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ON SPECTRAL CRITERIA AND INCLUSIONS FOR SOLUTIONS OF EVOLUTION EQUATIONS VIA REDUCED SPECTRA

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Abstract

We revisit the notion of reduced spectra $sp_{\mathcal{F}}(\phi)$ for bounded measurable functions $\phi \in L^{\infty}(J, X)$, $\mathcal{F} \subset L^1_{loc}(J, X)$. In Section 2, we give two examples which seem to be of independent interest for spectral theory. In Section 3, we prove a spectral inclusion result for bounded mild solutions of evolution equation

$$\frac{du(t)}{dt} = Au(t) + \phi(t), \qquad (*)$$

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where A is a closed linear operator on X, $\phi \in L^{\infty}(J, X)$ and $t \in J \in \{\mathbb{R}_+, \mathbb{R}\}$.

In this paper, we give two Examples 2.1 and 2.2 showing that the following criterion:

"$sp_{\mathcal{A}}(\phi) = \varnothing$ implies $\phi \in \mathcal{A}$ if $\phi \in BUC(J, X)$ [2, Theorem 4.2.1]"

becomes false if ϕ is only bounded continuous. Moreover, Example 2.1 shows that there is $\phi \in BC(\mathbb{R}, X)$ which is S^1-almost periodic, Bochner-almost automorphic but ϕ is not almost periodic and Example 2.2 shows that there is an almost periodic function ψ with derivative ψ' continuous and bounded but ψ' is not even recurrent or Poisson stable (see definitions (2.1) and (2.2) in Section 2). These examples are instructive for various conclusions concerning many classes of generalized almost periodic functions. They demonstrate that the assumption of uniform continuity introduced in [13] is essential for [12]. In Section 3, we prove a spectral inclusion $(isp_{\mathbb{F}}(u) \subset ((\sigma(A) \cap i\mathbb{R}) \cup isp_{\mathbb{F}}(\phi)))$ for the bounded mild solutions of

$$\frac{du(t)}{dt} = Au(t) + \phi(t), \quad t \in J, \tag{$*$}$$

where A is a closed linear operator on X and instead ϕ uniformly continuous bounded only $\phi \in L^{\infty}(J, X)$ is needed, $J \in \{\mathbb{R}_+, \mathbb{R}\}$.

This seems new even for uniformly continuous u (special cases are [3, Theorem 3.3, Corollary 3.4 (i)], [12, Lemma 4.2 for uniformly continuous u, ϕ] (see [13]); in [12] besides our (1.2), (1.3) additionally the restrictive (iv) of Definition 2.3 of [12] was needed). The criterion is particularly useful in the case when ϕ is not uniformly continuous (see Example 3.3 and [7, Theorem 4.2]).

1. Notation, Definitions and Preliminaries

In the following $J \in \{\mathbb{R}_+, \mathbb{R}\}$, where $\mathbb{R}_+ = [0, \infty)$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, X

is a complex Banach space, $L(X)$ is the Banach space of linear bounded operators $B : X \to X$; the elements of all the other spaces $(\subset X^J)$ considered are functions $\phi : J \to X$ (not equivalence classes), the =, +, scalar multiplication are pointwise on J (not a.e.), correspondingly, $L^\infty(J, X)$ has the norm $\|\phi\|_\infty := \sup\{\|f(t)\| : t \in J\}$ (not essentially supremum), $BC(J, X)$, $BUC(J, X)$, $C_0 = C_0(J, X)$, $AP = AP(J, X)$, $BAA = BAA(\mathbb{R}, X)$, respectively, $VAA = VAA(\mathbb{R}, X)$ denote the Banach space of $f : J \to X$ which are bounded continuous, bounded uniformly continuous, continuous vanishing at infinity, almost periodic [1, pp. 289, 305], Bochner almost periodic [9, Definition 2], respectively, Veech almost periodic [16, Definition 1.2.1], all with sup-norm $\|\cdot\|_\infty$. The Schwartz space of rapidly decreasing C^∞ functions on $\mathbb{R}$ will be denoted by $\mathcal{S}(\mathbb{R})$. The Fourier transform of $f \in L^1(\mathbb{R}, \mathbb{C})$ will be denoted by $\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t} f(t)dt$, γ_λ, respectively, $\mathfrak{g}$ will denote the functions $\gamma_\lambda(t) = e^{i\lambda t}$, respectively, $\mathfrak{g}(t) = e^{it^2}$, $t, \lambda \in \mathbb{R}$. The translate f_a of $f : J \to X$ is defined by $f_a(t) = f(t + a)$ for all $a \in \mathbb{R}$, $t + a \in J$, $|f|(t) := \|f(t)\|$. If $\phi \in L^1_{loc}(J, X)$, then $P\phi$, respectively, $M_h\phi$ will denote the indefinite integral, respectively, Friedrich's mollifier defined by $P\phi(t) = \int_0^t \phi(s)ds$, respectively, $M_h\phi(t) = (1/h)\int_0^h \phi(t + s)ds$ for $t \in J$, $h > 0$. The set of absolutely regular functions $AR := AR(J, X)$ is defined by

$$AR = \{\phi \in L^1_{loc}(J, X) : \phi/w_k \in L^1(J, X) \text{ for some } k \in \mathbb{N}_0\}.$$

Here $w_k(t) = (1 + t^2)^k$.

For the convenience of the reader, we collect some further definitions, assumptions and relevant earlier results for $\mathcal{F} \subset X^J$.

Invariant: $\phi_a \in \mathcal{F}$ if $\phi \in \mathcal{F}$, $a, a + t \in J$ with *translate* $\phi_a(t) := \phi(t + a)$.

BUC-invariant: $\phi \in BUC(\mathbb{R}, X)$ and $\phi | J \in \mathcal{F}$ imply $\phi_a | J \in \mathcal{F}$ for all $a \in \mathbb{R}$.

Uniformly closed: $\phi_n \in \mathcal{F}$, $n \in \mathbb{N}$, and $\phi_n \to \phi$ uniformly on *J* implies $\phi \in \mathcal{F}$.

(1.1)

$\mathcal{MF}(J, X) = \{\psi \in L^1_{loc}(J, X) : M_h\psi \in \mathcal{F}, h > 0\}$.

(1.2)

$\mathcal{F}$ linear $\subset L^1_{loc}(J, X)$, $\mathcal{F}$ uniformly closed,

$\mathcal{F}$ *BUC*-invariant ([6, (3.1)]).

(1.3)

(i) $\gamma_\lambda \phi \in \mathcal{F}$ for each $\gamma_\lambda(t) = e^{i\lambda t}$, $\phi \in \mathcal{F}$ ([3, (l_2), p. 60]),

(ii) $\mathcal{F}$ contains all constant functions ([3, (l_3), p. 60]),

(iii) $B \circ \phi \in \mathcal{F}$ for each $B \in L(X)$, $\phi \in \mathcal{F}$([3, (l_5), p. 60]).

The *spectrum* of a $\phi \in L^\infty(J, X)$ with respect to a class $\mathcal{F} \subset L^1_{loc}(J, X)$ is defined by ([2, Definition 4.1.2, p. 20], [4, p. 118], [6, Definition 3.1], [10, Definition 3]):

$$sp_{\mathcal{F}}(\phi) := sp_{\mathcal{F}}(\Phi), \tag{1.4}$$

$$sp_{\mathcal{F}}(\Phi) := \{\lambda \in \mathbb{R} : f \in L^1(\mathbb{R}, \mathbb{C}), \Phi * f \mid J \in \mathcal{F} \text{ implies } \hat{f}(\lambda) = 0\}.$$

$$\text{Here } \Phi = \phi \text{ on } \mathbb{J} \text{ and, if } \mathbb{J} = \mathbb{R}_+, \Phi|(-\infty, 0) = 0. \tag{1.5}$$

$sp_{\mathcal{F}}(\phi)$ is always closed in $\mathbb{R}$. The $sp_{\mathcal{F}}(\phi)$ of (1.4) coincides with the definitions in [2, 4, 6, 10] by (1.6) (see [4, Lemma 1.1 (C)]):

If $\mathcal{F} \subset L^1_{loc}(J, X)$ satisfies (1.2) and $\phi \in L^\infty(J, X)$, then

$$sp_{\mathcal{F}}(\phi) = sp_{\mathcal{F}}(\psi) \text{ for any } \psi \in L^\infty(\mathbb{R}, X) \text{ with } \psi = \phi \text{ on } J. \tag{1.6}$$

(1.7) If $\mathcal{F} \subset L^1_{loc}(J, X)$ satisfies (1.2), $\phi \in L^\infty(J, X)$, $f \in L^1(\mathbb{R}, \mathbb{C})$ and Φ is as in (1.5), then $sp_{\mathcal{F}}(\Phi * f) \subset sp_{\mathcal{F}}(\phi) \cap \operatorname{supp} \hat{f}$ ([4, Corollary 2.3 (C)]).

(1.8) If $\mathcal{F} \subset BUC(J, X)$ satisfies (1.2) or $\mathcal{F} \in \{AP, VAA, C_0\}$, then $\mathcal{F}$ satisfies $\mathcal{F} \subset \mathcal{MF}$ ([5, Proposition 3.5 (ii), p. 431]).

Proposition 1.1. *For any* $\mathcal{F} \subset X^J$, $\phi \in L^\infty(J, X)$, *if* $\phi \in \mathcal{MF}$, *then* $sp_{\mathcal{F}}(\phi) = \emptyset$.

Proof. For any $\lambda \in \mathbb{R}$, define $h = \pi/|\lambda|$ if $\lambda \neq 0$, else $h = 1$; then the step function $f = (1/h)\chi_{(-h, 0)} \in L^1(\mathbb{R}, \mathbb{R})$ and with $\Phi := 0$ outside J, $\Phi = \phi$ on J, one has $f * \Phi | J = M_h \phi \in \mathcal{F}$, with $\hat{f}(\lambda) \neq 0$, so $\lambda \notin sp_{\mathcal{F}}(\Phi)$. It follows $sp_{\mathcal{F}}(\Phi) = \emptyset$ and so $sp_{\mathcal{F}}(\phi) = \emptyset$ by (1.4). □

Corollary 1.2. *If* $\phi \in \mathcal{F} \subset \mathcal{MF}$ *and* $\phi \in L^\infty(J, X)$, *then* $sp_{\mathcal{F}}(\phi) = \emptyset$.

This is false without $\mathcal{F} \subset \mathcal{MF}$ *by Example* 3.3 (i).

In the following, we identify $L^1(I, \mathbb{C})$, respectively, $\mathcal{F} \subset L^\infty(I, X)$ with the sub-space $\{f \in L^1(\mathbb{R}, \mathbb{C}) : f(t) = 0, t \in \mathbb{R} \backslash I\}$, respectively,

$$\{\phi \in L^\infty(\mathbb{R}, X) : \phi | I \in \mathcal{F}, \phi = 0, t \in \mathbb{R} \backslash I\}.$$

Here $I \in \{\mathbb{R}_+, \mathbb{R}_-\}$, $\mathbb{R}_- = (-\infty, 0]$.

We study the following conditions:

(1.9)

(a) $\mathcal{F} \subset \mathcal{MF}$, (b) $\mathcal{F} * L^1(\mathbb{R}_-, \mathbb{C}) | J \subset \mathcal{F}$,

(c) $\mathcal{F} * L^1(\mathbb{R}, \mathbb{C}) | J \subset \mathcal{F}$.

Proposition 1.3. *Let $\mathcal{F}$ be a linear uniformly closed subset of $L^\infty(J, X)$. Then*

(i) *The conditions* (a), (b) *of* (1.9) *are equivalent*.

(ii) *If $\mathcal{F}$ satisfies* (1.2), *then* (a), (c) *of* (1.9) *are equivalent*.

(iii) $(\mathcal{F} * E)|J \subset \mathcal{F}$ *for some dense subset $E \subset L^1(\mathbb{R}, \mathbb{C})$ implies $\mathcal{F} \cap BUC(J, X)$ is BUC-invariant*.

Proof. (i) (a) $\Rightarrow$ (b) With $\phi \in L^\infty(J, X)$ and Φ as in (1.5), we have

$$M_h\phi = (\Phi * s_h)|J, \text{ where } s_h := 1/h \text{ on } (-h, 0)$$
$$\text{and } s_h|\mathbb{R}\backslash(-h, 0) = 0. \tag{1.10}$$

As $M_h\phi = (\Phi * s_h)|J \in \mathcal{F}$, $\phi \in \mathcal{F}$, $h > 0$, it follows $\Phi * \xi | J \in \mathcal{F}$ for all step functions ξ on $\mathbb{R}_-$; since these are dense in $L^1(\mathbb{R}_-, \mathbb{C})$ and $\mathcal{F}$ is uniformly closed, (b) follows.

(b) $\Rightarrow$ (a) Follows by (1.10) and $s_h \in L^1(\mathbb{R}_-, \mathbb{C})$ for each $h > 0$.

See [8, Proposition 3.2] for the proofs of parts (ii) and (iii). $\square$

Example 1.4. $\mathcal{F} := \{\phi \in C_0(\mathbb{R}, \mathbb{R}) : \phi = 0 \text{ on } \mathbb{R}_+\}$ is linear, uniformly closed, $\subset BUC(\mathbb{R}, \mathbb{R})$, $\phi_a | J \in \mathcal{F}$ if $\phi \in \mathcal{F}$, $a > 0$, with $\mathcal{F} \subset \mathcal{M}\mathcal{F}$, but $\mathcal{F}$ is not invariant.

2. Two Examples

For the benefit of the reader, we give the relevant definitions.

(2.1) By a *recurrent* function ϕ, we mean

$$\phi \in REC(\mathbb{R}, X) := \{\phi \in C(\mathbb{R}, X) : E(\phi, 1/n, n)$$

relatively dense in $\mathbb{R}$ for each $n \in \mathbb{N}\}$, with

$$E(\phi, \varepsilon, n) := \{\tau \in \mathbb{R} : \|\phi(t + \tau) - \phi(t)\| \leq \varepsilon \text{ for all } |t| \leq n\}.$$

$E(\phi, 1/n, n)$ is *relatively dense* means there is a compact set $K \subset \mathbb{R}$ such that $K + E(\phi, 1/n, n) = \mathbb{R}$ (see [15, Definition 2, p. 80], [5, p. 427]).

(2.2) A $\phi \in C(\mathbb{R}, X)$ is *Poisson stable* if it has at least one sequence $(t_m) \subset \mathbb{R}$ with $t_m \to \infty$ such that $\phi_{t_m} \to \phi$ locally uniformly in $\mathbb{R}$ (see [15, Definition 1, p. 80]).

Example 2.1. The function $\phi = \sin \dfrac{1}{p} \in BC(\mathbb{R}, \mathbb{R})$ with $p(t) = 2 + \cos t + \cos \sqrt{2}t$ is Stepanoff almost periodic $S^1\text{-}AP \subset \mathcal{M}AP$ and $sp_{AP}(\phi) = \varnothing$ but $\phi \notin BUC(\mathbb{R}, \mathbb{C})$ and so $\phi \notin AP = AP(\mathbb{R}, \mathbb{C})$. This ϕ is also Bochner almost automorphic (*B-aa*) [9] and so Veech almost automorphic (*V-aa*) [16] and *L-ap* [5, p. 430, (3.3)] (see [4, p. 119, (1.3), p. 118 above (1.2), (3.5), (3.8)] and the references therein).

Proof. First, we show that $\phi \in S^1\text{-}AP$. Set

$$\phi_n(t) := \sin \frac{1}{2 + \max\left\{\cos t, -1 + \frac{1}{n}\right\} + \cos \sqrt{2}t}.$$

Then $\phi_n \in AP$ for each $n \in \mathbb{N}$ and $\phi_n(t) = \phi(t)$ if $\max\left\{\cos t, -1 + \frac{1}{n}\right\} = \cos t$. It follows $\int_0^{2\pi} |\phi_n(t + s) - \phi(t + s)| ds \le 2\mu(E_n^t)$, where μ is the Lebesgue measure on $\mathbb{R}$ and

$$E_r^t = \left\{\tau \in [t, t + 2\pi] : \max\left\{\cos \tau, -1 + \frac{1}{r}\right\} = -1 + \frac{1}{r}\right\}, \quad r \ge 1.$$

Then $\mu(E_n^t) = \mu(E_n^0) = \mu([\pi - \delta_n, \pi + \delta_n])$ with $\cos \delta_n = 1 - 1/n$, $t \in \mathbb{R}$, with $\delta_n \to 0$ as $n \to \infty$. It follows

$$\lim_{n\to\infty} \int_0^1 |\phi_n(t + s) - \phi(t + s)| ds = 0$$

uniformly in $t \in \mathbb{R}$ and implies $\phi \in S^1\text{-}AP$ (see [4, p. 132]). So $M_h\phi(\cdot) = (1/h)\int_0^h \phi(\cdot + s)ds \in AP$ for each $h > 0$ by [4, (3.8)]. By Proposition 1.1, one gets $sp_{AP}(\phi) = \varnothing$.

Now, we show that ϕ is not uniformly continuous.

Indeed, for each $n \in \mathcal{F}$, by Kronecker's approximation theorem [11, p. 436, (d)] and continuity, there is $t_n > 0$ such that $p(t_n) = \frac{1}{n\pi}$ and thus range $R(p) = (0, 4]$. Choose t_n' nearest point to t_n with $p(t_n') = \frac{1}{\left(n - \frac{1}{2}\right)\pi}$.

We have $|t_n - t_n'| \le \mu\left(E^{t_n}_{\left(n-\frac{1}{2}\right)\pi}\right) \to 0$ as $n \to \infty$. Since $|\phi(t_n') - \phi(t_n)| = 1$, we get ϕ is not uniformly continuous. It follows $\phi \notin AP$.

Finally, we show that ϕ is B-aa. Indeed, since $(\cos t, \cos\sqrt{2}t)$ is almost periodic, for any $(t_{n'}) \subset \mathbb{R}$ there are $\alpha, \beta \in \mathbb{R}$ and a subsequence (t_n) such that

$$\cos(t + t_n) \to \cos(t + \alpha), \quad \cos(\sqrt{2}(t + t_n)) \to \cos(\sqrt{2}(t + \beta)),$$

$$p(t + t_n) \to (2 + \cos(t + \alpha) + \cos(\sqrt{2}(t + \beta))) =: q(t),$$

$$\text{uniformly in } t \in \mathbb{R}. \tag{2.3}$$

Since q is entire $\not\equiv 0$, $C := \{s \in \mathbb{R} : q(s) = 0\}$ is at most countable. So, there is a (diagonal) subsequence (s_n) and $\psi : \mathbb{R} \to [-1, 1]$ with $\phi(t + s_n) = \sin\frac{1}{p(t + s_n)} \to \psi(t)$ pointwise for each $t \in \mathbb{R}$. Now, (2.3) implies that $q(t - s_n) \to p(t)$, $p(t + s_m - s_n) \to q(t - s_n)$ and then $p(t + s_m - s_n)$

$\to p(t)$ as $(n, m) \to \infty$ for each $t \in \mathbb{R}$. This yields $\phi(t + s_m - s_n) \to \phi(t)$ as $(n, m) \to \infty$ pointwise in $t \in \mathbb{R}$; $m \to \infty$ and the definition of ψ gives therefore $\psi(t - s_n) \to \phi(t)$. By Definition 2 of [9], ϕ is *B-aa*. □

(See also [14, pp. 212, 213] for another proof that $\phi \in S^1\text{-}AP$ but $\phi \notin AP$; and [5, Example 3.3] that ϕ is *B-aa*).

Example 2.2. There is $\psi \in BC(\mathbb{R}, \mathbb{R})$ which is not *ap* or *B-aa* or *V-aa* or recurrent or uniformly continuous (not even Poisson stable (see (2.1), (2.2), respectively, Example 2.1), also $\Delta_1\psi(\cdot) := \psi(\cdot + 1) - \psi(\cdot)$ and so ψ are not Stepanoff S^1-almost periodic, but $P\psi(t) := \int_0^t \psi(s)ds$ is almost periodic and so $sp_{AP}(\psi) = sp_{BAA}(\psi) = \varnothing$.

Proof. Take $\psi = \sum_{n=1}^{\infty} h_n$, h_n periodic with period 2^{n+1},

$$h_n(t) = 0, \quad t \in [-2^n, 2^n - 1], \quad h_n(t) = \sin(2^n \pi t), \quad t \in [2^n - 1, 2^n] =: I_n.$$

One has $\operatorname{supp} h_n = I_n + 2^{n+1}\mathbb{Z}$ and for each $n \neq m$, $\operatorname{supp} h_n \cap \operatorname{supp} h_m = \varnothing$; the right endpoints of the translations of I_n are all even, so if $n = m + k$, $k \in \mathbb{N}_0$, with $(I_n + 2^{n+1}u) \cap (I_m + 2^{m+1}v) \neq \varnothing$, then $2^n + 2^{n+1}u = 2^m + 2^{m+1}v$; this implies $2^k(1 + 2u) = (1 + 2v)$ and then $k = 0$, $u = v$. It follows $\psi \in BC(\mathbb{R}, \mathbb{R})$ and with $I = [-2, 0]$ for each $\tau \geq 2$, $r \in \mathbb{N}$,

$$\sup_{t \in I} |\psi(t + \tau) - \psi(t)| = \sup_{t \in I} |\psi(t + \tau)| \geq \sup_{t \in I} |h_r(t + \tau)|. \tag{2.4}$$

Since $\int_{I_n} h_n(t)dt = 0$, Ph_n is periodic with period 2^{n+1} and $\| Ph_n \|_\infty \leq 2^{-n}$. It follows that $P\psi \in AP(\mathbb{R}, \mathbb{R})$. This implies that $M_h\psi \in AP$ for $h > 0$, and so $sp_{BAA}(\psi) \subset sp_{AP}(\psi) = \varnothing$ by Proposition 1.1.

With $\delta = 2^{-n}$, one has

$$\int_{I_n} |\Delta_1\psi(t+\delta) - \Delta_1\psi(t)|\,dt$$

$$\geq \int_{I_n} |\psi(t+\delta) - \psi(t)|\,dt - \int_{I_n} |\psi(t+1+\delta) - \psi(t+1)|\,dt$$

$$= \int_{I_n} |h_n(t+\delta) - h_n(t)|\,dt$$

$$- \int_{I_n} |\psi(t+1+\delta)|\,dt \geq 2/\pi - 2^{-n} > 0.1 \text{ for all } n \in \mathbb{N}.$$

It follows $\Delta_1\psi$ and so ψ are not uniformly continuous even in the S^1-norm (see [4, p. 132] for the definition). Hence ψ, $\Delta_1\psi$ are not S^1-almost periodic.

Since to each even $n \in \mathbb{N}$, there exist unique $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ such that $n = (1+2k)2^m$, we get

$$n = 2^m + k2^{m+1}. \tag{2.5}$$

We show that for each $\tau \geq 2$, there is $r \in \mathbb{N}$ with $\sup_{t\in I}|h_r(t+\tau)| = 1$. Indeed, let $\tau \in 2\mathbb{N} + y$ for some $y \in [0, 2]$. Then, by (2.5), since $2n + y = 2(n+1) + y'$ with $y' = y - 2$,

$$\tau = 2^m + k2^{m+1} + y \text{ for unique } m \in \mathbb{N},\ k \in \mathbb{N}_0 \text{ and } y \in \left[0, \frac{3}{2}\right], \text{ or}$$

$$\tau = 2^{m'} + k'2^{m'+1} + y' \text{ for unique } m' \in \mathbb{N},\ k' \in \mathbb{N}_0 \text{ and } y' \in \left[-\frac{1}{2}, 0\right].$$

With $t = -y - 2^{-m-1}$, respectively, $t = -y' - 1 + 2^{-m'-1}$, we get

$$\sup_{t\in I}|h_m(t + 2^m + k2^{m+1} + y)| = 1 \text{ for each } y \in \left[0, \frac{3}{2}\right],$$

$$\sup_{t\in I}|h_{m'}(t + k'2^{m'+1} + 2^{m'} + y')| = 1 \text{ for each } y' \in \left[-\frac{1}{2}, 0\right].$$

By (2.4), it follows $\sup_{t\in I}|\psi(t+\tau)-\psi(t)|\geq 1$ for all $\tau\geq 2$. Since *B-aa* and *V-aa* functions are always recurrent (see [5, (3.3)]), we conclude ψ is not *B-aa* or *V-aa* or recurrent or Poisson stable by definitions (2.1) and (2.2). □

3. Reduced Spectrum of Solutions of Evolution Equations

In this section, we study the reduced spectrum with respect to a class $\mathcal{F}\subset L^1_{loc}(J, X)$ of bounded solutions of evolution equations

$$\frac{du(t)}{dt} = Au(t)+\phi(t), \quad u(0) = x\in X, \quad t\in J, \tag{3.1}$$

where A is a closed linear operator on X and $\phi\in L^{\infty}(J, X)$.

The half-line (Laplace) spectrum denoted by $sp_L(\psi)$ for $\psi\in L^{\infty}(\mathbb{R}_+, X)$ is introduced in [1, p. 275]. If $\mathcal{F}\subset L^1_{loc}(\mathbb{R}_+, (X))$ satisfies (1.2), then $sp_{\mathcal{F}}(\psi)\subset sp_w(\psi)\subset sp_L(\psi)$, by [6, (3.12), (3.14)]. Here $sp_w(\psi)$ is the weak half-line (Laplace) spectrum [1, Definition 4.9.1, p. 324]. The reduced spectrum and the half-line spectrum of solutions of (3.1) when $u, \phi\in BUC(J, X)$ have been investigated by many authors, see for example, [3], [1, Proposition 5.6.7, Theorem 5.6.8] and lists of references therein. In this section, we prove inclusions (3.2), (3.3) for (3.1) which are known for the half-line spectrum of solutions of (3.1) in the case $u, \phi\in BUC(\mathbb{R}_+, X)$, see [1, Proposition 5.6.7 (b), pp. 380-381].

Definition 3.1. A function $u\in C(J, X)$ is called a *mild solution* of (3.1) if $\int_0^t u(s)ds\in D(A)$, $x\in X$ and $u(t)-x = A\int_0^t u(s)ds+\int_0^t \phi(s)ds$, $t\in J$ (see [1, pp. 120, 121, 380 for $J=\mathbb{R}_+$]).

Theorem 3.2. *Let* $\mathcal{F}\subset L^1_{loc}(J, X)$ *satisfy* (1.2), (1.3) *and let* $\phi\in L^{\infty}(J, X)$, $J\in\{\mathbb{R}_+, \mathbb{R}\}$. *If* $u\in BC(J, X)$ *is a mild solution of* (3.1), *then*

$$isp_{\mathcal{F}}(u) \subset ((\sigma(A) \cap i\mathbb{R}) \cup isp_{\mathcal{F}}(\phi)). \tag{3.2}$$

If moreover $\phi \in \mathcal{MF}$, *then*

$$isp_{\mathcal{F}}(u) \subset \sigma(A) \cap i\mathbb{R}. \tag{3.3}$$

Proof. First, we prove the case $\phi \in BUC(J, X)$ and $u \in BUC(J, X) \cap C^1(J, X)$ with $u(0), u'(0) \in D(A)$, u classical solution of (3.1) on J ([1, p. 120] for $\mathbb{R}_+$). Denote by $U : \mathbb{R} \to \mathbb{X}$ the function defined by

$U = u$ on J and, if $J = \mathbb{R}_+$, $U(t) = u(0)\cos t + u'(0)\sin t$ when $t \le 0$ (see [3, Lemma 3.2]). Then $U \in BUC(\mathbb{R}, X) \cap C^1(\mathbb{R}, X)$, $U(\mathbb{R}) \subset D(A)$ and U is a classical solution of the equation $v'(t) - Av(t) = F(t)$, where $F(t) = U'(t) - AU(t)$, $t \in \mathbb{R}$.

Note that $F = \phi$ on J and so $sp_{\mathcal{F}}(u) = sp_{\mathcal{F}}(\phi)$ by (1.6). Let $\rho(A)$ be the resolvent set of A and let $i\lambda_0 \in O = (\rho(A) \cap i\mathbb{R}) \cap (i\mathbb{R} \backslash isp_{\mathcal{F}}(\phi))$. Since O is an open set, there is $1 > \delta > 0$ and $\varphi \in \mathcal{S}(\mathbb{R})$ such that $i(\lambda_0 - \delta, \lambda_0 + \delta) \subset O$ with $\hat{\varphi}(\lambda_0) = 1$ and $\operatorname{supp} \hat{\varphi} \subset (\lambda_0 - \delta, \lambda_0 + \delta)$. By [3, Proposition 2.5 (d) for $\mathcal{F} \cap BUC(J, X)$], $sp_{\mathcal{F}} = sp_{\mathcal{F} \cap BUC}$, $sp_{\mathcal{F}}(F * \varphi) \subset sp_{\mathcal{F}}(F) \cap \operatorname{supp} \hat{\varphi} \subset sp_{\mathcal{F}}(\phi) \cap O = \varnothing$. Since $F * \varphi \in BUC(\mathbb{R}, X)$, $F * \varphi | J \in \mathcal{F} \cap BUC(J, X)$, by [2, Theorem 4.2.1]. By [3, Corollary 3.4], $isp_{\mathcal{F}}(U * \varphi) \subset \sigma(A) \cap i\mathbb{R}$, $isp_{\mathcal{F}}(U * \varphi) \subset i(\lambda_0 - \delta, \lambda_0 + \delta) \subset O$ by [3, Proposition 2.5(d)]. As $O \cap (\sigma(A) \cap i\mathbb{R}) = \varnothing$, we get $isp_{\mathcal{F}}(U * \varphi) = \varnothing$. Since $U * \varphi \in BUC(\mathbb{R}, X)$, we conclude that $U * \varphi | J \in \mathcal{F}$ by [2, Theorem 4.2.1] or [4, Corollary 2.3 (A)], and so $\lambda_0 \notin sp_{\mathcal{F}}(U) = sp_{\mathcal{F}}(u)$ by (1.6). This proves (3.2). If $\phi \in \mathcal{MF}$, then $sp_{\mathcal{F}}(\phi) = \varnothing$ by Proposition 1.1. This and (3.2) give (3.3).

The case $\phi \in L^\infty(J, X)$ and $u \in BC(J, X)$. Let $u \in BC(J, X)$ be a mild solution of equation (3.1) and let $k, h > 0$. With Definition 3.1 and an extension of [1, Proposition 3.1.15, p. 120] for $M_h u$ and $M_k M_h u,$ one can show that $v_{k,h} = M_k M_h u$ is a classical solution of

$$\frac{dv(t)}{dt} = Av(t) + \psi_{k,h}(t), \quad v(0) = v_{k,h}(0), \quad t \in J \tag{3.4}$$

with

$$v_{k,h}(0),\, v'_{k,h}(0) \in D(A), \quad \psi_{k,h} = M_k M_h \phi. \tag{3.5}$$

Moreover, $v_{k,h}, \psi_{k,h} \in BUC(J, X) \cap C^1(J, X).$ Applying the above, we get

$$isp_{\mathcal{F}}(v_{k,h}) \subset ((\sigma(A) \cap i\mathbb{R}) \cup isp_{\mathcal{F}}(\psi_{k,h})). \tag{3.6}$$

With [6, Lemma 4.2, (3.11)] (or [8, Proposition 3.4(ii)]), (1.4), (1.5) and (1.6), one gets, for $w \in L^\infty(J, X)$ and $\mathcal{F}$ with (1.2),

$$sp_{\mathcal{F}}(w) = \bigcup_{h>0} sp_{\mathcal{F}}(M_h w). \tag{3.7}$$

This gives

$$\begin{aligned} isp_{\mathcal{F}}(u) &= \bigcup_{k>0,\, h>0} isp_{\mathcal{F}}(v_{k,h}) \subset \bigcup_{k>0,\, h>0}((\sigma(A) \cap i\mathbb{R}) \cup isp_{\mathcal{F}}(\psi_{h,k})) \\ &= (\sigma(A) \cap i\mathbb{R}) \cup (\bigcup_{k>0,\, h>0}(isp_{\mathcal{F}}(M_h M_k \phi))) \\ &= (\sigma(A) \cap i\mathbb{R}) \cup isp_{\mathcal{F}}(\phi). \end{aligned}$$

This proves (3.2). If $\phi \in \mathcal{MF},$ then $sp_{\mathcal{F}}(\phi) = \varnothing$ and so (3.3) follows from (3.2). □

In the following example, we consider the case $X = \mathbb{C}$ and $A : \mathbb{C} \to \mathbb{C}$ defined by $Ac = ic.$ We have $\sigma(A) = \{i\}.$ Examples 3.3 (i), (ii) show that the condition $\phi \in \mathcal{MF}$ in Proposition 1.1 and (3.3) cannot be replaced by $\phi \in \mathcal{F}.$ Also, it supports the suspicion that (3.2) might be trivial without the condition $\mathcal{F} \subset \mathcal{MF}.$ Examples 3.3 (iii), (iv) show that (3.2) and (3.3) can be valid though conditions (1.3) (i), (ii) are not satisfied. One can optimize conditions (1.3) (i), (ii) using [6, Theorem 4.3].

Example 3.3. All complex valued solutions of the equation $y'(t) = iy(t) + \mathfrak{g}(t),\ t \in J$, are bounded uniformly continuous and given by

$$y(t) = e^{it}\left(c + \int_0^t e^{-is}\mathfrak{g}(s)\,ds\right) =: \gamma_1(t)(c + y_1(t)),$$

where $c \in \mathbb{C}$ and $\mathfrak{g}(t) = e^{it^2}$.

(i) If $\mathcal{F}_1 = \mathfrak{g} \cdot AP(\mathbb{R}, \mathbb{C})$, then $sp_{\mathcal{F}_1}(\mathfrak{g}) = \mathbb{R}$, so (3.2) holds trivially.

(ii) If $\mathcal{F}_2 = \mathfrak{g} \cdot AP(\mathbb{R}, \mathbb{C}) \oplus AP(\mathbb{R}, \mathbb{C})$, then $\mathfrak{g} \in \mathcal{F}_2$, $\mathcal{F}_2$ satisfies (1.2) and (1.3) but $\mathcal{F}_2 \not\subset \mathcal{MF}_2$. Here, $sp_{\mathcal{F}_2}(\mathfrak{g}) = sp_{\mathcal{F}_2}(y) = \mathbb{R}$ and $\sigma(A) \cap i\mathbb{R} = \{i\}$. So, (3.3) is not satisfied.

(iii) If $\mathcal{F}_3 = (\mathfrak{g} \cdot AP(\mathbb{R}_+, \mathbb{C})) \oplus (C_0(\mathbb{R}_+, \mathbb{C}) \oplus (\gamma_1 \cdot \mathbb{C}))$, then $\mathcal{F}_3 \subset \mathcal{MF}_3$, $sp_{\mathcal{F}_3}(\mathfrak{g}) = \varnothing$ and $sp_{\mathcal{F}_3}(y) = \varnothing$, $c \in \mathbb{C}$.

(iv) If $\mathcal{F}_4 = (\mathfrak{g} \cdot AP(\mathbb{R}, \mathbb{C})) \oplus (\gamma_1 \cdot C(\overline{\mathbb{R}}, \mathbb{C}))$, then $sp_{\mathcal{F}_4}(\mathfrak{g}) = sp_{\mathcal{F}_4}(y) = \varnothing$, $c \in \mathbb{C}$. Here,

$$C(\overline{\mathbb{R}}, \mathbb{C}) = \{\phi \in BUC(\mathbb{R}, \mathbb{C}) : \lim_{t\to\infty}\phi(t),\ \lim_{t\to-\infty}\phi(t) \text{ exist}\}.$$

Proof. (i) We have $\mathfrak{g} \in \mathcal{F}_1 = \mathfrak{g} \cdot AP$ but $sp_{\mathcal{F}_1}(\mathfrak{g}) = sp_{\mathcal{F}_1 \cap BUC(\mathbb{R}, \mathbb{C})}(\mathfrak{g}) = sp_{\{0|\mathbb{R}\}}(\mathfrak{g}) = \operatorname{supp}\hat{\mathfrak{g}} = \mathbb{R}$, since $\hat{\mathfrak{g}}(s) = (\pi)^{1/2}e^{i\pi/4}e^{-is^2/4}$ (see [4, (1.3)], [6, (3.3)], [8, Example 4.5]).

(ii) $\mathcal{F}_2$ satisfies (1.2) and (1.3), but we omit the proof that $\mathcal{F}_2$ is uniformly closed. We have $\mathfrak{g} \in \mathcal{F}_2$ but $sp_{\mathcal{F}_2}(\mathfrak{g}) = sp_{\mathcal{F}_2 \cap BUC(\mathbb{R}, \mathbb{C})}(\mathfrak{g}) = sp_{AP}(\mathfrak{g}) = \mathbb{R}$, since $\mathfrak{g} * f \in AP(\mathbb{R}, \mathbb{C})$ implies $\mathfrak{g} * f \in AP(\mathbb{R}, \mathbb{C}) \cap C_0(\mathbb{R}, \mathbb{C}) = \{0\}$, $f \in L^1(\mathbb{R}, \mathbb{C})$, and $sp_{\mathcal{F}_2}(\mathfrak{g}) = sp_{\{0|\mathbb{R}\}}(\mathfrak{g}) = \mathbb{R}$.

As

$$sp_{\mathcal{F}_2}(y) = sp_{\mathcal{F}_2 \cap BUC(\mathbb{R},\, \mathbb{C})}(y) = sp_{AP}(y),$$

$$sp_{AP}(\gamma_1 y_1) \subset sp_{AP}(y) \cup sp_{AP}(-c\gamma_1)$$

and $sp_{AP}(-c\gamma_1) = sp_{AP}(\gamma_1) = \varnothing$ ([2, Theorem 4.1.4, (4.1.7), Theorem 4.2.1]), one gets

$$sp_{AP}(\gamma_1 y_1) = 1 + sp_{AP}(y_1),\;\; sp_{AP}(u') \subset sp_{AP}(u)\; (u \in BUC,\, u' \in BC),$$

and $sp_{AP}(y_1') = -1 + sp_{AP}(\mathfrak{g})$, $sp_{AP}(\mathfrak{g}) = \mathbb{R}$ shown already, it follows $sp_{\mathcal{F}_2}(y) = \mathbb{R}$.

(iii) We have $y,\, \mathfrak{g}|\mathbb{R}_+ \in \mathcal{F}_3$ and $\mathcal{F}_3$ satisfies $\mathcal{F}_3 \subset \mathcal{M}\mathcal{F}_3$. So, $sp_{\mathcal{F}_3}(\mathfrak{g}) = sp_{\mathcal{F}_3}(y) = \varnothing$ by Corollary 1.2.

(iv) We have $\mathfrak{g},\, y \in \mathcal{F}_4$ and $\mathcal{F}_4 \subset \mathcal{M}\mathcal{F}_4$. The result follows by Corollary 1.2.

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