Far East Journal of Applied Mathematics
Volume 66, Number 1, 2012, Pages 51-67
Published Online: June 2012
Available online at http://pphmj.com/journals/fjam.htm
Published by Pushpa Publishing House, Allahabad, INDIA

NEW EXACT TRAVELLING WAVE SOLUTIONS OF THE GENERALIZED ITO EQUATION AND BBMB EQUATION

Ming Song*, **Jianshe Zhou and Jun Cao**

Department of Mathematics
Faculty of Sciences
Yuxi Normal University
Yuxi, 653100, P. R. China
e-mail: songming12_15@163.com

Abstract

In this paper, the $\left(\frac{G'}{G}\right)$-expansion method is employed to solve the generalized Ito equation and Benjamin-Bona-Mahony-Burgers (BBMB) equation. New exact travelling wave solutions are obtained. The travelling wave solutions are expressed by hyperbolic functions, the trigonometric functions and the rational functions.

1. Introduction

The investigation of the exact travelling wave solutions of nonlinear evolutions equations plays an important role in the study of nonlinear

2010 Mathematics Subject Classification: 34G20, 35L05, 83C15.

Keywords and phrases: $\left(\frac{G'}{G}\right)$-expansion method, Ito equation, Benjamin-Bona-Mahony-Burgers equation, travelling wave solutions.

Research is supported by the Natural Science Foundation of Yunnan Province (no. 2010ZC154).

*Corresponding author

Received March 15, 2012

physical phenomena. In recent years, both mathematicians and physicists have devoted considerable effort to the study of exact solutions of the nonlinear ordinary or partial differential equations corresponding to the nonlinear problems. Many powerful methods have been presented. For instance, inverse scattering method [1], Baklund/Darboux transformation [2, 3], bilinear transformation [4], Exp-function method [5, 6], the sine-cosine method [7, 8], the Jacobi elliptic function method [9, 10], F-expansion method [11, 12], auxiliary equation method [13, 14] and bifurcation method of planar dynamical systems [15-18].

In [19], Wang et al. introduced the $\left(\frac{G'}{G}\right)$-expansion method for a reliable treatment of the nonlinear wave equations. The useful $\left(\frac{G'}{G}\right)$-expansion method is widely used by many authors in [20-25].

2. Description of the $\left(\frac{G'}{G}\right)$-expansion Method

Wang et al. summarized the main steps for using $\left(\frac{G'}{G}\right)$-expansion method, as follows:

Step 1. Suppose a nonlinear PDE

$$P(u,\ u_t,\ u_x,\ u_{tt},\ u_{xt}u_{xx},\ ...) = 0, \tag{2.1}$$

can be converted to an ODE

$$O(\varphi,\ -c\varphi',\ \varphi',\ c^2\varphi'',\ -c\varphi'',\ \varphi'',\ ...) = 0, \tag{2.2}$$

using a travelling wave variable $u(x,\ t) = \varphi(\xi),\ \xi = x - ct.$

Step 2. Suppose that the solution of the ODE (2.2) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u = \sum_{i=0}^{n} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{2.3}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{2.4}$$

where α_i $(i = 0, 1, 2, ..., n)$, λ and μ are constants to be determined later, $\alpha_n \neq 0$, the unwritten part in (2.3) is also a polynomial in $\left(\frac{G'}{G}\right)$, but the degree of which is generally equal to or less than $n - 1$, the positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (2.2).

Step 3. By substituting (2.3) into equation (2.2) and using second order LODE (2.4), collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left-hand side of equation (2.2) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $\alpha_0, \alpha_1, ..., \alpha_n, c, \lambda$ and μ.

Step 4. Assuming that the constants $\alpha_0, \alpha_1, ..., \alpha_n, c, \lambda$ and μ can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second order LODE (2.4) have been well known for us, then substituting $\alpha_0, \alpha_1, ..., \alpha_n, c$ and the general solutions of equation (2.4) into (2.3), we have more travelling wave solutions of the nonlinear evolution equation (2.1).

3. Application to the Generalized $(1+1)$-dimensional Ito Equation

In this section, we apply the $\frac{G'}{G}$-expansion method to the $(1+1)$-dimensional Ito equation [26, 27],

$$u_{tt} + u_{xxxt} + 3(2u_x u_t + uu_{xt}) + 3u_{xx}\int_{-\infty}^{x} u_t dx' = 0, \tag{3.1}$$

that can be reduced to

$$v_{ttx} + v_{xxxxt} + 6v_{xx}v_{xt} + 3v_x v_{xxt} + 3v_{xxx}v_t = 0, \tag{3.2}$$

upon using the potential $u = v_x$.

We introduce the travelling wave variable $v(x, t) = \varphi(\xi),\ \xi = x - ct$ into (3.2) to find

$$c^2\varphi''' - c\varphi^{(5)} - 6c(\varphi'')^2 - 6c\varphi'\varphi''' = 0. \tag{3.3}$$

Integrating (3.3) twice and letting the integral constants be zero, we have

$$c\varphi' - \varphi''' - 3(\varphi')^2 = 0. \tag{3.4}$$

Considering the homogeneous balance between φ''' and $(\varphi')^2$ in equation (3.4) gives

$$n + 3 = 2n + 2, \tag{3.5}$$

so that

$$n = 1. \tag{3.6}$$

We suppose that the solutions $\varphi(\xi)$ of equation (3.4) is of the form

$$\varphi(\xi) = \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0. \tag{3.7}$$

By using equation (2.4), from equation (3.7), we have

$$\varphi' = -\alpha_1\left(\frac{G'}{G}\right)^2 - \alpha_1\lambda\left(\frac{G'}{G}\right) - \alpha_1\mu, \tag{3.8}$$

$$\varphi''' = -6\alpha_1\left(\frac{G'}{G}\right)^4 - 12\alpha_1\lambda\left(\frac{G'}{G}\right)^3 - (8\alpha_1\mu + 7\alpha_1\lambda^2)\left(\frac{G'}{G}\right)^2$$
$$- (8\alpha_1\lambda\mu + \alpha_1\lambda^3)\left(\frac{G'}{G}\right) - (2\alpha_1\mu^2 + 2\alpha_1\lambda^2\mu). \tag{3.9}$$

By substituting equations (3.8) and (3.9) into equation (3.4) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left-hand side of equation (3.4) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for α_1, c, λ and μ as follows:

$$\left(\frac{G'}{G}\right)^4 : \alpha_1(-6 + 3\alpha_1) = 0, \tag{3.10}$$

$$\left(\frac{G'}{G}\right)^3 : \alpha_1(-12\lambda + 6\lambda\alpha_1) = 0, \tag{3.11}$$

$$\left(\frac{G'}{G}\right)^2 : \alpha_1(-7\lambda^2 - 8\mu + 3\lambda^2\alpha_1 + 6\mu\alpha_1) = 0, \tag{3.12}$$

$$\left(\frac{G'}{G}\right)^1 : \alpha_1(c\lambda - \lambda^3 - 8\lambda\mu + 3\lambda^2\alpha_1 + 6\mu\alpha_1) = 0, \tag{3.13}$$

$$\left(\frac{G'}{G}\right)^0 : \alpha_1(c\mu - \lambda^2\mu - 2\mu^2 + 3\mu^2\alpha_1) = 0. \tag{3.14}$$

Solving the algebraic equations above, yields

$$\alpha_1 = 2, \quad c = \lambda^2 - 4\mu, \tag{3.15}$$

where λ and μ are arbitrary constants.

By using (3.15), expression (3.7) can be written as

$$\varphi(\xi) = 2\left(\frac{G'}{G}\right) + \alpha_0, \tag{3.16}$$

where $\xi = x - (\lambda^2 - 4\mu)t$, α_0 is arbitrary constant.

Substituting the general solutions of equation (2.4) into (3.16) we have three types of travelling wave solutions of equation (3.4) as follows:

When $\lambda^2 - 4\mu > 0$,

$$\varphi(\xi) = \sqrt{\lambda^2 - 4\mu}\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right) - \frac{\lambda}{2} + \alpha_0. \tag{3.17}$$

Recall that $u(x, t) = v_x(x, t) = \varphi(\xi)$. Consequently, we obtain the travelling wave solution

$$u(x, t) = \frac{\lambda^2 - 4\mu}{2} \frac{(C_1^2 - C_2^2)\left(\cosh^2 \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi - \sinh^2 \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi\right)}{\left(C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi\right)^2}, \tag{3.18}$$

where $\xi = x - (\lambda^2 - 4\mu)t$, C_1 and C_2 are arbitrary constant.

When $\lambda^2 - 4\mu = 0$,

$$\varphi(\xi) = \frac{2C_2}{C_1 + C_2\xi} - \frac{\lambda}{2} + \alpha_0. \tag{3.19}$$

Recall that $u(x, t) = v_x(x, t) = \varphi(\xi)$. Consequently, we obtain the travelling wave solution

$$u(x, t) = -\frac{2C_2^2}{(C_1 + C_2\xi)^2}, \tag{3.20}$$

where $\xi = x$, C_1 and C_2 are arbitrary constant.

When $\lambda^2 - 4\mu < 0$,

$$\varphi(\xi) = \sqrt{4\mu - \lambda^2}\left(\frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right) - \frac{\lambda}{2} + \alpha_0. \tag{3.21}$$

Recall that $u(x, t) = v_x(x, t) = \varphi(\xi)$. Consequently, we obtain the travelling wave solution

$$u(x, t) = \frac{\lambda^2 - 4\mu}{2} \frac{C_1^2 + C_2^2}{\left(C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi\right)^2}, \tag{3.22}$$

where $\xi = x - (\lambda^2 - 4\mu)t$, C_1 and C_2 are arbitrary constant.

Remark 1. We note that the solutions (3.20) and (3.22) do not appear in [26, 27].

4. Application to the Generalized $(2+1)$-dimensional Ito Equation

In this section, we apply the $\frac{G'}{G}$-expansion method to the $(2+1)$-dimensional Ito equation [26, 27],

$$u_{tt} + u_{xxxt} + 3(2u_x u_t + uu_{xt}) + 3u_{xx}\int_{-\infty}^{x} u_t dx' + au_{yt} + bu_{xt} = 0, \tag{4.1}$$

that can be reduced to

$$v_{ttx} + v_{xxxxt} + 6v_{xx}v_{xt} + 3v_x v_{xxt} + 3v_{xxx}v_t + av_{xyt} + bv_{xxt} = 0, \tag{4.2}$$

upon using the potential $u = v_x$, where a and b are arbitrary constants.

We introduce the wave variable $v(x, y, t) = \varphi(\xi), \xi = x + y - ct$ into (4.2) to find

$$c^2\varphi''' - c\varphi^{(5)} - 6c(\varphi'')^2 - 6c\varphi'\varphi''' - ac\varphi''' - bc\varphi''' = 0. \tag{4.3}$$

Integrating (4.3) twice and letting the integral constants be zero, we have

$$(c - (a + b))\varphi' - \varphi''' - 3(\varphi')^2 = 0. \tag{4.4}$$

Considering the homogeneous balance between φ''' and $(\varphi')^2$ in equation (4.4) gives

$$n + 3 = 2n + 2, \tag{4.5}$$

so that

$$n = 1. \tag{4.6}$$

We suppose that the solutions $\varphi(\xi)$ of equation (4.4) is of the form

$$\varphi(\xi) = \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_1 \neq 0. \tag{4.7}$$

By using equation (2.4), from equation (4.7), we have

$$\varphi' = -\alpha_1\left(\frac{G'}{G}\right)^2 - \alpha_1\lambda\left(\frac{G'}{G}\right) - \alpha_1\mu, \tag{4.8}$$

$$\varphi''' = -6\alpha_1\left(\frac{G'}{G}\right)^4 - 12\alpha_1\lambda\left(\frac{G'}{G}\right)^3 - (8\alpha_1\mu + 7\alpha_1\lambda^2)\left(\frac{G'}{G}\right)^2$$
$$- (8\alpha_1\lambda\mu + \alpha_1\lambda^3)\left(\frac{G'}{G}\right) - (2\alpha_1\mu^2 + 2\alpha_1\lambda^2\mu). \tag{4.9}$$

By substituting equations (4.8) and (4.9) into equation (4.4) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left-hand side of equation (4.4) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for α_1, c, λ and μ as follows:

$$\left(\frac{G'}{G}\right)^4 : \alpha_1(-6 + 3\alpha_1) = 0, \tag{4.10}$$

$$\left(\frac{G'}{G}\right)^3 : \alpha_1(-12\lambda + 6\lambda\alpha_1) = 0, \tag{4.11}$$

$$\left(\frac{G'}{G}\right)^2 : \alpha_1(-7\lambda^2 - 8\mu + 3\lambda^2\alpha_1 + 6\mu\alpha_1) = 0, \tag{4.12}$$

$$\left(\frac{G'}{G}\right)^1 : \alpha_1((c - (a + b))\lambda - \lambda^3 - 8\lambda\mu + 3\lambda^2\alpha_1 + 6\mu\alpha_1) = 0, \tag{4.13}$$

$$\left(\frac{G'}{G}\right)^0 : \alpha_1((c - (a + b))\mu - \lambda^2\mu - 2\mu^2 + 3\mu^2\alpha_1) = 0. \tag{4.14}$$

Solving the algebraic equations above, yields

$$\alpha_1 = 2, \quad c = a + b + \lambda^2 - 4\mu, \tag{4.15}$$

where λ and μ are arbitrary constants.

By using (4.16), expression (4.7) can be written as

$$\varphi(\xi) = 2\left(\frac{G'}{G}\right) + \alpha_0, \tag{4.16}$$

where $\xi = x + y - (a + b + \lambda^2 - 4\mu)t$, α_0 is arbitrary constant.

Substituting the general solutions of equation (2.4) into (4.16) we have three types of travelling wave solutions of equation (4.3) as follows:

When $\lambda^2 - 4\mu > 0$,

$$\varphi(\xi) = \sqrt{\lambda^2 - 4\mu}\left(\frac{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right) - \frac{\lambda}{2} + \alpha_0. \tag{4.17}$$

Recall that $u(x, y, t) = v_x(x, y, t) = \varphi(\xi)$. Consequently, we obtain the travelling wave solution

$$u(x, y, t) = \frac{\lambda^2 - 4\mu}{2} \frac{(C_1^2 - C_2^2)\left(\cosh^2 \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi - \sinh^2 \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi\right)}{\left(C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi\right)^2}, \tag{4.18}$$

where $\xi = x + y - (a + b + \lambda^2 - 4\mu)t$, α_0, C_1 and C_2 are arbitrary constant.

When $\lambda^2 - 4\mu = 0$,

$$\varphi(\xi) = \frac{2C_2}{C_1 + C_2 x} - \frac{\lambda}{2} + \alpha_0. \tag{4.19}$$

Recall that $u(x, y, t) = v_x(x, y, t) = \varphi(\xi)$. Consequently, we obtain the travelling wave solution

$$u(x, y, t) = -\frac{2C_2^2}{(C_1 + C_2\xi)^2}, \tag{4.20}$$

where $\xi = x + y - (a + b)t$, C_1 and C_2 are arbitrary constant.

When $\lambda^2 - 4\mu < 0$,

$$\varphi(\xi) = \sqrt{4\mu - \lambda^2}\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right) - \frac{\lambda}{2} + \alpha_0. \tag{4.21}$$

Recall that $u(x, y, t) = v_x(x, y, t) = \varphi(\xi)$. Consequently, we obtain the travelling wave solution

$$u(x, y, t) = \frac{\lambda^2 - 4\mu}{2}\frac{C_1^2 + C_2^2}{\left(C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi\right)^2}, \tag{4.22}$$

where $\xi = x + y - (a + b + \lambda^2 - 4\mu)t$, C_1 and C_2 are arbitrary constant.

Remark 2. We note that the solutions (4.20) and (4.22) do not appear in [26, 27].

5. Application to the Benjamin-Bona-Mahony-Burgers (BBMB) Equation

In this section, we apply the $\frac{G'}{G}$-expansion method to the Benjamin-Bona-Mahony-Burgers (BBMB) equation [28],

$$u_t - u_{xxt} + u_x + uu_x = 0. \tag{5.1}$$

We introduce the wave variable $u(x, t) = \varphi(\xi)$, $\xi = x - ct$ into (5.1) to find

$$-c\varphi' + c\varphi''' + \varphi' + \varphi\varphi' = 0. \tag{5.2}$$

Integrating (5.2) once and letting the integral constant be zero, we have

$$-c\varphi + \varphi + c\varphi'' + \frac{1}{2}\varphi^2 = 0. \tag{5.3}$$

Considering the homogeneous balance between φ'' and φ^2 in equation (5.3) gives

$$n + 2 = 2n, \tag{5.4}$$

so that

$$n = 2. \tag{5.5}$$

We suppose that the solutions $\varphi(\xi)$ of equation (5.3) is of the form

$$\varphi(\xi) = \alpha_2\left(\frac{G'}{G}\right)^2 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_2 \neq 0. \tag{5.6}$$

By using equation (2.4), from equation (5.6), we have

$$\varphi^2 = \alpha_2^2\left(\frac{G'}{G}\right)^4 + 2\alpha_1\alpha_2\left(\frac{G'}{G}\right)^3 + (\alpha_1^2 + 2\alpha_2\alpha_0)\left(\frac{G'}{G}\right)^2 + 2\alpha_1\alpha_0\left(\frac{G'}{G}\right) + \alpha_0^2, \tag{5.7}$$

$$\varphi'' = 6\alpha_2\left(\frac{G'}{G}\right)^4 + (2\alpha_1 + 10\alpha_2\lambda)\left(\frac{G'}{G}\right)^3 (8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)\left(\frac{G'}{G}\right)^2$$

$$+ (6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)\left(\frac{G'}{G}\right) + 2\alpha_2\mu^2 + \alpha_1\lambda\mu. \tag{5.8}$$

By substituting equations (5.6)-(5.8) into equation (5.3) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left-hand side of equation

(5.3) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for $\alpha_2,\ \alpha_1,\ \alpha_0,\ c,\ \lambda$ and μ as follows:

$$\left(\frac{G'}{G}\right)^4 : 6c\alpha_2 + \frac{1}{2}\alpha_2^2 = 0, \tag{5.9}$$

$$\left(\frac{G'}{G}\right)^3 : \alpha_1\alpha_2 + 2c\alpha_1 + 10c\lambda\alpha_2 = 0, \tag{5.10}$$

$$\left(\frac{G'}{G}\right)^2 : \frac{1}{2}\alpha_1^2 + \alpha_2 - c\alpha_2 + \alpha_0\alpha_2 + 3c\lambda\alpha_1 + 4c\lambda^2\alpha_2 + 8c\mu\alpha_2 = 0, \tag{5.11}$$

$$\left(\frac{G'}{G}\right)^1 : \alpha_1 - c\alpha_1 + \alpha_0\alpha_1 + c\lambda^2\alpha_1 - 2c\mu\alpha_1 + 6c\lambda\mu\alpha_2 = 0, \tag{5.12}$$

$$\left(\frac{G'}{G}\right)^0 : \alpha_0 - c\alpha_0 + \frac{1}{2}\alpha_0^2 + c\lambda\mu\alpha_1 + 2c\mu^2\alpha_2 = 0. \tag{5.13}$$

Solving the algebraic equations above, yields

$$\begin{aligned} \alpha_2 &= -\frac{12}{1-\lambda^2+4\mu}, \quad \alpha_1 = -\frac{12\lambda}{1-\lambda^2+4\mu}, \\ \alpha_0 &= -\frac{12\mu}{1-\lambda^2+4\mu}, \quad c = \frac{1}{1-\lambda^2+4\mu} \end{aligned} \tag{5.14}$$

or

$$\begin{aligned} \alpha_2 &= -\frac{12}{1+\lambda^2-4\mu}, \quad \alpha_1 = -\frac{12\lambda}{1+\lambda^2-4\mu}, \\ \alpha_0 &= -\frac{12\mu}{1+\lambda^2-4\mu}, \quad c = \frac{1}{1+\lambda^2-4\mu}. \end{aligned} \tag{5.15}$$

By using (5.14) and (5.15), expression (5.6) can be written as

$$\varphi(\xi) = -\frac{12}{1-\lambda^2+4\mu}\left(\frac{G'}{G}\right)^2 - \frac{12\lambda}{1-\lambda^2+4\mu}\left(\frac{G'}{G}\right) - \frac{12\mu}{1+\lambda^2+4\mu}, \quad (5.16)$$

where $\xi = x - \frac{1}{1+\lambda^2+4\mu}t$ or

$$\varphi(\xi) = -\frac{12}{1-\lambda^2-4\mu}\left(\frac{G'}{G}\right)^2 - \frac{12\lambda}{1-\lambda^2-4\mu}\left(\frac{G'}{G}\right) - \frac{12\mu}{1+\lambda^2-4\mu}, \quad (5.17)$$

where $\xi = x - \frac{1}{1+\lambda^2-4\mu}t.$

Substituting the general solutions of equation (2.4) into (5.16) and (5.17), we have three types of travelling wave solutions of equation (5.1) as follows:

When $\lambda^2 - 4\mu > 0,$

$$\varphi(\xi) = -\frac{6\sqrt{\lambda^2-4\mu}}{1-\lambda^2+4\mu}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2$$

$$+\frac{3\lambda-12\mu}{1-\lambda^2+4\mu}, \quad (5.18)$$

where $\xi = x + \frac{1}{1-\lambda^2+4\mu}t,$ C_1 and C_2 are arbitrary constant or

$$\varphi(\xi) = -\frac{6\sqrt{\lambda^2-4\mu}}{1+\lambda^2-4\mu}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right)^2$$

$$+\frac{3\lambda-12\mu}{1+\lambda^2-4\mu}, \quad (5.19)$$

where $\xi = x + \frac{1}{1+\lambda^2-4\mu}t,$ C_1 and C_2 are arbitrary constant.

When $\lambda^2 - 4\mu = 0$,

$$\varphi(\xi) = -\frac{12C_2^2}{(C_1 + C_2\xi)^2} + 3\lambda - 12\mu, \tag{5.20}$$

where $\xi = x + t$, C_1 and C_2 are arbitrary constant.

When $\lambda^2 - 4\mu < 0$,

$$\varphi(\xi) = -\frac{6\sqrt{4\mu - \lambda^2}}{1 - \lambda^2 + 4\mu}\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right)^2$$

$$+ \frac{3\lambda - 12\mu}{1 - \lambda^2 + 4\mu}, \tag{5.21}$$

where $\xi = x + \frac{1}{1 - \lambda^2 + 4\mu}t$, C_1 and C_2 are arbitrary constant or

$$\varphi(\xi) = -\frac{6\sqrt{4\mu - \lambda^2}}{1 + \lambda^2 - 4\mu}\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right)^2$$

$$+ \frac{3\lambda - 12\mu}{1 + \lambda^2 - 4\mu}, \tag{5.22}$$

where $\xi = x + \frac{1}{1 + \lambda^2 - 4\mu}t$, C_1 and C_2 are arbitrary constant.

Remark 3. We note that the solutions (5.18)-(5.22) do not appear in [28].

6. Conclusions

The $\frac{G'}{G}$-expansion method was successfully used to establish travelling wave solutions. Comparing the other methods in the literature, the $\frac{G'}{G}$-expansion method appears to be easier and faster by means of a symbolic computation system. This paper confirms that the method is direct, concise and effective. The method can be used for many other nonlinear partial differential equations of mathematical physics.

References

[1] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge Univ. Press, Cambridge, 1991.

[2] C. Rogers and W. F. Shadwick, Backlund Transformations, Academic Press, New York, 1982.

[3] V. A. Matveev and M. A. Salle, Darboux Transformation and Solitons, Springer, Berlin, 1991.

[4] R. Hirota, Exact *N*-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices, J. Math. Phys. 14 (1973), 810.

[5] J. H. He and X. H. Wu, Exp-function method for nonlinear wave equations, Chaos Solitons Fractals 30 (2006), 700-708.

[6] X. H. (Benn) Wu and J. H. He, Solitary solutions, periodic solutions and compacton like solutions using the Exp-function method, Comput. Math. Appl. 54 (2007), 966-986.

[7] A. M. Wazwaz, Distinct variants of the KdV equation with compact and noncompact structures, Appl. Math. Comput. 150 (2004), 365-377.

[8] A. M. Wazwaz, Variants of the generalized KdV equation with compact and noncompact structures, Comput. Math. Appl. 47 (2004), 583-591.

[9] S. K. Liu, Z. T. Fu, S. D. Liu and Q. Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Lett. A 289 (2001), 69-74.

[10] Z. Y. Yan, Abundant families of Jacobi elliptic function solutions of the (2 + 1)-dimensional integrable Davey-Stewartson-type equation via a new method, Chaos Solitons Fractals 18 (2003), 299-309.

[11] M. L. Wang and Y. B. Zhou, The periodic wave solutions for the Klein-Gordon-Schrodinger equations, Phys. Lett. A 318 (2003), 84-92.

[12] M. L. Wang and X. Z. Li, Extended *F*-expansion method and periodic wave solutions for the generalized Zakharov equations, Phys. Lett. A 343 (2005), 48-54.

[13] Sirendaoreji and J. Sun, Auxiliary equation method for solving nonlinear partial differential equations, Phys. Lett. A 309 (2003), 387-396.

[14] Sirendaoreji, Auxiliary equation method and new solutions of Klein-Gordon equations, Chaos Solitons Fractals 31 (2007), 943-950.

[15] J. B. Li and Z. R. Liu, Smooth and non-smooth traveling waves in a nonlinearly dispersive equation, Appl. Math. Model. 25(1) (2000), 41-56.

[16] M. Song and C. X. Yang, Exact traveling wave solutions of the Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation, Appl. Math. Comput. 216 (2010), 3234-3243.

[17] M. Song and J. H. Cai, Solitary wave solutions and kink wave solutions for a generalized Zakharov-Kuznetsov equation, Appl. Math. Comput. 217 (2010), 1455-1462.

[18] M. Song, C. X. Yang and B. G. Zhang, Exact solitary wave solutions of the Kadomtsov-Petviashvili-Benjamin-Bona-Mahony equation, Appl. Math. Comput. 217 (2010), 1334-1339.

[19] M. L. Wang, X. Z. Li and J. L. Zhang, The $\left(\frac{G'}{G}\right)$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. A 372 (2008), 417-423.

[20] M. L. Wang, J. L. Zhang and X. Z. Li, Application of the $\left(\frac{G'}{G}\right)$-expansion to travelling wave solutions of the Broer-Kaup and the approximate long water wave equations, Appl. Math. Comput. 206 (2008), 321-326.

[21] A. Bekir, Application of the $\left(\frac{G'}{G}\right)$-expansion method for nonlinear evolution equations, Phys. Lett. A 372 (2008), 3400-3406.

[22] D. D. Ganji and M. Abdollahzadeh, Exact traveling solutions of some nonlinear evolution equation $\left(\frac{G'}{G}\right)$-expansion method, J. Math. Phys. 50 (2009), 013519.

[23] I. Aslan, Discrete exact solutions to some nonlinear differential-difference equations via the $\frac{G'}{G}$-expansion method, Appl. Math. Comput. 215 (2009), 3140-3147.

[24] I. Aslan and T. Ozis, On the validity and reliability of the $\frac{G'}{G}$-expansion method by using higher-order nonlinear equations, Appl. Math. Comput. 211 (2009), 531-536.

[25] M. Song and Y. L. Ge, Application of $\left(\frac{G'}{G}\right)$-expansion method for $(3+1)$-dimensional nonlinear evolution equations, Comput. Math. Appl. 60 (2010), 1220-1227.

[26] A. M. Wazwaz, Multiple-soliton solutions for the generalized $(1+1)$-dimensional and the generalized $(2+1)$-dimensional Ito equations, Appl. Math. Comput. 202 (2008), 840-849.

[27] F. Khani, Analytic study on the higher order Ito equations: new solitary wave solutions using the Exp-function method, Chaos Solitons Fractals 41 (2009), 2128-2134.

[28] Z. Z. Ganji, D. D. Ganji and H. Bararnia, Approximate general and explicit solutions of nonlinear BBMB equations by Exp-function method, Appl. Math. Model. 33 (2009), 1836-1841.