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STABILITY OF DISCRETE-TIME TWO SPECIES COMPETING MODEL

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Abstract

In this paper, we study the stability analysis of discrete-time two species competing model. Forward Euler method is applied to the continuous model to obtain the discrete-time model. All the critical points of the continuous model have been identified and the stability criterion of the discrete-model at critical points has been discussed.

1. Introduction

Mathematical modelling of ecosystem can be broadly classified as prey-predation, competition, mutualism, and commensalisms, etc. The dynamic relationship between competing species will continue to be one of the dominating themes in both ecology and mathematical biology due to its universal existence and importance. Many mathematicians and ecologists

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have studied and contributed to the growth of continuous mathematical models of competing species. Danca et al. [3] and Leach and Miritzis [6] studied a model of competition between populations of two species and discussed the stability at various equilibrium points. Mimura et al. [8] have studied the pattern formation of the competing system and shown that coexistence is possible by the effect of cross-population where the inter-specific competition is stronger than the intra-specific. Reddy et al. [9] have analyzed a model of two mutually interacting species with limited resources of one species and unlimited resources of other species and identified two equilibrium points and described their stability criteria. Tsokos and Hinkley [10] formulated a general stochastic bivariate model without specifying the nature of the relationship between two species. The studies so far reported are suitable for large size population. Many authors [1, 2, 4, 7] have also suggested that the discrete-time models are more appropriate and provide efficient results as compared to the continuous models when the size of the population is small.

In this paper, we have analyzed the stability criteria of discrete-time two species competing model at all the critical points. At the first stage, the rescaling of the population parameters and reproduction rate parameters of both the species were carried out in the classical model of two competing species and forward Euler method is applied to the system to obtain the discrete-time model. At the second stage, all the critical points of the continuous model have been identified. The Jacobian matrices of the discrete model at all the critical points have been determined and the stability criterion at each critical point has been discussed.

2. Mathematical Model

The simplest form of two species competing model is given by

$$\begin{cases} \dfrac{dN_1}{d\tau} = r_1 N_1\left(1 - \dfrac{N_1}{K_1} - b_{12}\dfrac{N_2}{K_1}\right), \\ \dfrac{dN_2}{d\tau} = r_2 N_2\left(1 - \dfrac{N_2}{K_2} - b_{21}\dfrac{N_1}{K_2}\right), \end{cases} \tag{1}$$

where (N_1, N_2), (K_1, K_2) and (r_1, r_2) represent the population, carrying capacity and the rates of reproduction of first and second species, respectively, and b_{12} and b_{21} give the measure of effects of competition between two species. The system (1) has been analyzed by Hsu et al. [5] by means of rescaling the parameters as:

$$x = \frac{N_1}{K_1}, \quad y = \frac{N_2}{K_2}, \quad t = r_1\tau, \quad \frac{r_2}{r_1} = \rho, \quad b_{12}\frac{K_2}{K_1} = \alpha\rho, \quad b_{21}\frac{K_1}{K_2} = \frac{\beta}{\rho}$$

and it yielded the following system of equations:

$$\begin{cases} \dfrac{dx}{dt} = x(1 - x - \alpha\rho y), \\ \dfrac{dy}{dt} = \rho y(1 - y) - \beta xy. \end{cases} \tag{2}$$

By applying the forward Euler method to system (2), we obtain the discrete-time two species competing model as follows:

$$\begin{cases} x_{n+1} = x_n + \delta x_n(1 - x_n - \alpha\rho y_n), \\ y_{n+1} = y_n + \delta y_n(\rho - \rho y_n - \beta x_n), \end{cases} \tag{3}$$

where δ is the step size and all the parameters α, β and ρ are positive.

3. Existence and Stability of Critical Points

Critical points of the system of equations (2) are $E_1(0, 0)$, $E_2(0, 1)$, $E_3(1, 0)$ and $E_4(x^*, y^*)$, where $x^* = \frac{1-\alpha\rho}{1-\alpha\beta}$, $y^* = \frac{\rho - \beta}{\rho(1-\alpha\beta)}$ and x^* is positive when

$$\beta \text{ and } \rho \text{ both are less than } \frac{1}{\alpha} \tag{4}$$

or

$$\beta \text{ and } \rho \text{ both are greater than } \frac{1}{\alpha}. \tag{5}$$

Lemma 3.1. *Let* $F(\lambda) = \lambda^2 - B\lambda + C$. *Suppose that* $F(1) > 0$, λ_1 *and* λ_2 *are roots of* $F(\lambda) = 0$. *Then*

(i) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $C < 1$;

(ii) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ (*or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$) *if and only if* $F(-1) < 0$;

(iii) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $C > 1$;

(iv) $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *if and only if* $F(-1) = 0$ *and* $B \neq 0, 2$;

(v) λ_1 *and* λ_2 *are complex and* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $B^2 - 4AC < 0$ *and* $C = 1$.

Let λ_1 and λ_2 be eigen values of Jacobian matrix at the critical point $E(x, y)$. Then $E(x, y)$ is called a *sink* if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. A sink is locally asymptotically stable. $E(x, y)$ is called a *saddle* if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$). $E(x, y)$ is called a *source* if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. A source is locally unstable. $E(x, y)$ is called *non-hyperbolic* if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Proposition 3.2(a). *The critical point* $E_1(0, 0)$ *is a source*.

The Jacobian matrix of (3) at $E_1(0, 0)$ is given by

$$J_1 = \begin{bmatrix} 1+\delta & 0 \\ 0 & 1+\delta\rho \end{bmatrix}.$$

The eigen values of Jacobian matrix J_1 are $\lambda_1 = 1 + \delta$ and $\lambda_2 = 1 + \delta\rho$. Here $|\lambda_1| > 1$ and $|\lambda_2| > 1$, therefore $E_1(0, 0)$ is a source.

Proposition 3.2(b). *The critical point* $E_2(0, 1)$ *is sink when* $\frac{1}{\alpha} < \rho$ *and is saddle when* $\frac{1}{\alpha} > \rho$.

The Jacobian matrix of (3) at $E_2(0, 1)$ is given by

$$J_2 = \begin{bmatrix} 1+\delta(1-\alpha\rho) & 0 \\ -\delta\beta & 1-\delta\rho \end{bmatrix}.$$

The eigen values of Jacobian matrix J_2 are $\lambda_1 = 1+\delta(1-\alpha\rho)$ and $\lambda_2 = 1-\delta\rho.$ Here

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1,$ when $\frac{1}{\alpha} < \rho.$ Therefore $E_2(0, 1)$ is sink when $\frac{1}{\alpha} < \rho.$

(ii) $|\lambda_1| > 1$ and $|\lambda_2| < 1,$ when $\frac{1}{\alpha} > \rho.$ Therefore $E_2(0, 1)$ is saddle when $\frac{1}{\alpha} > \rho.$

Proposition 3.2(c). *The critical point* $E_3(1, 0)$ *is sink when* $\rho < \beta$ *and is saddle when* $\rho > \beta.$

The Jacobian matrix of (3) at $E_3(1, 0)$ is given by

$$J_3 = \begin{bmatrix} 1-\delta & -\alpha\rho\delta \\ 0 & 1+\delta(\rho-\beta) \end{bmatrix}.$$

The eigen values of Jacobian matrix J_3 are $\lambda_1 = 1-\delta$ and $\lambda_2 = 1+\delta(\rho-\beta).$ Here

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1,$ when $\rho < \beta.$ Therefore $E_3(1, 0)$ is sink when $\rho < \beta.$

(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1,$ when $\rho > \beta.$ Therefore $E_3(1, 0)$ is saddle when $\rho > \beta.$

Proposition 3.2(d). *The critical point* $E_4(x^*, y^*)$ *is a sink when* $\rho > \beta + \frac{(2\delta x^* - 4)}{\left[\delta^2 x^* - \frac{2\delta}{(1-\alpha\beta)}\right]}$ *and is a saddle when* $\beta < \rho < \beta + \frac{(2\delta x^* - 4)}{\left[\delta^2 x^* - \frac{2\delta}{(1-\alpha\beta)}\right]}.$

The Jacobian matrix of (3) at $E_4(x^*, y^*)$ is given by

$$J_4 = \begin{bmatrix} 1 - \delta x^* & -\alpha\rho\delta x^* \\ \dfrac{\beta\delta(-\rho + \beta)}{\rho(1 - \alpha\beta)} & 1 + \dfrac{\delta(-\rho + \beta)}{(1 - \alpha\beta)} \end{bmatrix}.$$

The corresponding characteristic equation can be written as

$$\lambda^2 - (trJ_4)\lambda + \det J_4 = 0, \tag{6}$$

where

$$trJ_4 = 2 - \delta\left[x^* + \frac{(\rho - \beta)}{(1 - \alpha\beta)}\right] \tag{7}$$

and

$$\det J_4 = (1 - \delta x^*)\left[1 + \frac{\delta(-\rho + \beta)}{(1 - \alpha\beta)}\right] + \alpha\beta\delta^2 \frac{(-\rho + \beta)}{(1 - \alpha\beta)} x^*. \tag{8}$$

Let

$$F(\lambda) = \lambda^2 - (trJ_4)\lambda + \det J_4. \tag{9}$$

From (9), we have

$$F(1) = 1 - (trJ_4) + \det J_4. \tag{10}$$

Using (7), (8) in (10) and solving (10), we get $F(1) = \delta^2(\rho - \beta)x^*$.

Now

$$F(1) \text{ is positive when } \rho > \beta. \tag{11}$$

Combining (4), (5) and (11), we observe that

$$F(1) \text{ is positive when } \beta < \rho < \frac{1}{\alpha} \text{ or } \frac{1}{\alpha} < \beta < \rho.$$

From (9), we have

$$F(-1) = 1 + (trJ_4) + \det J_4. \tag{12}$$

Using (7), (8) in (12) and solving (12), we get

$$F(-1) = 4 - 2\delta x^* - \frac{2\delta(\rho - \beta)}{(1 - \alpha\beta)} + \delta^2(\rho - \beta)x^*.$$

Now

$$F(-1) > 0 \text{ when } \rho > \beta + \frac{(2\delta x^* - 4)}{\left[\delta^2 x^* - \frac{2\delta}{(1 - \alpha\beta)}\right]} \tag{13}$$

and

$$\det J_4 < 1 \text{ when } \rho > \beta - \frac{1}{\left[\frac{1}{(1 - \alpha\beta)} - \delta x^*\right]} x^*. \tag{14}$$

It is easy to see from (13) and (14) that critical point $E_4(x^*, y^*)$ is a sink when $\rho > \beta + \frac{(2\delta x^* - 4)}{\left[\delta^2 x^* - \frac{2\delta}{(1 - \alpha\beta)}\right]}$. Now

$$F(-1) < 0 \text{ when } \rho < \beta + \frac{(2\delta x^* - 4)}{\left[\delta^2 x^* - \frac{2\delta}{(1 - \alpha\beta)}\right]}. \tag{15}$$

From (11) and (15), we observe that critical point $E_4(x^*, y^*)$ is a saddle when

$$\beta < \rho < \beta + \frac{(2\delta x^* - 4)}{\left[\delta^2 x^* - \frac{2\delta}{(1 - \alpha\beta)}\right]}.$$

4. Numerical Simulations

In this section, we presented the variation of x and y versus time t in the interval $[0, 10]$ taking initial values of x and y as 5 and 3, respectively, and for various values of parameters α, β and ρ as given in Table 1.

Table 1

S. No.	α	β	ρ	Figure No. showing variation of x and y versus time t
1	0.3	0.1	0.5	4.1
2	0.2	0.2	0.5	4.2
3	0.1	0.3	0.5	4.3
4	0.3	0.1	1	4.4
5	0.2	0.2	1	4.5
6	0.1	0.3	1	4.6
7	0.3	0.1	2	4.7
8	0.2	0.2	2	4.8
9	0.1	0.3	2	4.9

In case 1, y dominates x within the time interval $t_1^* = 0.445$ to $t_2^* = 5.947$ and y recedes x up to a time $t_1^* = 0.445$ and after a time $t_2^* = 5.947$ as seen in Figure 4.1. In case 2, y recedes x and becomes equal once at a time $t^* = 1.208$ as seen in Figure 4.2. In case 3, y recedes x and continuous to do so as seen in Figure 4.3. Therefore it is clear that of x dominates y after a certain time in all the three cases when rate of reproduction of first species is greater than that of second species.

When rates of reproduction of both the species are equal, we see that in case 4, y recedes x up to a time $t^* = 0.699$ and then after x recedes y as illustrated in Figure 4.4. In case 5, y recedes x up to a time $t^* = 3.721$ and after that y and x become equal as seen in Figure 4.5. In case 6, y recedes x always as seen in Figure 4.6.

In case 7, y recedes x up to a time $t^* = 0.858$ and after that x recedes y and continuous to do so as seen in Figure 4.7. In case 8, y recedes x up to time $t^* = 1.526$ and after that *x* recedes *y* as seen in Figure 4.8. In case 9, *y* recedes *x* up to a time $t^* = 3.371$ and after that *x* recedes *y* as seen in Figure 4.9. Therefore, it is clear that *x* recedes *y* after a certain time t^* in all the three cases when rate of reproduction of second species is greater than that of first species.

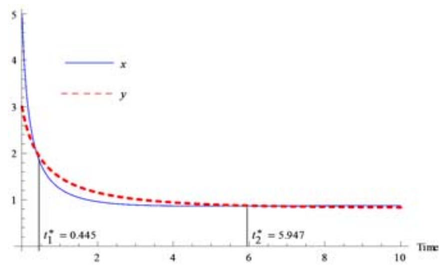

Figure 4.1

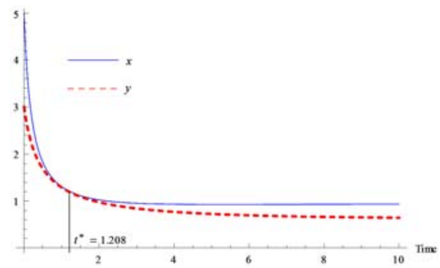

Figure 4.2

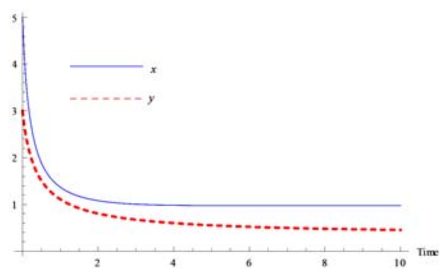

Figure 4.3

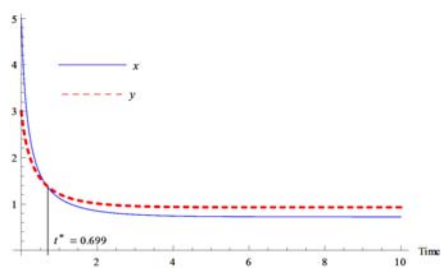

Figure 4.4

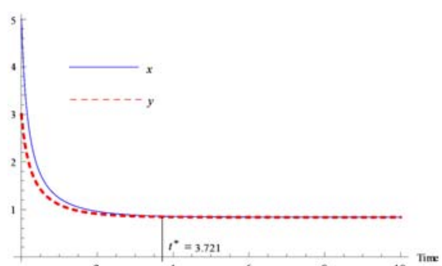

Figure 4.5

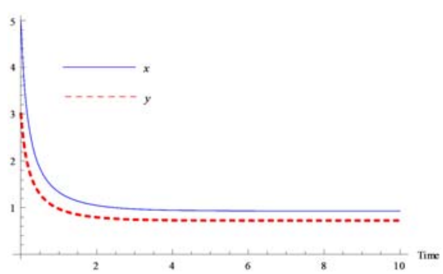

Figure 4.6

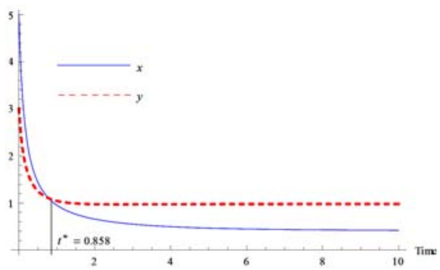

Figure 4.7

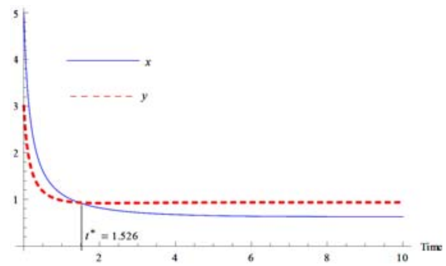

Figure 4.8

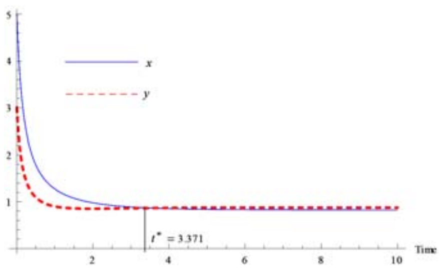

Figure 4.9

Conclusions

We have obtained the stability criteria of the discrete-time two species competing model at critical points of the continuous model. We observed that the critical point:

(1) $E_1(0, 0)$ is a source.

(2) $E_2(0, 1)$ is sink when $\frac{1}{\alpha} < \rho$ and is saddle when $\frac{1}{\alpha} > \rho$.

(3) $E_3(1, 0)$ is sink when $\rho < \beta$ and is saddle when $\rho > \beta$.

(4) $E_4(x^*, y^*)$ is a sink when $\rho > \beta + \frac{(2\delta x^* - 4)}{\left[\delta^2 x^* - \frac{2\delta}{(1-\alpha\beta)}\right]}$ and is a saddle when

$$\beta < \rho < \beta + \frac{(2\delta x^* - 4)}{\left[\delta^2 x^* - \frac{2\delta}{(1-\alpha\beta)}\right]}.$$

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