_i = c_j = \{1, 2\}$ and $c_l = c_s = \{1, 3\})$ or $(c_i = c_j = \{1, 2\}$ and $c_l = c_s = \{3, 4\})$. First, let $c_i = c_j = \{1, 2\}$ and $c_l = c_s = \{1, 3\}$. To distinguish all the vertices of H_i, H_j and H_l, H_s, one of $\{x_i, x_j\}$ must be in either S_1 or S_2, and one of $\{x_l, x_s\}$ must be in either S_1 or S_3. By symmetry, we may assume that $x_i, x_l \in S_1$. Now, consider x_j and x_s. If $x_j \in S_2$, then x_i and x_j are dominant vertices. Vertex x_s cannot be in S_3, since otherwise x_l becomes dominant (too many dominant in S_1, namely, more than one vertices in S_1 are dominant). Thus, x_s is in either S_2 or S_t for $t \geq 4$. If $x_s \in S_2$, then consider the vertex x_{r_1} adjacent to H_{r_1} with $c_{r_1} = \{2, 3\}$. For sure, x_{r_1} cannot be in $S_1 \cup S_2$, since otherwise its representation will be the same with the one of x_l or x_s. But, x_{r_1} cannot be in S_3 to avoid x_l and x_i becoming dominant vertices from the same set. Therefore, x_{r_1} must be in

S_t, $t \geq 4$, say $x_{r_1} \in S_4$. Now, consider x_{r_2} adjacent to H_{r_2} with $c_{r_2} = \{2, 4\}$. Then x_{r_2} must be a dominant vertex since it is adjacent to x_j and x_{r_1}. Thus, w.l.o.g., $c_{r_2} \in S_4$ or S_5. We do this process for all x_h in K_m to obtain that all these vertices are dominant. Therefore, we have more than $n + 2$ dominant vertices, a contradiction.

Now, consider $x_j \in S_2$ and $x_s \in S_4$. In this case, x_i, x_j are dominant. Next, consider x_{r_1} adjacent to H_{r_1} with $c_{r_1} = \{1. 4\}$. This vertex x_{r_1} is also dominant, since adjacent to x_i and x_s. Therefore, x_{r_1} must in either S_t, $t \geq 4$, say $x_{r_1} \in S_4$. We do this process for all x_h in K_m to obtain that all these vertices are dominant. Therefore, we have more than $n + 2$ dominant vertices, a contradiction.

Next, consider x_i, $x_l \in S_1$ and $x_j \in S_3$. In this case, x_l is dominant. For sure, x_s cannot be in S_2 (since x_i and x_l become both dominant) or S_3 (by symmetry argument above). Therefore, x_s must be in S_t, $t \geq 4$, say w.l.o.g., $x_s \in S_4$. Now, consider the vertex x_{r_1} adjacent to H_{r_1} with $c_{r_1} = \{1, 4\}$. Thus, x_{r_1} must be a dominant vertex. Therefore, x_{r_1} must be in either S_t, $t \geq 3$, say $x_{r_1} \in S_3$. But, now x_s is also dominant. Let us now consider vertex x_{r_2} adjacent to H_{r_2} with $c_{r_2} = \{3, 4\}$. Then x_{r_2} must be a dominant vertex since it is adjacent to x_s and x_{r_1}. Thus, w.l.o.g., $c_{r_2} \in S_5$. We do this process for all x_h in $V(K_m)$ to obtain that all these vertices are dominant. Therefore, we have more than $n + 2$ dominant vertices, a contradiction.

Second, consider $c_i = c_j = \{1, 2\}$ and $c_l = c_s = \{3, 4\}$. To distinguish all the vertices of H_i, H_j and H_l, H_s, then x_i and x_j must be in different partitions and one of $\{x_i, x_j\}$ is in either S_1 or S_2, and one of $\{x_l, x_s\}$ must be in either S_3 or S_4 and they are in different partitions. By symmetry, we may assume that $x_i \in S_1$ and $x_l \in S_3$. Now, consider x_j and x_s. If one of either $x_j \notin S_3$ or $x_s \notin S_1$ holds, then we have three partitions holding $x_i,\ x_j,\ x_l,\ x_s$. Any two combinations will give another a dominant vertex x_{r_1} by similar method above. We do this process for all x_h in $V(K_m)$ to obtain that all these vertices are dominant. Therefore, we have more than $n + 2$ dominant vertices, a contradiction.

Now, the only remaining case is $x_i \in S_1,\ x_l \in S_3,\ x_j \in S_3$ and $x_s \in S_1$. Let us consider x_{r_1} adjacent to H_{r_1} with $c_{r_1} = \{1, 3\}$. Since it is also adjacent to x_i and x_l, then x_{r_1} must be a dominant vertex. If $x_{r_1} \notin S_1 \bigcup S_3$, then we have three partitions holding $x_i,\ x_j,\ x_l,\ x_s$ and x_{r_1}. Therefore, by the similar method above, we will have too many dominant vertices, a contradiction. Thus, $x_{r_1} \in S_1$. But, now consider vertex x_{r_2} adjacent to H_{r_2} with $c_{r_2} = \{2, 3\}$. This vertex cannot be in $S_1 \bigcup S_2 \bigcup S_3$. Therefore, $x_{r_2} \in S_t$, $t \geq 4$. Thus, we have three partitions holding these vertices so far. This implies that there will be too many dominant vertices, a contradiction. This completes the proof of the second statement.

Case 3. $\binom{n+k-1}{n} + 1 \leq m \leq \binom{n+k}{n}$, and $k \geq 3$.

Let $T = \{\text{all } n\text{-combinations from } n + k \text{ distinct numbers}\}$.

Let $\Pi = \{S_1,\ S_2,\ ...,\ S_{n+k}\}$. Since all vertices of each H_i must be in n different partitions, each H_i can be associated with an n-combination in T. Then define Π as follows:

a. Assign H_i, for $i = 1, 2, ..., m$ to a member in T so that no two H_i, H_j have been assigned to the same member of T.

b. Put all vertices $x_i's$ is into S_1.

It is clear that Π is a resolving partition of G. Therefore, $pd(G) \leq n + k$ in this case. □

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