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EXCEPTIONAL SETS OF LAURICELLA HYPERGEOMETRIC FUNCTIONS

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Abstract

In this paper, we study values at algebraic points of Lauricella hypergeometric functions in n complex variables, $n \geq 1$. We are mostly interested in criteria for the transcendence of these values. The combined results of [P. Cohen and G. Wüstholz, Application of the André-Oort conjecture to some questions in transcendence, A panorama of number theory or the view from Baker's garden (Zürich, 1999), pp. 89-106, Cambridge Univ. Press, Cambridge, 2002], [Invent. Math. 92 (1988), 187-216] determine necessary and sufficient conditions on the parameters a, b, c for finiteness of the exceptional set of the classical hypergeometric function of one complex variable $(n = 1)$: $\mathcal{E} = \{x \in \overline{\mathbb{Q}} : F(a; b; c; x) \in \overline{\mathbb{Q}}^*\}$. These results rely on Wüstholz's analytic subgroup theorem [G. Wüstholz, Algebraic groups, Hodge theory and transcendence, Proc. of the Intern. Congress of Math., Berkeley, California, U.S.A., 1986], [Ann. Math. 129 (1989), 501-517] and on a known particular case, proved in [Ann. Math. 157 (2003), 621-645], of the André-Oort conjecture on the distribution of complex multiplication points on curves in Shimura varieties. The results of [P. Cohen and G.

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Wüstholz, Application of the André-Oort conjecture to some questions in transcendence, A panorama of number theory or the view from Baker's garden (Zürich, 1999), pp. 89-106, Cambridge Univ. Press, Cambridge, 2002] were generalized to the two $(n = 2)$ variable Appell hypergeometric function by the author [Ramanujan J. 8(3) (2004), 331-355], subject this time to the André-Oort conjecture for surfaces in Shimura varieties. In the present paper, we treat the case of several $(n \geq 3)$ complex variables. The main contribution of the present paper is the construction for the Lauricella function of the appropriate exceptional set that allows for the application of the André-Oort conjecture for n-dimensional subvarieties of Shimura varieties. Some additional results on transcendence of values of Lauricella functions are given, as well as a new counterexample to a conjecture of Coleman.

0. Introduction

Wolfart [27] studied values of the classical (Gauss) hypergeometric function of a single complex variable at algebraic points. In particular, using Wüstholz's analytic subgroup theorem [30], he investigated conditions on the parameters a, b, c that ensure infiniteness of the *exceptional set*: $\mathcal{E} = \{x \in \overline{\mathbb{Q}} : F(a; b; c; x) \in \overline{\mathbb{Q}}^*\}$. Wolfart also correctly predicted conditions on the parameters that ensure finiteness, although, as first noticed by Walter Gubler, his proof of this prediction contains a serious error. Cohen and Wüstholz [11] corrected this error and completely solved the problem of criteria for the finiteness of $\mathcal{E}$, subject to a weak form of the André-Oort conjecture for curves in Shimura varieties. This particular case of the André-Oort conjecture was proved by Edixhoven and Yafaev in [15]. For a statement of the André-Oort conjecture see [1], [2], [19], [20]. The results of [11] were generalized to the two variable Appell hypergeometric function by the author [14], subject this time to the as yet unproved André-Oort conjecture for surfaces in Shimura varieties. In this paper, we study the transcendence of values at algebraic points of Lauricella hypergeometric functions in n complex variables, $n \geq 1$. We again apply the analytic subgroup theorem, and our results are again subject to the André-Oort conjecture, this time for n-dimensional subvarieties of Shimura varieties. In order to link the analytic subgroup theorem to the André-Oort conjecture, we rely as in [11], [14] on the construction of certain analytic families of abelian varieties appearing in [27], [9] (for $n = 1$), in [10] (for $n = 2$) and in [5] (for all $n \geq 1$). The main new contribution is the construction of the appropriate exceptional set for Lauricella functions that allows one to apply the André-Oort conjecture. As first observed in [14], the passage to several variables requires that we utilize a proper subset of the set of algebraic points at which the function takes algebraic values. The definition of this smaller exceptional set requires quite subtle conditions. We also give a new counterexample to a conjecture of Coleman, thereby extending the list of examples in [16], [11], [14]. Our last result shows how algebraic values of Lauricella functions, at algebraic points, imply the transcendence of the values of other such functions at related points.

The plan of this article is as follows. In Section 1, we recall some classical properties of the Lauricella hypergeometric functions: more details can be found in [3], [12] and [32]. In Section 2 and in the first part of Section 3, we summarize results needed for the sequel that appear in [5], [9], [10], [24]: namely, the construction of an analytic family of "hypergeometric" abelian varieties associated to the Lauricella hypergeometric functions and the identification of the associated Shimura variety V. This determines a morphism from the space $\mathcal{Q}$ of regular points of the Lauricella hypergeometric functions to the complex points $V(\mathbb{C})$ of the Shimura variety. As we discuss in the remainder of Section 3, this morphism extends to the space of semi-stable points described in [12]. In Section 4, we state our main new result as Theorem 1. This theorem shows how the exceptional set for the Lauricella hypergeometric function must be constructed in order for its elements to correspond to hypergeometric abelian varieties of CM type. The exact definition of the exceptional set is given at the end of Section 4: the conditions defining this exceptional set are quite involved. Our definition reduces to the one of [14] in the case $n = 2$ and to the one of [27] in the case $n = 1$. We state in Section 5 a weak form of the André-Oort conjecture for n-dimensional subvarieties in Shimura varieties sufficient for our purposes. This result is still only known for the case $n = 1$, see [15]. As in [11] $(n = 1)$ and [14] $(n = 2)$, this allows us to make a link between the Zariski density of the points of the exceptional set and the nature of the image of $\mathcal{Q}$ in the Shimura variety V: the points of the exceptional set will be Zariski-dense if and only if the Zariski closure of the image of $\mathcal{Q}$ is of Hodge type. This leads us to the result of Theorem 2, which states a criterion for the arithmeticity of the monodromy group of the Lauricella hypergeometric function which generalizes to all $n \geq 1$ that of [11], [27] $(n = 1)$ and [14] $(n = 2)$. We also show that known results in the spirit of the André-Oort conjecture give a counterexample of Coleman's conjecture. Other counterexamples obtained in a similar way were given in [16], [14]. In Section 6, we state a new result as Theorem 3. This result shows how algebraic values of Lauricella functions, at algebraic points, imply the transcendence of the values of other such functions at related points.

1. Lauricella Hypergeometric Functions

The Lauricella hypergeometric function is a generalization to n complex variables of the classical (Gauss) hypergeometric function to the case of the one complex variable, see [3, Chapter VII]. It is a solution of a system of linear partial differential equations $E_n(a;\, b_2;\, \ldots;\, b_{n+1};\, c)$ which has an $(n+1)$-dimensional solution space. The regular singularities are located along the hyperplanes $x_i = 0;\, 1;\, \infty;\, x_j$ for $i;\, j \in \{2;\, \ldots;\, n+1\}$ and $j \neq i$. These hyperplanes are often referred to as characteristic surfaces. Denote the space of regular points by,

$$\mathcal{Q} := \{(x_2;\, \ldots;\, x_{n+1}) \in \mathbb{P}_1(\mathbb{C})^n : x_i \neq 0, 1, \infty, x_j, \text{ for all } i, j \in \{2;\, \ldots;\, n+1\}, i \neq j\}.$$

For all $(x_2;\, \ldots;\, x_{n+1}) \in \mathcal{Q}$, a basis of solutions of the equations $E_n(a;\, b_2;\, \ldots;\, b_{n+1};\, c)$, can be represented by Euler integrals, as follows:

$$\int_{\gamma_{gh}} u^{\sum_{i=2}^{n+1} b_i - c}(u-1)^{c-a-1} \prod_{i=2}^{n+1} (u - x_i)^{-b_i}\, du = \int_{\gamma_{gh}} \omega(x_2;\, \ldots;\, x_{n+1}),$$

where the γ_{gh} are Pochhammer cycles around $g, h \in \{0; 1; x_2; \ldots; x_{n+1}; \infty\}$, $g \neq h$. Under certain natural conditions (see Section 2), in particular when the parameters are rational numbers, these integrals are equal, up to multiplication by a non-zero algebraic number, to the Euler line integrals:

$$\int_g^h \omega(x_2; \ldots; x_{n+1}).$$

It is convenient at this stage to introduce another system of parameters. Let

$$\begin{cases} \mu_0 = c - \sum_{i=2}^{n+1} b_i \\ \mu_1 = a + 1 - c \\ \mu_i = b_i, \text{ for } i = 2; \ldots; n+1 \\ \mu_{n+2} = 1 - a. \end{cases}$$

We then have

$$\omega(x_2; \ldots; x_{n+1}) = u^{-\mu_0}(u-1)^{-\mu_1} \prod_{i=2}^{n+1} (u - x_i)^{-\mu_i} du.$$

The Lauricella hypergeometric function is the unique solution of the system of linear partial differential equations which extends to a holomorphic solution equalling 1 at the point $(0; \ldots; 0)$. For $(x_2; \ldots; x_{n+1}) \in \mathcal{Q}$, its expression is the following,

$$F(a; b_2; \ldots; b_{n+1}; c; x_2; \ldots; x_{n+1}) = B(a; c-a)^{-1} \int_1^\infty \omega(x_2; \ldots; x_{n+1}),$$

where $B(\cdot, \cdot)$ is the Bêta function.

2. Hypergeometric Abelian Varieties

To ensure that the hypergeometric function is a transcendental function and that the Euler line integrals of Section 1 correspond to periods of differentials of the first kind on algebraic curves, we assume from now on the following conditions on the parameters:

$$a; b_i, c \in \mathbb{Q}, \quad i = 2, \ldots, n+1; \quad 0 < a < c; \quad 0 < c < 1; \quad 0 < \sum_{i=2}^{n+1} b_i < c.$$

Alternatively,

$$\mu_i \in \mathbb{Q} \cap]0; 1[, \quad i = 2, \ldots, n+1; \quad \mu_1 + \mu_{n+2} > 1.$$

Note that the condition $\sum_{i=0}^{n+2} \mu_i = 2$ always holds. Let N be the least common denominator of the μ_i. Under these conditions, the Euler line integrals $\int_g^h \omega(x_2; \ldots; x_{n+1})$ of Section 1 are, up to a non-zero algebraic scalar, periods of the first kind on abelian varieties in an analytic family: these families were constructed in [27], [9] $(n = 1)$, in [10] $(n = 2)$, and in [24], [5] $(n \geq 1)$. We briefly recall this construction for use in later sections. Consider the family of algebraic curves:

$$\mathcal{X}_N(x_2; \ldots; x_{n+1}) : w^N = u^{\mu_0 N}(u-1)^{\mu_1 N} \prod_{i=2}^{n+1} (u - x_i)^{\mu_i N}, (x_2; \ldots; x_{n+1}) \in \mathcal{Q}.$$

Next, we consider a subvariety $T(x_2; \ldots; x_{n+1})$ of the Jacobian of each such curve, the so-called "new part of the Jacobian". It is defined as follows: for each proper divisor f of N, let

$$\mathcal{X}_f(x_2; \ldots; x_{n+1}) : w^f = u^{\mu_0 N}(u-1)^{\mu_1 N} \prod_{i=2}^{n+1} (u - x_i)^{\mu_i N}.$$

The morphism defined by

$$\mathcal{X}_N(x_2; \ldots; x_{n+1}) \to \mathcal{X}_f(x_2; \ldots; x_{n+1})$$

$$(u; w) \mapsto (u; w^{\frac{N}{f}})$$

induces a morphism of Jacobians m_f from $\mathrm{Jac}\mathcal{X}_N(x_2; \ldots; x_{n+1})$ to $\mathrm{Jac}\mathcal{X}_f(x_2; \ldots; x_{n+1})$. Then $T(x_2; \ldots; x_{n+1})$ is given by $\bigcap_{f \mid N} Ker\, m_f$, the connected component of the identity of the intersection of the kernels of these morphisms.

Remark. We have the following decomposition, up to isogeny:

$$\mathrm{Jac}\mathcal{X}_N(x_2; \ldots; x_{n+1}) \triangleq T(x_2; \ldots; x_{n+1}) \oplus \sum_{f \mid N} \mathrm{Jac}\mathcal{X}_f(x_2; \ldots; x_{n+1}).$$

In [9], [10], [24], [5] the authors show that $T(x_2; ...; x_{n+1})$ satisfies the following properties:

(a) It is a principal polarized abelian variety of dimension $(n+1)\frac{\varphi(N)}{2}$ (where φ is the Euler's function).

(b) It is of type IV (in the sense of Shimura-Taniyama [25]), with generalized complex multiplication by the cyclotomic field $\mathbb{Q}(\zeta_N)$, where ζ_N is a certain primitive root of unity. Therefore $\mathbb{Q}(\zeta_N)$ can be embedded in the endomorphism algebra of $T(x_2; ...; x_{n+1})$, denoted by $\mathrm{End}_0(T(x_2; ...; x_{n+1}))$, and given by $\mathrm{End}(T(x_2; ...; x_{n+1})) \otimes_{\mathbb{Z}} \mathbb{Q}$. This embedding is unitary.

(c) The cyclotomic field $\mathbb{Q}(\zeta_N)$ acts on the vector space $H^0(T(x_2; ...; x_{n+1}); \Omega)$ of differential forms of the first kind. This action is induced by the automorphism of $\mathcal{X}_N(x_2; ...; x_{n+1})$ sending $(u; w)$ to $(u; \zeta_N^{-1} \cdot w)$.

The action in (c) induces a decomposition of $H^0(T(x_2; ...; x_{n+1}); \Omega)$ into eigenspaces V_s associated to the eigenvalues ζ_N^s, $s \in (\mathbb{Z}/N\mathbb{Z})^{\times}$. Namely, let $\langle t \rangle$ be the fractional part of the real number t ($\langle t \rangle = t - E(t)$, for $E(t)$ the integer part of t). The following differential form has eigenvalue ζ_N^s:

$$\omega_s(x_2; ...; x_{n+1}) = u^{-\langle s\mu_0 \rangle}(u-1)^{-\langle s\mu_1 \rangle}\prod_{i=2}^{n+1}(u - x_i)^{-\langle s\mu_i \rangle} du.$$

The field $\mathbb{Q}(\zeta_N)$ acts on V_s via multiplication by $\sigma_s(\mathbb{Q}(\zeta_N))$, where σ_s is the embedding of $\mathbb{Q}(\zeta_N)$ in $\mathbb{C}$ sending ζ_N to ζ_N^s. The eigensubspace related to this eigenvalue ζ_N^s is of dimension:

$$\dim V_s = r_s = -1 + \sum_{i=0}^{n+2}\langle s\mu_i \rangle$$

and for all $s \in (\mathbb{Z}/N\mathbb{Z})^{\times}$,

$$r_s + r_{-s} = n + 1.$$

Let M_k be the number of eigensubspaces of dimension k, and denote these subspaces by $V^k_{i_1^{(k)}}; \dots; V^k_{i^{(k)}_{M_k}}$. Notice that $i_1^{(1)} = 1$ since $\sum_{i=0}^{n+2} \mu_i = 2.$ There are also M_k eigensubspaces (for the conjugate eigenvalues) of dimension $n+1-k$, which we denote by $V^{n+1-k}_{-i_1^{(k)}}; \dots; V^{n+1-k}_{-i^{(k)}_{M_k}}$. Then,

$$\begin{aligned} &H^0(T(x_2; \dots; x_{n+1}); \Omega) \\ &= V^{n+1}_{-i_1^{(0)}} \oplus \cdots \oplus V^{n+1}_{-i^{(0)}_{M_{n+1}}} \oplus V^1_{i_1^{(1)}} \oplus \cdots \oplus V^1_{i^{(1)}_{M_1}} \oplus V^n_{-i_1^{(1)}} \\ &\oplus \cdots \oplus V^n_{-i^{(1)}_{M_1}} \oplus \cdots \oplus V^{E\left(\frac{n}{2}\right)}_{i_1^{\left(E\left(\frac{n}{2}\right)\right)}} \\ &\oplus \cdots \oplus V^{E\left(\frac{n}{2}\right)}_{i^{\left(E\left(\frac{n}{2}\right)\right)}_{M_{\left(E\left(\frac{n}{2}\right)\right)}}} \oplus V^{n+1-E\left(\frac{n}{2}\right)}_{-i_1^{\left(E\left(\frac{n}{2}\right)\right)}} \oplus \cdots \oplus V^{n+1-E\left(\frac{n}{2}\right)}_{-i^{\left(E\left(\frac{n}{2}\right)\right)}_{M_{E\left(\frac{n}{2}\right)}}}. \end{aligned}$$

The CM type of $T(x_2; \dots; x_{n+1})$ is given by:

$$\Phi = \sum_{s \in (\mathbb{Z}/N\mathbb{Z})^\times} r_s \cdot \sigma_s.$$

Shimura [25] has shown that the complex isomorphism classes of principally polarized abelian varieties of dimension $(n+1)\varphi(N)/2$, of type IV, with complex multiplication by the cyclotomic field $\mathbb{Q}(\zeta_N)$ of type Φ and lattice isomorphic to $\mathbb{Z}[\zeta_N]^{(n+1)}$ are parameterized by the complex points of a quasi-projective variety V, defined over $\overline{\mathbb{Q}}$, which is the quotient by a certain arithmetic group Γ acting discontinuously on the following product of spaces:

$$H := \prod_{r+t=n+1,\, r,\, t \geq 0}^{E\left(\frac{n}{2}\right)} (\mathcal{H}^3_{r;\, t})^{M_r},$$

where for $r = 0$ or $t = 0$ the corresponding factor is trivial and for $r, t \geq 1$,

$$\mathcal{H}^3_{r;t} := \{Z : Z \text{ complex matrix with } r \text{ rows and}$$
$$t \text{ columns such that } 1 - Z^t\overline{Z} \text{ is positive hermitian}\}.$$

Remarks. (i) If for all $s \in (\mathbb{Z}/N\mathbb{Z})^{\times}$, we have $r_s \in \{0; 1; n; n+1\}$, then $H = B_n^{M_1}$, where B_n is the unit ball of dimension n. The space H is then of dimension

$$nM_1 = \sum_{s \in (\mathbb{Z}/N\mathbb{Z})^{\times}/\{\pm 1\}} r_s \cdot r_{-s}.$$

(ii) The condition in (i) is always true for $n = 1$ or 2.

(iii) The $T(x_2; ...; x_{n+1})$ for $(x_2; ...; x_{n+1}) \in \mathcal{Q}$ form a subfamily of dimension n of this space.

3. Maps between Moduli Spaces

In the one and two variable case, previous authors (in [9], [10], [11] and [14]) have studied the relation of the Shimura variety V to the discontinuous monodromy groups $\Delta(\mu_i)$ of the respective systems $E_1(a; b; c)$, acting on $H = \mathcal{H}$ (the upper half plane), and $E_2(a; b; b'; c)$, acting on $H = B_2$ (the unit ball). When the action of these groups is discontinuous, these authors construct a modular embedding which is in general non-trivial because the groups considered are not necessary arithmetic. This embedding is a holomorphic map from the covering spaces H of these groups to the covering space of the Shimura variety parametrizing the analytic family of polarized abelian varieties to which the "new part" $T(x)$ or $T(x; y)$ belong. The map satisfies an equivariance property with respect to $\Delta(\mu_i)$ which allows one to pass to the quotient to obtain a morphism from $H/\Delta(\mu_i)$ to the Shimura variety. Moreover, in these papers the authors obtained transcendence results on the covering radius of the quotient spaces $\mathcal{H}/\Delta(\mu_i)$ and $B_2/\Delta(\mu_i)$.

For one variable, there is an infinity of such discontinuous groups. However, for two variables, there are only 58 cases. When the number of variables is between 3 and 12, there are 32 cases of such groups and only 1 is non-arithmetic. There are no such groups if the number of variables is at least 13 (see [12], [18] and [22]).

From now on, we assume that

$$s \in (\mathbb{Z}/N\mathbb{Z})^{\times}, \quad r_s \in \{0; 1; n; n+1\}$$

so that the modular space is $H = B_n^{M_1}$, according to the remarks in Section 2. Recall from Section 1 that $\mathcal{Q}$ denotes the space of regular points of $E_n(a; b_2; \ldots; b_{n+1}; c)$ and that V is the Shimura variety of Section 2.

Theorem [24]**.** *There exists a birational morphism of quasi-projective varieties defined over* $\overline{\mathbb{Q}}$*:*

$$\Phi : \begin{array}{ccc} \mathcal{Q} & \to & V(\mathbb{C}) \\ (x_2; \ldots; x_{n+1}) & \mapsto & J([T(x_2; \ldots; x_{n+1})]), \end{array}$$

where $J([T(x_2; \ldots; x_{n+1})])$ *is the point of* $V(\mathbb{C})$ *that corresponds to the isomorphism class of the abelian variety* $T(x_2; \ldots; x_{n+1})$*. This map is induced by the composition of a map*

$$\psi : \mathcal{Q} \to B_n$$

and a map

$$F : \begin{array}{ccc} B_n & \to & B_n^{M_1} \\ \psi(x_2; \ldots; x_{n+1}) & \mapsto & Z([T(x_2; \ldots; x_{n+1})]), \end{array}$$

followed by passage to the quotient by a certain arithmetic group Γ *acting on* $B_n^{M_1}$*, with* $V(\mathbb{C})$ *isomorphic to* $B_n^{M_1}/\Gamma$*.*

The orbit of $Z([T(x_2; \ldots; x_{n+1})])$ *under* Γ *corresponds to the point* $J([T(x_2; \ldots; x_{n+1})])$ *of* $V(\mathbb{C})$*.*

It is sometimes useful to consider the $n+3$-tuple $(x_0; x_1; x_2; ...; x_{n+2})$ in the space $\mathbb{P}_1(\mathbb{C})^{n+3}$ modulo the action of $PSL_2(\mathbb{C})$ on $\mathbb{P}_1(\mathbb{C})^{n+3}$, instead of $(0; 1; x_2; ...; x_{n+1}; \infty)$ in $\mathbb{P}_1(\mathbb{C})^n$. Then, the differential form ω can be written as

$$\omega(x_0; ...; x_{n+2}) = \prod_{i=0}^{n+2} (u - x_i)^{-\mu_i} du.$$

Fix an $n+3$-tuple $\mu = (\mu_0; ...; \mu_{n+2})$.

Definition. A point $(x_0; ...; x_{n+2})$ in $\mathbb{P}_1(\mathbb{C})^{n+3}$ is called:

μ-*stable* when for all subsets $T \subset \{0; ...; n+2\}$, $\sum_{x_t = x_{t'},\, t, t' \in T} \mu_t < 1$.

μ-*semistable* when for all subsets $T \subset \{0; ...; n+2\}$, $\sum_{x_t = x_{t'},\, t, t' \in T} \mu_t \leq 1$.

Here, when $x_t \neq x_{t'}$ for all $t, t' \in T$, there is no condition.

Let M_{st} be the set of the stable points, M_{sst} be the set of the semistable points and $M_{cusp} = M_{sst} \setminus M_{st}$. Each element of M_{cusp} is determined by a partition $\{I, J\}$ of $\{0; ...; n+2\}$ with $\sum_{i \in I} \mu_i = \sum_{j \in J} \mu_j = 1$, with the x_i equal for $i \in I$, and the x_j equal for $j \in J$, but with $x_i \neq x_j$.

The group $\mathrm{Aut}(\mathbb{P}_1(\mathbb{C})) = \mathrm{PGL}_2(\mathbb{C})$ acts freely on $\mathbb{P}_1(\mathbb{C})$ and consequently on $\mathbb{P}_1(\mathbb{C})^{n+3}$ by the diagonal action. Using [12, Paragraph 4.1], we can define on M_{sst} an equivalence relation $\mathcal{R}$ as follows: we have $x \equiv x'(\mathcal{R})$ either when $x, x' \in M_{st}$ and there is a $\gamma \in \mathrm{PGL}_2(\mathbb{C})$ such that $x' = \gamma x$, or when $x, x' \in M_{cusp}$ and x, x' are defined by the same partition $I \cup J = \{0; ...; n+2\}$, $I \cap J = \varnothing$.

Now, consider the quotient spaces:

$$\mathcal{Q}_{sst} = M_{sst}/\mathcal{R} \qquad \mathcal{Q}_{st} = M_{st}/\mathcal{R} \quad \text{and} \quad \mathcal{Q}_{cusp} = M_{cusp}/\mathcal{R}.$$

The space $\mathcal{Q}_{sst}$ is Hausdorff and compact and can be given the structure of an algebraic variety [12, Paragraph 4]. The space $\mathcal{Q}$ can be realized as a subset of $\mathcal{Q}_{sst}$ using the diagonal action of $\mathrm{PGL}_2(\mathbb{C})$, and $\mathcal{Q}_{sst}$ is in fact a compactification of $\mathcal{Q}$.

We can now study the action of the morphism Φ on the stable points located on the boundary of $\mathcal{Q}$. Using [26], we see that Φ can be extended to a birational map from $\mathcal{Q}_{st}$ into $V(\mathbb{C})$. Indeed, in [26, Theorem 1, Section 1], it is shown that along the characteristic surfaces $S_{st}(ij) : x_i = x_j,\ \mu_i + \mu_j < 1$, there are $n+1$ solutions (n holomorphic and one of the form $(x_i - x_j)^{1-\mu_i-\mu_j} \times$ a holomorphic function) of a linear system of differential equations of the type (E_n) with the same monodromy as $(E_n(a; b_2; \ldots; b_{n+1}; c))$. This enables us to extend the application $\psi : \mathcal{Q} \to B_n$ to $\mathcal{Q}_{st}$ of the above theorem. Now consider also the application $F : B_n \to B_n^{M_1}$ of the above theorem. For $I \not\subseteq \{0; \ldots; n+2\}$, denote by E_I the subspace of $\mathbb{P}_1(\mathbb{C})^{n+2}$ given by the equations:

$$E_I : \begin{cases} x_{i_1} = x_{j_1}, & i_1, j_1 \in I_1 \\ \quad\vdots & \quad\vdots \\ x_{i_k} = x_{j_k}, & i_k, j_k \in I_k \end{cases} \quad \text{where } I = I_1 \cup \cdots \cup I_k.$$

Each stable subspace is of this type.

Consider $F \circ \psi(E_I)$. Recall that $s \in R_1$ when $r_s = 1$ and that $M_1 = \operatorname{Card} R_1$.

Two kinds of images are possible, depending on the value of $s \in R_1$.

Case (i) for which

$$\sum_{i \in I} \langle s\mu_i \rangle < 1.$$

Then, there exist elements $\theta_s^{(1)}, \ldots, \theta_s^{(n)} \in \overline{\mathbb{Q}}$ such that the s-th projection of the image of E_I by $F \circ \psi$ is given by

$$\left(\theta_s^{(1)}\int_{\gamma_1}\omega'_{(s)} : \cdots : \theta_s^{(n)}\int_{\gamma_n}\omega'_{(s)} : \int_{\gamma_0}\omega'_{(s)}\right) \in B_n,$$

where

$$\omega'_{(s)} = \omega'_{(s)}(x_{i_1}; \ldots; x_{i_k}; x_j,\ j \in J)$$

$$= \prod_{l=1}^{k}(u - x_{i_l})^{-\sum_{i\in I_l}\langle s\mu_i\rangle}\prod_{j\in J}(u - x_j)^{-\langle s\mu_j\rangle}du.$$

This differential is of the first kind by our assumptions.

Case (ii) for which

$$\sum_{i\in I}\langle s\mu_i\rangle > 1.$$

Then, using a blow-up σ (see [23, Chapter VI.2]) one has to consider the subvariety $E_J^{(s)}$ given by the equations:

$$E_J^{(s)} : \begin{cases} x_{i_1} = x_{j_1}, & i_1, j_1 \in J_1 \\ \quad\vdots & \quad\vdots \\ x_{i_k} = x_{j_k}, & i_k, j_k \in J_l \end{cases}$$

where $J = J_1 \cup \cdots \cup J_l$, $\{I, J\}$ partition of $\{0; \ldots; n+2\}$

which is stable because of the hypothesis $s \in R_1$, i.e., $\sum_{i=0}^{n+2}\langle s\mu_i\rangle = 2$.

But, along these subvarieties (modified by a blow-up σ) one has $E_J^{(s)} = \sigma(E_I^{(s)})$. Terada [26] shows, in the demonstration of his Proposition 3, Section 3, that one can also find $n+1$ linearly independent solutions of a Lauricella hypergeometric system of differential equations E_n. More precisely, there exist elements $\theta_s^{(1)}, \ldots, \theta_s^{(n)} \in \overline{\mathbb{Q}}$ such that the s-th projection of the image of E_I by $F \circ \psi$ is given by

$$\left(\theta_s^{(1)}\int_{\gamma_1}\omega''_{(s)} : \cdots : \theta_s^{(n)}\int_{\gamma_n}\omega''_{(s)} : \int_{\gamma_0}\omega''_{(s)}\right) \in B_n,$$

where

$$\omega''_{(s)} = \omega''_{(s)}(x_{j_1}; \ldots; x_{j_p}; x_i, i \in I)$$

$$= \prod_{l=1}^{p} (u - x_{j_l})^{-\sum_{j \in J_l} \langle s\mu_j \rangle} \prod_{i \in I} (u - x_i)^{-\langle s\mu_i \rangle} du.$$

These are differential forms of the first kind by our assumptions.

We give here a geometric meaning to the image of the stable points by Φ. Each stable element E_I is an intersection of hypersurfaces, denoted $S_{st}(ij) : x_i = x_j$, where $\mu_i + \mu_j < 1$, $i \neq j \in I$. Consider a "coherent" system of hypersurfaces characterizing E_I. That means: make (between the $C^2_{card I}$-choices) the right choice of hypersurfaces whose intersection is E_I (there can be several possible choices).

This decomposition is useful, because by [10, Paragraph 5], along $S_{st}(ij)$, the abelian variety $T(x_2; \ldots; x_{n+1})$ can be decomposed, up to isogeny, as

$$T(x_2; \ldots; x_{n+1}) \triangleq A_{ij} \times T,$$

where A_{ij} is an abelian variety of dimension $\varphi(N)/2$ with CM by a subfield of $\mathbb{Q}(\zeta_N)$ of type

$$\Phi^{(ij)} = \sum_{s \in (\mathbb{Z}/N\mathbb{Z})^\times} (\langle s\mu_i \rangle + \langle s\mu_j \rangle - \langle s(\mu_i + \mu_j) \rangle)\sigma_s.$$

This abelian variety is characterized by the period

$$B(\mu_i; \mu_j) = \int_{x_i}^{x_j} (u - x_i)^{-\mu_i} (u - x_j)^{-\mu_j} (u - x_k)^{-\sum_{k=0, k \neq i, j}^{n+2} \mu_k} du.$$

The abelian variety T is of dimension $\eta\varphi(N)/2$, of type IV with CM by the cyclotomic field $\mathbb{Q}(\zeta_N)$, of type

$$\Phi^{(T)} = \sum_{s \in (\mathbb{Z}/N\mathbb{Z})^\times} \left(\sum_{k=0,\ k \neq i, j}^{n+2} \langle s\mu_k \rangle - \left\langle s \sum_{k=0,\ k \neq i, j}^{n+2} \mu_k \right\rangle \right) \sigma_s.$$

This abelian variety is characterized by the periods

$$\int_g^h (u - x_i)^{-\mu_i - \mu_j} \prod_{k=0,\ k \neq i,\ j}^{n+2} (u - x_k)^{-\mu_k} du,\ g,\ h \in \{x_0;\ ...;\ x_{n+2}\} \setminus \{x_j\},\ g \neq h.$$

Corollary. *Each stable point corresponds to an abelian variety with CM.*

A stable point P_{st} is the intersection of n hypersurfaces $S_{st}(ij)$ of $\mathbb{P}_1(\mathbb{C})^n = \{(x_2;\ ...;\ x_{n+1})/x_i \in \mathbb{P}_1(\mathbb{C})\}$. Along each hypersurface $S_{st}(ij)$, we extract from $T(x_2;\ ...;\ x_{n+1})$ an abelian subvariety, denoted by A_{ij}, of dimension $\varphi(N)/2$ with CM by a subfield of $\mathbb{Q}(\zeta_N)$. Thus, at the point P_{st}, the decomposition of $T(x_2;\ ...;\ x_{n+1})$ is, up to isogeny, the following:

$$T(x_2;\ ...;\ x_{n+1}) \triangleq A_{i_1,\, j_1} \times \cdots \times A_{i_n,\, j_n} \times A.$$

We can deduce that A has CM with dimension equal to $\varphi(N)/2$, and we can moreover deduce its CM type.

4. Construction of the Exceptional Set

We now construct the appropriate generalization of the exceptional set, considered in [27] in the 1-variable case and in [14] in the 2-variable case. We will fix an abelian variety T_0 and then determine conditions that imply that an abelian variety $T(x_2;\ ...;\ x_{n+1})$ is of the same isogeny class. The abelian variety T_0 used here is the one corresponding to the stable point $P_{st}(0;\ ...;\ 0)$. To ensure stability we make the hypothesis $\mu_1 + \mu_{n+2} > 1$. This turns out not to be a strong restriction as can be seen by the discussion of the counterexamples of the next paragraph, where other hypotheses are used. Corollary in Section 3 proves that the point $P_{st}(0;\ ...;\ 0)$ corresponds to an abelian variety with complex multiplication having, up to isogeny, the following decomposition:

$$A'_{1,\, n+2} \times \prod_{k=2}^{n+1} A_{0,\, k} =: T_0.$$

Here $A'_{1,n+2}$ (resp. $A_{0,k}$) denotes the abelian variety of dimension $\varphi(N)/2$, with complex multiplication (by a subfield of $\mathbb{Q}(\zeta_N)$), extracted from the Jacobian of the Fermat curve and characterized by the period of the first kind $B(1-\mu_1; 1-\mu_{n+2}) = B(c-a; a)$ $\left(\text{resp. } B(\mu_0; \mu_k) = B\left(c - \sum_{i=2}^{n+1} b_i; b_k\right)\right)$, for more details see [17]. The following theorem relates the arithmetic and geometric aspects of the problem. In order to use the analytic subgroup theorem, which is the Haupsatz in [30], we must avoid the loci of zeros of the hypergeometric functions that intervene in our arguments. Let Z be the zero set of the hypergeometric functions appearing in the statement of the hypotheses of Theorem 1.

Theorem 1. *For* $(x_2; \ldots; x_{n+1}) \in \mathcal{Q} \cap \overline{\mathbb{Q}}^n$, *and not in* Z, *the abelian variety* $T(x_2; \ldots; x_{n+1})$ *is isogenous to* T_0 *if and only if the following hypotheses are true*:

Hypothesis (h):

$$F(a; b_2; \ldots; b_{n+1}; c; x_2; \ldots; x_{n+1}) \in \overline{\mathbb{Q}}^*,$$

and for all $l \in \{0; \ldots; n-2\}$, *there exists a* $k \in \{2; \ldots; n+1\}$, *such that*:

Hypothesis $(h_l^{(k)})$:

$$F\left(b_k; 1-b_2; \ldots; c-a; \ldots; 1-b_{n+1}; \right.$$
$$\left. c + l - \sum_{i=2,\, i\neq k}^{n+1} b_i; \frac{x_k}{x_k - x_2}; \cdots; \frac{x_k}{x_k - 1}; \cdots; \frac{x_k}{x_k - x_{n+1}}\right) \in \overline{\mathbb{Q}}^*.$$

Here the k*-th parameter is given by* $c-a$ *and the* $(k-1)$*-th variable is given by* $\frac{x_k}{x_k - 1}$.

This theorem is a generalization of Theorem 2.3 in [14] (the 2-dimensional case). To prove this theorem, one needs some other results, including the following proposition that is proved later:

Proposition 1. *For all* $(x_2; \ldots; x_{n+1}) \in \overline{\mathbb{Q}}^n \cap \mathcal{Q}$, *and not in* Z, *the hypothesis* (h) *of Theorem* 1:

$$F(a; b_2; \ldots; b_{n+1}; c; x_2; \ldots; x_{n+1}) \in \overline{\mathbb{Q}}^*$$

implies that $T(x_2; \ldots; x_{n+1})$ *admits the subvariety* $A'_{1, n+2}$ *in its decomposition up to isogeny. Moreover, if* l *is an integer between* 0 *and* $n-1$, *for all* $k \in \{2; \ldots; n+1\}$, *the hypothesis* $(h_l^{(k)})$ *in Theorem* 1:

$$F\left(b_k; 1-b_2; \ldots; c-a; \ldots; 1-b_{n+1}; \right.$$

$$\left. c + l - \sum_{i=2,\, i \neq k}^{n+1} b_i; \frac{x_k}{x_k - x_2}; \ldots; \frac{x_k}{x_k - 1}; \ldots; \frac{x_k}{x_k - x_{n+1}} \right) \in \overline{\mathbb{Q}}^*,$$

where the k-th parameter is given by $c - a$ *and the* $(k-1)$*-th variable is given by* $\frac{x_k}{x_k - 1}$, *implies that* $T(x_2; \ldots; x_{n+1})$ *admits the subvariety* $A_{0, k}$ *in its decomposition up to isogeny.*

We have the following result.

Lemma 1. *In a neighborhood of the point* $x_2 = \cdots = x_{n+1} = 0$ *there exist* $(n+1)$ *solutions of* $E_n(a; b_2; \ldots; b_{n+1}; c)$ *given by integrals of Euler type and with power series developments in* $c \cdot \overline{\mathbb{Q}[[x_2; \ldots; x_{n+1}]]}$.

For the first solution

$$c = B(1 - \mu_1; 1 - \mu_{n+2})$$

and for the n other solutions

$$c = B(1 - \mu_0; 1 - \mu_k), \quad k = 2; \ldots; n+1.$$

As opposed to the one variable case (where the dimension of $T(x)$ is $\varphi(N)$), a problem arises here due to the dimension of $T(x_2; \ldots; x_{n+1})$, which equals to $(n+1)\varphi(N)/2$. Namely, to deduce that $T(x_2; \ldots; x_{n+1})$ is of CM type, is not enough to assume that the value

$$F(a;\, b_2;\, \ldots;\, b_{n+1};\, c;\, x_2;\, \ldots;\, x_{n+1})$$

is algebraic, for a fixed $(x_2; \ldots; x_{n+1}) \in \overline{\mathbb{Q}}^n \cap \mathcal{Q}$ with algebraic coordinates, which is the natural condition generalizing Wolfart's condition. Even though this assumption enables us to extract a CM subvariety in the decomposition of $T(x_2; \ldots; x_{n+1})$, we do not have enough information on the remaining factor, which is of dimension $\eta\varphi(N)/2$. Recall that we want a condition ensuring that $T(x_2; \ldots; x_{n+1})$ is of CM type and in a given isogeny class.

In a neighborhood of $(x_2; \ldots; x_{n+1}) = (0; \ldots; 0)$, the $n+1$ solutions considered in Lemma 1 are the following:

$$\begin{aligned}\varphi_1 &:= \int_1^{\infty} \omega(x_2; \ldots; x_{n+1})\\ &= B(1-\mu_1; 1-\mu_{n+2})F(a; b_2; \ldots; b_{n+1}; c; x_2; \ldots; x_{n+1}).\end{aligned}$$

For $k = 2; \ldots; n+1$,

$$\begin{aligned}\varphi_k &:= \int_0^{x_k} \omega(x_2; \ldots; x_{n+1})\\ &= B(1-\mu_0; 1-\mu_k) \cdot x_k^{1-\mu_0-\mu_k}(x_k-1)^{-\mu_1} \prod_{i=2,\, i\neq k}^{n+1} (x_k - x_i)^{-\mu_i}\\ &\quad \times F\Bigg(1-b_k; b_2; \ldots; a+1-c; \ldots; b_{n+1}; 2 - \sum_{i=2,\, i\neq k}^{n+1} b_i - c;\\ &\qquad \frac{x_k}{x_k - x_2}; \ldots; \frac{x_k}{x_k-1}; \ldots; \frac{x_k}{x_k - x_{n+1}}\Bigg),\end{aligned}$$

where the k-th parameter is given by $a+1-c$ and the $(k-1)$-th variable is given by $\dfrac{x_k}{x_k-1}$.

The hypothesis $c < 1$, that is, $\mu_1 + \mu_{n+2} > 1$, implies that the period $B(1-\mu_1; 1-\mu_{n+2})$ (appearing in the first solution φ_1) is of the first kind; while the periods $B(1 - \mu_0; 1 - \mu_k)$ (appearing in the other solutions φ_k) are of the second kind, since $\sum_{i=0}^{n+2} \mu_i = 2$. Therefore we cannot directly use algebraic values of the Lauricella functions appearing in the other solutions in the definition of a useful exceptional set.

The functions appearing in the statement of our Theorem 1, have their origin in the following result due to P. B. Cohen, H. Shiga and J. Wolfart:

Recall that $\dim V_s + \dim V_{-s} = 1 + n$. Let $\gamma_0, \ldots, \gamma_n$ be generators of $H_1(T(x_2; \ldots; x_{n+1}); \mathbb{Z})$.

Lemma 2 [24, Corollary 6]**.** *For all*

$$(x_2; \ldots; x_{n+1}) \in \overline{\mathbb{Q}}^n \cap \mathcal{Q},$$

the abelian variety $T(x_2; \ldots; x_{n+1})$ *has CM if and only if there exists a basis* $\omega^{(2)}; \ldots; \omega^{(n+1)}$ *over* $\overline{\mathbb{Q}}$ *of the space of differential forms in* V_{-1} *such that, for all* $j = 1; \ldots; n+1$, *the periods*

$$\int_{\gamma_0} \omega^{(j)}, \int_{\gamma_1} \omega^{(j)}, \ldots, \int_{\gamma_n} \omega^{(j)}$$

generate a $\overline{\mathbb{Q}}$*-vector space of dimension* 1*. Here, we suppose* $\omega^{(1)} = \omega$.

The differential forms used correspond to the eigenvalue ζ_N^{-1}. They have been computed explicitly in Section 6 of [5]. We recall this construction in the next lemma.

Lemma 3 (see [5, Section 6])**.** *Let*

$$\omega_{N-1}(x_2; \ldots; x_{n+1}) := u^{\mu_0 - 1}(u-1)^{\mu_1 - 1} \prod_{i=2}^{n+1} (u - x_i)^{\mu_i - 1}.$$

Then

$$\{\omega_{N-1}(x_2; \ldots; x_{n+1}); u \cdot \omega_{N-1}(x_2; \ldots; x_{n+1}); \ldots; u^{n-1} \cdot \omega_{N-1}(x_2; \ldots; x_{n+1})\}$$

is a basis of V_{-1}.

The calculations justifying Lemma 3 are a direct application of Section 6 in [5]: we now summarize them. Denote $\alpha \wedge \beta$ the greatest common divisor of α and β. In order to use regularity conditions in [5] for differential forms of the algebraic curve $\mathcal{X}_N(x_2; \ldots; x_{n+1})$, we let

$$\omega_{-1}(x_2; \ldots; x_{n+1}) = \omega_{N-1}(x_2; \ldots; x_{n+1})$$

$$= \frac{u^{a_0}(u-1)^{a_1}\prod_{i=2}^{n+1}(u-x_i)^{a_i}}{w^{N-1}}\, du \text{ with } a_i \in \mathbb{Z}.$$

We study necessary conditions on the a_i to have differential forms of the first kind on the algebraic curve $\mathcal{X}_N(x_2; \ldots; x_{n+1})$. Denote $A = N\mu_0$; $B = N\mu_1$, $C_i = N\mu_i$ for $i = 2; \ldots; n+1$, $D = N\mu_{n+2}$. At the points $(u; w) \in \{(0; 0); (1; 0); (x_i; 0); (\infty; \infty)\}$, $i = 2; \ldots; n+1$, regularity conditions of the differential form $\omega_{N-1}(x_2; \ldots; x_{n+1})$ on $\mathcal{X}_N(x_2; \ldots; x_{n+1})$ are given in Section 6 of [5] by

(1) $a_0 \geq \dfrac{(N-1)A + N \wedge A}{N} - 1$

(2) $a_1 \geq \dfrac{(N-1)B + N \wedge B}{N} - 1$

(3_i) $a_i \geq \dfrac{(N-1)C_i + N \wedge C_i}{N} - 1$, for $i = 2; \ldots; n+1$

(4) $a_0 + \cdots + a_{n+1}$

$$\leq \frac{(N-1)\left(A + B + \sum_{i=2}^{n+1} C_i + D\right) - N \wedge \left(A + B + \sum_{i=2}^{n+1} C_i + D - N\right)}{N} - 1.$$

Consider the differential form obtained by using the minimal values $(a_i)_{\min}$ for the a_i, $i = 0; \ldots; n+2$. For instance, in condition (1) we have

$$\frac{(N-1)A + N \wedge A}{N} - 1 = (A-1) + \frac{N \wedge A - A}{N}.$$

Thus, when $A \mid N$, we have $N \wedge A - A = 0$ and $(a_0)_{\min} = A - 1$. When $A \nmid N$, we have $N \wedge A - A < 0$ and

$$(a_0)_{\min} = E\left(\frac{(N-1)A + N \wedge A}{N} - 1\right) + 1 = (A-1) + E\left(\frac{N \wedge A - A}{N}\right) + 1.$$

Moreover, since $-1 < \frac{N \wedge A - A}{N} < 0,$ we have $E\left(\frac{N \wedge A - A}{N}\right) = -1$ and $(a_0)_{\min} = A - 1.$ Reasoning in a similar way with conditions (2) and (3_i), we obtain the following differential form constructed using the $(a_i)_{\min}$:

$$\omega_{N-1}(x_2; \dots; x_{n+1}) = \frac{u^{A-1} \cdot (u-1)^{B-1} \cdot \prod_{i=2}^{n+1} (u - x_i)^{C_i - 1}}{w^{N-1}} \cdot du$$

$$= u^{\mu_0 - 1} \cdot (u-1)^{\mu_1 - 1} \cdot \prod_{i=2}^{n+1} (u - x_i)^{\mu_i - 1} \cdot du.$$

It remains to verify the regularity condition (4). This condition is equivalent to

$$A + B + \sum_{i=2}^{n+1} C_i + D - 4$$

$$\leq A + B + \sum_{i=2}^{n+1} C_i + D - 1$$

$$- \frac{A + B + \sum_{i=2}^{n+1} C_i + D + N \wedge \left(A + B + \sum_{i=2}^{n+1} C_i + D - N\right)}{N},$$

which is in turn equivalent to

$$\frac{A + B + \sum_{i=2}^{n+1} C_i + D + N \wedge \left(A + B + \sum_{i=2}^{n+1} C_i + D - N\right)}{N} \leq 3.$$

Note that

$$\frac{A + B + \sum_{i=2}^{n+1} C_i + D + N \wedge \left(A + B + \sum_{i=2}^{n+1} C_i + D - N\right)}{N}$$

$$\leq \frac{2\left(A + B + \sum_{i=2}^{n+1} C_i + D\right) - N}{N}.$$

The hypothesis $\mu_{n+2} > 0$ is equivalent to $3 - 2\mu_{n+2} \leq 3$. This is in turn equivalent to $2(\sum_{i=0}^{n+1}\mu_i) - 1 \leq 3$, that is, $\frac{2(A + B + \sum_{i=2}^{n+1} C_i + D) - N}{N} < 3$. Thus, the hypothesis $\mu_{n+2} > 0$ implies condition (4). All the regularity conditions are true for the differential form $\omega_{N-1}(x_2; \ldots; x_{n+1})$ so that it is of the first kind on the algebraic curve $\mathcal{X}(x_2; \ldots; x_{n+1})$. Clearly,

$$\omega_{N-1}(x_2; \ldots; x_{n+1}) = \frac{u^{A-1} \cdot (u-1)^{B-1} \cdot \prod_{i=2}^{n+1}(u - x_i)^{C_i - 1}}{w^{N-1}} \cdot du$$

is an eigen-differential form for the eigenvalue $\zeta_N^{N-1} = \zeta_N^{-1}$. Therefore, it is an element of V_{-1}. The same method of proof can be applied to the $u^l \cdot \omega_{N-1}(x_2; \ldots; x_{n+1})$, $l = 1; \ldots; n-1$. Moreover, the differential forms $\omega_{\min} = \omega_{N-1}(x_2; \ldots; x_{n+1})$ we have constructed using the $(a_i)_{\min}$ form a basis of V_{-1}. For more details on all of the above arguments for Lemma 3, see [5].

Let α and β be two non-zero complex numbers. From now on, we write $\alpha \sim \beta$ when there exists a non-zero algebraic number δ such that $\alpha = \delta\beta$. For two abelian varieties, we write $A \triangleq B$ when A is isogenous to B. In order to prove Proposition 1 stated above, we need the following consequence of the analytic subgroup theorem of Wüstholz [30] (see also [11] and [24]):

Lemma 4. *Let A and B be abelian varieties defined over $\overline{\mathbb{Q}}$. Denote by V_A the $\overline{\mathbb{Q}}$-vector space generated by the numbers,*

$$\left\{\int_\gamma \omega : \omega \in H^0(A; \Omega_{\overline{\mathbb{Q}}}); \gamma \in H_1(A; \mathbb{Z})\right\}$$

and the same for B. Then $V_A \cap V_B \neq \{0\}$ if and only if there exist a non-trivial simple subvariety A' of A, and a non-trivial simple subvariety B' of B such that $A' \triangleq B'$.

When $(x_2; \ldots; x_{n+1}) \in \overline{\mathbb{Q}}^n \cap \mathcal{Q}$, not in Z, we have $F(a; b_2; \ldots; b_{n+1}; c; x_2; \ldots; x_{n+1}) \in \overline{\mathbb{Q}}^*$ if and only if we have the relation between non-zero numbers,

$$\int_1^\infty \omega(x_2; \ldots; x_{n+1}) \sim B(1 - \mu_1; 1 - \mu_{n+2}).$$

Therefore,

$$V_{T(x_2; \ldots; x_{n+1})} \cap V_{A'_{1; n+2}} \supseteq \{B(1 - \mu_1; 1 - \mu_{n+2})\} \neq \{0\}.$$

Using Lemma 4, the abelian varieties $T(x_2; \ldots; x_{n+1})$ and $A'_{1; n+2}$ admit, up to isogeny, a common subvariety, noted E, and related to the period $B(1 - \mu_1; 1 - \mu_{n+2})$. The abelian variety $A'_{1; n+2}$ is stable under the cyclotomic field $\mathbb{Q}(\zeta_N)$. Using an argument of Bertrand (see [6]), there can therefore be two possibilities:

- either $A'_{1; n+2}$ is simple and then $E \triangleq A'_{1; n+2}$,

- or $A'_{1; n+2} \triangleq F^\varepsilon$, where $\varepsilon \in \mathbb{N}$, $\varepsilon > 1$, and F is a subvariety of dimension $\varphi(N)/2\varepsilon$, with CM by a subfield k of $\mathbb{Q}(\zeta_N)$ such that $[\mathbb{Q}(\zeta_N) : k] = \varepsilon$. Thus $T(x_2; \ldots; x_{n+1}) \triangleq F^\lambda \times G$. But the same argument implies that: either $T(x_2; \ldots; x_{n+1})$ admits no proper subvariety stable under $\mathbb{Q}(\zeta_N)$ and then $T(x_2; \ldots; x_{n+1}) \triangleq F^{(n+1)\varepsilon}$; or $T(x_2; \ldots; x_{n+1})$ can be decomposed by proper subvarieties stable under $\mathbb{Q}(\zeta_N)$ and then $\lambda = 2p\varepsilon$, $p \in \mathbb{N}^*$.

In both cases, the variety $T(x_2; \ldots; x_{n+1})$ contains the factor $A'_{1; n+2}$ in its decomposition up to isogeny. This proves the first part of Proposition 1. Let $l = 0; \ldots; n-1$ and $k \in \{2; \ldots; n+1\}$. We use the same proof, applied to the following formula, for the second part of Proposition 1:

$$\int_0^{x_k} u^l \omega_{N-1}(x_2; \ldots; x_{n+1})$$

$$= x_k^{\mu_0+\mu_k+l-1}(x_k-1)^{\mu_1-1}\prod_{i=2, i\neq k}^{n+1}(x_k-x_i)^{\mu_i-1}B(\mu_k;\,\mu_0+l)$$

$$\times F\Bigg(b_k;\,1-b_2;\,\ldots;\,c-a;\,\ldots;\,1-b_{n+1};$$

$$c+l-\sum_{i=2, i\neq k}^{n+1}b_i;\,\frac{x_k}{x_k-x_2};\,\ldots;\,\frac{x_k}{x_k-1};\,\ldots;\,\frac{x_k}{x_k-x_{n+1}}\Bigg)$$

$$\sim B(\mu_k;\,\mu_0)F\Bigg(b_k;\,1-b_2;\,\ldots;\,c-a;\,\ldots;\,1-b_{n+1};\,c+l-\sum_{i=2, i\neq k}^{n+1}b_i;$$

$$\frac{x_k}{x_k-x_2};\,\ldots;\,\frac{x_k}{x_k-1};\,\ldots;\,\frac{x_k}{x_k-x_{n+1}}\Bigg),$$

where, in the Lauricella functions, the k-th parameter is given by $c-a$ and the $(k-1)$-th variable is given by $\frac{x_k}{x_k-1}$. It follows that for $(x_2;\,\ldots;\,x_{n+1})\in\overline{\mathbb{Q}}^n\cap\mathcal{Q}$, not in Z, the condition

$$F\Bigg(b_k;\,1-b_2;\,\ldots;\,c-a;\,\ldots;\,1-b_{n+1};$$

$$c+l-\sum_{i=2, i\neq k}^{n+1}b_i;\,\frac{x_k}{x_k-x_2};\,\ldots;\,\frac{x_k}{x_k-1};\,\ldots;\,\frac{x_k}{x_k-x_{n+1}}\Bigg)\in\overline{\mathbb{Q}}^*,$$

with notation as in the statement of Theorem 1, is equivalent to

$$\int_0^{x_k}u^l\omega_{N-1}(x_2;\,\ldots;\,x_{n+1})\sim B(\mu_0;\,\mu_k),$$

and the second part of Proposition 1 follows.

Proposition 1 explains hypotheses of Theorem 1. Indeed, when the n conditions are true, this implies that $T(x_2;\,\ldots;\,x_{n+1})\triangleq A'_{1;\,n+2}\times A_{0;\,2}\times\cdots\times A_{0;\,n+1}$ which is exactly the abelian variety T_0. Conversely, assume that $(x_2;\,\ldots;\,x_{n+1})\in\overline{\mathbb{Q}}^n\cap\mathcal{Q}$, not in Z, and that,

$$T(x_2;\ ...;\ x_{n+1}) \triangleq A'_{1;\,n+2} \times A_{0;\,2} \times \cdots \times A_{0;\,n+1} =: T_0,$$

that is,

$$T(x_2;\ ...;\ x_{n+1}) \triangleq B \times C_2 \times \cdots \times C_{n+1},$$

where $B \triangleq A'_{1;\,n+2}$ and $C_k \triangleq A_{0;\,k}$, $k \in \{2;\ ...;\ n+1\}$. This is equivalent to

$$\begin{cases} \mathrm{End}_0(B) \simeq \mathrm{End}_0(A'_{1;\,n+2}) \\ \text{For all } k \in \{2;\ ...;\ n+1\},\ \mathrm{End}_0(C_k) \simeq \mathrm{End}_0(A_{0;\,k}). \end{cases}$$

Let Λ_F be the lattice of periods of the abelian variety F. This is equivalent to

$$\begin{cases} \Lambda_B \otimes_{\mathbb{Z}} \mathbb{Q} = \Lambda_{A'_{1;\,n+2}} \otimes_{\mathbb{Z}} \mathbb{Q} \\ \text{For all } k \in \{2;\ ...;\ n+1\},\ \Lambda_{C_k} \otimes_{\mathbb{Z}} \mathbb{Q} = \Lambda_{A_{0;\,k}} \otimes_{\mathbb{Z}} \mathbb{Q}. \end{cases}$$

Since the lattice of periods of an abelian variety is stable under its type Φ, this is in turn equivalent to

$$(*) \begin{cases} \Phi(\Lambda_B \otimes_{\mathbb{Z}} \mathbb{Q}) = \Lambda_{A'_{1;\,n+2}} \otimes_{\mathbb{Z}} \mathbb{Q} \\ \text{For all } k \in \{2;\ ...;\ n+1\},\ \Phi(\Lambda_{C_k} \otimes_{\mathbb{Z}} \mathbb{Q}) = \Lambda_{A_{0;\,k}} \otimes_{\mathbb{Z}} \mathbb{Q}. \end{cases}$$

The type Φ of the abelian variety $T(x_2;\ ...;\ x_{n+1})$ can be decomposed using the action of the cyclotomic field $\mathbb{Q}(\zeta_N)$ on the vector space $H^0(T(x_2;\ ...;\ x_{n+1});\ \Omega)$ of differential forms of the first kind as noted before. Thus, for all $a \in \mathbb{Q}(\zeta_N) \subseteq \mathrm{End}_0(T(x;\ y))$,

$$\Phi(a) = \begin{pmatrix} \Phi_B(a) & & & 0 \\ & \Phi_{C_2}(a) & & \\ & & \ddots & \\ 0 & & & \Phi_{C_{n+1}}(a) \end{pmatrix},$$

where

$$\Phi_B(a) = \begin{pmatrix} \sigma_{i_1^{(1)}}(a) & & & & & & 0 \\ & \ddots & & & & & \\ & & \sigma_{i_{M_1}^{(1)}}(a) & & & & \\ & & & \ddots & & & \\ & & & & \sigma_{i_1^{\left(E\left(\frac{n}{2}\right)\right)}}(a) & & \\ & & & & & \ddots & \\ 0 & & & & & & \sigma_{i_{M_{\left(E\left(\frac{n}{2}\right)\right)}}^{\left(E\left(\frac{n}{2}\right)\right)}}(a) \end{pmatrix}$$

and where, for $k \in \{2; \dots; n+1\}$,

$$\Phi_{C_k}(a) = \begin{pmatrix} \sigma_{-i_1^{(1)}}(a) & & & 0 \\ & \ddots & & \\ & & \sigma_{-i_{M_1}^{(1)}}(a) & \\ 0 & & & \ddots \end{pmatrix}.$$

We do not make explicit the coefficients on the diagonal of $\Phi_{C_k}(a)$ because we do not need them and the corresponding notation is cumbersome.

We have $\omega(0; \dots; 0) = u^{-c}(u-1)^{c-a-1}du$. Changing the Pochhammer cycles to line integrals, one shows that

$$\int_{\gamma_0} \omega(0; \dots; 0) \sim \int_1^{\infty} \omega(0; \dots; 0) = B(1-\mu_1; 1-\mu_{n+2}).$$

Moreover, for $l = 0; \dots; n-1$, we have

$$u^l \omega_{N-1}(0; \dots; 0) = u^{l+c-n-1}(u-1)^{a-c}du,$$

and therefore, for $k \in \{1; \dots; n\}$, we have

$$\int_{\gamma_k} u^l \omega_{N-1}(0; \dots; 0) \sim \int_0^{x_{k+1}} u^l \omega_{N-1}(0; \dots; 0) \sim B(\mu_0; \mu_k)$$

because $B(p; q) \sim B(p; q+1)$, for $p, q \notin -\mathbb{N}$.

We see that the condition $(*)$ is equivalent to

$$\begin{cases} \forall i \in \{0; \ldots; n\}, \exists a \in \mathbb{Q}(\zeta_N), \Phi_B(a) \cdot \int_{\gamma_i} \eta_B(x_2; \ldots; x_{n+1}) \sim \int_{\gamma_i} \eta_B(0; \ldots; 0), \\ \forall k \in \{2; \ldots; n+1\}, \\ \forall j \in \{0; \ldots; n\}, \exists a \in \mathbb{Q}(\zeta_N), \Phi_{C_k}(a) \cdot \int_{\gamma_j} \eta_{C_k}^{(l)}(x_2; \ldots; x_{n+1}) \sim \int_{\gamma_j} \eta_{C_k}^{(l)}(0; \ldots; 0), \end{cases}$$

where $\eta_B(x_2; \ldots; x_{n+1}) =^t (\omega_{i_1}^{(1)}; \ldots; \omega_{M_1}^{(1)}; \ldots; \omega_{i_1}^{\left(E\left(\frac{n}{2}\right)\right)}; \ldots; \omega_{i_{\left(E\left(\frac{n}{2}\right)\right)}}^{\left(E\left(\frac{n}{2}\right)\right)})$.

Moreover, for $k \in \{2; \ldots; n+1\}$, we have

$$\eta_{C_k}^{(l)}(x_2; \ldots; x_{n+1}) =^t (u^l \omega_{-i_1^{(1)}}; \ldots; u^l \omega_{-i_{M_1}^{(1)}}; \ldots).$$

As $i_1^{(1)} = 1$, this implies for $i = 0$ that

$$\int_{\gamma_0} \omega(x_2; \ldots; x_{n+1}) \sim \int_{\gamma_0} \omega(0; \ldots; 0),$$

that is,

$$\int_1^{\infty} \omega(x_2; \ldots; x_{n+1}) \sim B(1 - \mu_1; 1 - \mu_{n+2})$$

which is the hypothesis (h). For $j = k - 1$,

$$\int_{\gamma_{k-1}} \omega_{N-1}(x_2; \ldots; x_{n+1}) \sim \int_{\gamma_{k-1}} \omega_{N-1}(0; \ldots; 0),$$

that is,

$$\int_0^{x_k} \omega_{N-1}(x_2; \ldots; x_{n+1}) \sim B(\mu_0; \mu_k),$$

which is the hypothesis $(h_l^{(k)})$. This ends the proof of Theorem 1.

Remark. We can change the hypothesis (h) and n integers l for which the condition $(h_l^{(k)})$ is true, by (h) and $n-1$ integers l for which the condition $(h_l^{(k)})$ is true; because the remaining subvariety F in the decomposition, up to isogeny, $T(x_2; \ldots; x_{n+1}) \triangleq F \times A$, is determined by comparison of dimension, type and stability under $\mathbb{Q}(\zeta_N)$ of these varieties.

In the light of Theorem 1, we make the following definition.

Definition. The exceptional set related to the Lauricella hypergeometric function is defined as

$$\mathcal{E}_n = \Bigg\{(x_2; \ldots; x_{n+1}) \in \mathcal{Q} \cap \overline{\mathbb{Q}}^n :$$

$$\left.\begin{cases} (h) : F(a; b_2; \ldots; b_{n+1}; c; x_2; \ldots; x_{n+1}) \in \overline{\mathbb{Q}} \\ \forall l \in \{0; \ldots; n-2\}, \exists k \in \{2; \ldots; n+1\}, (h_l^{(k)}) \text{ is true} \end{cases}\right\},$$

where $(h_l^{(k)})$ is the hypothesis

$$F\Bigg(b_k; 1-b_2; \ldots; c-a; \ldots; 1-b_{n+1};$$

$$c + l - \sum_{i=2,\, i\neq k}^{n+1} b_i; \frac{x_k}{x_k - x_2}; \ldots; \frac{x_k}{x_k - 1}; \ldots; \frac{x_k}{x_k - x_{n+1}}\Bigg) \in \overline{\mathbb{Q}}$$

with the k-th parameter given by $c - a$ and the $(k-1)$-th variable given by $\frac{x_k}{x_k - 1}$. By Theorem 1, it is a Zariski-dense subset of

$$\overline{\mathcal{E}_n} = \{(x_2; \ldots; x_{n+1}) \in \mathcal{Q} \cap \overline{\mathbb{Q}}^n : T(x_2; \ldots; x_{n+1}) \triangleq T_0\}.$$

The above definition generalizes the definition of the exceptional set for the Gauss hypergeometric function by Wolfart in [27] and for the Appell hypergeometric function by the author in [14].

5. Application of a Weak Version of the André-Oort Conjecture

This paragraph deals with the Zariski density of complex multiplication points. The exceptional set creates a link between complex multiplication points and the assuming of algebraic values by certain Lauricella hypergeometric functions at algebraic points. In [8], Paula B. Cohen shows how the following weak version of the André-Oort conjecture can be used for several transcendence results, see also [11]. The present paper gives an application of this conjecture for transcendence results on the Lauricella hypergeometric functions (see [14] for the case of Appell hypergeometric functions).

Weak version of the André-Oort conjecture [8]**.** Let Z be an algebraic irreducible subvariety of $V(\mathbb{C})$. If there exists a Zariski dense subset of points of Z, whose corresponding abelian varieties are of complex multiplication type and are in the same isogeny class, then Z is of Hodge type.

Remarks.

- The converse of this conjecture is known.

- In the case of dimension 1, this conjecture has been proven by Edixhoven and Yafaev, see [15].

- Recall that a subvariety of a Shimura variety is a union of varieties of Hodge type when it is a Shimura subvariety or the image, under a Hecke correspondence of a Shimura subvariety, see [8].

The following results show how this geometric conjecture can be used for transcendence results.

Corollary. *Assume the conditions on the* μ_i *given in Section* 2 *and the weak André-Oort conjecture. Let* $Z(\mathbb{C})$ *be the Zariski closure in* $V(\mathbb{C})$ *of the image* $\Phi(\mathcal{Q})$ *of the map of the Theorem in Section* 3. *Then* $Z(\mathbb{C})$ *is of Hodge type if and only if the image* $\Phi(\mathcal{E}_n)$ *of the exceptional set is Zariski dense in* $Z(\mathbb{C})$.

This corollary relies on the geometric description of the exceptional set as a set of abelian varieties in the same isogeny class as an abelian variety T_0 with complex multiplication.

Theorem 2. *Assume the conditions on the* μ_i *given in Section* 2 *and the weak André-Oort conjecture. Assume further that the monodromy group* $\Delta(\mu)$ *acts discontinuously on* B_n. *Then, the image of* $\mathcal{E}_n$ *is Zariski dense in* $Z(\mathbb{C})$ *if and only if* $\Delta(\mu)$ *is arithmetic.*

Indeed, when the monodromy group $\Delta(\mu)$ acts discontinuously on B_n, one knows, using [12, Proposition 12.7], that $\Delta(\mu)$ is arithmetic if and only if $M_1 = 1$. Then, as it is a Shimura variety, $B_n/\Delta(\mu)$ is of Hodge type. Using the converse (and known) sense of the preceding conjecture, this variety therefore contains a Zariski dense subset of CM points corresponding to the isogeny class of any fixed CM abelian variety. Therefore the image of the exceptional set $\mathcal{E}_n$ is Zariski dense in $Z(\mathbb{C})$.

Conversely, let us assume that the image of the exceptional set $\mathcal{E}_n$ is Zariski dense in $Z(\mathbb{C})$. Then, using the conjecture, we deduce that $Z(\mathbb{C})$ is of Hodge type. As a modular group, $\Delta(\mu)$ preserves a lattice, and thus is an arithmetic group.

Recalling the remarks at the beginning in Section 3, this theorem deals with a special kind (and a finite list for $n \geq 2$) of monodromy groups. Nonetheless, this leads to a list of counterexamples of a conjecture of Coleman. This conjecture predicts the finiteness of the number of isomorphism classes of algebraic curves, with genus greater than or equal to 4, for which Jacobian has CM.

Counterexamples 1 and 2. By a result of [16], which was revisited in [11] with techniques in the spirit of the present paper, there are infinitely many $x \in \overline{\mathbb{Q}}$ for which the following algebraic curves, with genus 4, correspond to a Jacobian with CM by the cyclotomic field $\mathbb{Q}(\zeta_5)$, or a subfield of $\mathbb{Q}(\zeta_5)$,

$$\mathcal{Y}_5(x) : v^5 = u(u-1)(u-x).$$

Moreover, there are infinitely many $x \in \overline{\mathbb{Q}}$ for which the following algebraic curves, with genus 6, correspond to a Jacobian with CM by the cyclotomic field $\mathbb{Q}(\zeta_7)$ or a subfield of $\mathbb{Q}(\zeta_7)$,

$$\mathcal{Y}_7(x) : v^7 = u(u-1)(u-x).$$

Counterexample 3 [14]**.** There are infinitely many $(x;\, y) \in \overline{\mathbb{Q}}^2$ for which the following algebraic curves, with genus 4, correspond to a Jacobian with CM by the cyclotomic field $\mathbb{Q}(\zeta_5)$ or a subfield of $\mathbb{Q}(\zeta_5)$,

$$\mathcal{Y}_5(x) : v^5 = u(u-1)(u-x)(u-y).$$

Counterexample 4. This example is new, but uses the same type of reasoning as in [14]. There are infinitely many $(x;\ y;\ z) \in \overline{\mathbb{Q}}^3$ for which the following algebraic curves, with genus 6, correspond to a Jacobian with CM by the cyclotomic field $\mathbb{Q}(\zeta_3)$ or a subfield of $\mathbb{Q}(\zeta_3)$,

$$\mathcal{Y}_3(x) : v^3 = u(u-1)(u-x)(u-y)(u-z).$$

In cases of 3 and 4, the main part of the proof is based on the method of [11]. Nonetheless, the proof has to be modified because, unlike in the dimension 1 case, the construction of the exceptional set of Section 4 cannot be used directly. As counterexample 3 was treated in [14], we focus on the counterexample 4:

Consider the family of algebraic curves $\mathcal{Y}_3(x;\ y;\ z)$, parameterized by $(x;\ y;\ z) \in \mathcal{Q}$

$$\mathcal{Y}_3(x;\ y;\ z) : v^3 = u(u-1)(u-x)(u-y)(u-z).$$

Each of them is birationally isomorphic to the algebraic curve

$$\mathcal{X}_3(x;\ y;\ z) : w^3 = [u(u-1)(u-x)(u-y)(u-z)]^2.$$

This isomorphism is given by $(u;\ v) \mapsto (u;\ w = v^2)$ with inverse $(u;\ w) \mapsto \left(u;\ v = \dfrac{u(u-1)(u-x)(u-y)(u-z)}{w^2}\right)$.

We then have to use the exceptional set, but in this case the condition $\mu_1 + \mu_{n+2} < 1$ is not true. This means that the point $P(0;\ 0;\ 0)$ is not a stable point and we cannot use the abelian variety $T_0 = T(0;\ 0;\ 0)$ to describe the exceptional set. We can adapt the previous construction as follows. Consider the stable point $P_{st}(1;\ 0;\ 0)$, intersection of the stable surfaces $S_{st}(12) : x = 1\ (\mu_1 + \mu_2 < 1)$, $S_{st}(03) : y = 0\ (\mu_0 + \mu_3 < 1)$ and $S_{st}(04) : z = 0\ (\mu_0 + \mu_4 < 1)$. Applying the Corollary, this point corresponds to an abelian variety with complex multiplication, described as follows:

$$T_0' := T(1;\ 0;\ 0) \triangleq A_{0,3} \times A_{0,4} \times A_{1,2} \times A',$$

where A' is an algebraic curve with complex multiplication by the field $\mathbb{Q}(\zeta_3)$ and characterised by the period $B\left(\frac{1}{3};\frac{1}{3}\right)$. Indeed, around the point $(1;\ 0;\ 0)$ there are 4 solutions: $\int_0^1 \omega$, $\int_1^x \omega$, $\int_0^y \omega$ and $\int_0^z \omega$ of the differential system $E_4\left(\frac{2}{3};\frac{1}{3};\frac{1}{3};\frac{1}{3};\frac{4}{3}\right)$ which can be written as series in $c \cdot \overline{\mathbb{Q}[[x-1;\ y;\ z]]}$. For the first solution

$$c = B(1-(\mu_0+\mu_3);\ 1-(\mu_0+\mu_4)) = B\left(\frac{1}{3};\frac{1}{3}\right),$$

for the second

$$c = B(1-\mu_1;\ 1-\mu_2) = B\left(\frac{2}{3};\frac{2}{3}\right),$$

and for the others

$$c = B(1-\mu_0;\ 1-\mu_k) = B\left(\frac{2}{3};\frac{2}{3}\right),\ k = 3 \text{ or } 4.$$

(See [10, Paragraph 6, Theorem 3] or [14, pp. 52-54] for more details.) Thus we can construct, in a similar way, an exceptional set related to the base point $P_{st}(1;\ 0;\ 0)$, for which the geometric description is

$$\mathcal{E}_3' = \{(x;\ y;\ z) \in \overline{\mathbb{Q}}^3 \cap \mathcal{Q}' : T(x;\ y;\ z) \triangleq T_0'\}.$$

This enables us to finish the proof: the monodromy group of the system of differential equations $E_4\left(\frac{2}{3};\frac{1}{3};\frac{1}{3};\frac{1}{3};\frac{4}{3}\right)$ is $\Delta\left(\frac{1}{3};\ldots;\frac{1}{3}\right)$. This is an arithmetic group, as we can check using Proposition 12.7 in [12]. We can therefore use the known part of Theorem 2, that is the known sense direction of the weak André-Oort Conjecture. The image of $\mathcal{E}_3'$ is Zariski dense in $Z(\mathbb{C})$, so *card* $\mathcal{E}_3' = \infty$. As $N = 3$ is a prime number, the Jacobian decomposes as follows (see the remark in Section 2):

$$\mathrm{Jac}\mathcal{X}_3(x;\ y;\ z) \triangleq T(x;\ y;\ z)$$

and the genus is $g = \dim \mathrm{Jac}\mathcal{X}_3(x;\ y;\ z) = \dim T(x;\ y;\ z) = \frac{4\varphi(3)}{2} = 4.$

There are infinitely many values $(x;\ y;\ z) \in \overline{\mathbb{Q}}^3$ for which $\mathrm{Jac}\mathcal{X}_3(x;\ y;\ z) \triangleq T_0'$, and therefore for which $\mathrm{Jac}\mathcal{X}_3(x;\ y;\ z)$ has complex multiplication by the cyclotomic field $\mathbb{Q}(\zeta_3)$ or by a subfield of $\mathbb{Q}(\zeta(3))$. This ends the proof.

6. A Transcendence Result

In the following results, each hypergeometric function is written as a non-zero quotient of periods of the first or of the second kind on the same abelian variety. Essential to our arguments is a result about linear independence over $\overline{\mathbb{Q}}$ of values of the Beta function. This result is due to Wolfart and Wüstholz (see [28, Satz 4]), and is a corollary of the analytic subgroup theorem (the Haupsatz of [30]).

Theorem 3. *For all n-tuples* $(x_2;\ ...;\ x_{n+1})$ *in the exceptional set* $\mathcal{E}_n$, *the following* $2n$ *numbers are zero or transcendental*:

$$F(n-l-a;\ 1-b_2;\ ...;\ 1-b_{n+1};\ n+1-l-c;\ x_2;\ ...;\ x_{n+1}),\quad l = 0,\ ...,\ n-1,$$

and

$$F\Bigg(1-b_k;\ b_2;\ ...;\ a+1-c;\ ...;\ b_{n+1};$$

$$2+\sum_{i=2,\, i\neq k}^{n+1} b_i - c;\ \frac{x_k}{x_k-x_2};\ ...;\ \frac{x_k}{x_k-1};\ ...;\ \frac{x_k}{x_k-x_{n+1}}\Bigg),\ k = 2,\ ...,\ n+1,$$

where the k-th parameter is given by $a+1-c$ *and the* $(k-1)$*-th variable is given by* $\dfrac{x_k}{x_k-1}$.

This is proven using the same tools as in the preceding parts of this paper, in particular we work with the differential forms

$$\omega(x_2;\ ...;\ x_{n+1});\ \omega_{N-1}(x_2;\ ...;\ x_{n+1});\ ...;\ u^{n-1}\omega_{N-1}(x_2;\ ...;\ x_{n+1})$$

which form a basis of $V_1 \cup V_{-1}$; and with their corresponding periods.

Denote

$$\delta_{CM}^{k} = \frac{B(\mu_1; \mu_{n+2})}{B(\mu_k; \mu_0)}, \quad k = 2; \ldots; n+1.$$

We can translate Lemma 2 into the following statement about Lauricella hypergeometric functions. Let Z' be the zero set of the functions appearing in Lemma 5.

Lemma 5. *For all* $(x_2; \ldots; x_{n+1}) \in \overline{\mathbb{Q}}^n \cap \mathcal{Q}$, *not in* Z', *the abelian variety* $T(x_2; \ldots; x_{n+1})$ *has CM if and only if for all* $k \in \{2; \ldots; n+1\}$, *we have*

$$F(a; b_2; \ldots; b_{n+1}; c; x_2; \ldots; x_{n+1}) \sim \delta_{CM}^{k}$$

$$\times F\left(1 - b_k; b_2; \ldots; a + 1 - c; \ldots; b_{n+1};\right.$$

$$\left. 2 + \sum_{i=2, i \neq k}^{n+1} b_i - c; \frac{x_k}{x_k - x_2}; \ldots; \frac{x_k}{x_k - 1}; \ldots; \frac{x_k}{x_k - x_{n+1}}\right),$$

where the k-th parameter is given by $a + 1 - c$ *and the* $(k-1)$*-th variable is given by* $\frac{x_k}{x_k - 1}$, *and we also have for all* $l \in \{0; \ldots; n-1\}$,

$$F(n - l - a; 1 - b_2; \ldots; 1 - b_{n+1}; n + 1 - l - c; x_2; \ldots; x_{n+1}) \sim \delta_{CM}^{k}$$

$$\times F\left(b_k; 1 - b_2; \ldots; c - a; \ldots; 1 - b_{n+1};\right.$$

$$\left. c + l - \sum_{i=2, i \neq k}^{n+1} b_i; \frac{x_k}{x_k - x_2}; \ldots; \frac{x_k}{x_k - 1}; \ldots; \frac{x_k}{x_k - x_{n+1}}\right),$$

where the k-th parameter is given by $c - a$ *and the* $(k-1)$*-th variable is given by* $\frac{x_k}{x_k - 1}$.

Changing integration around Pochhammer cycles to line integration, we write the periods as $\int_g^h \omega_s(x_2; \ldots; x_{n+1})$ and express these as hypergeometric Lauricella functions multiplied by a $B(p; q)$.

For instance, when $T(x_2; \ldots; x_{n+1})$ has CM $(x_2; \ldots; x_{n+1}) \in \overline{\mathbb{Q}}^n \cap \mathcal{Q}$, not in Z', we have

$$\int_{\gamma_0} \omega(x_2; \ldots; x_{n+1}) \sim \cdots \sim \int_{\gamma_n} \omega(x_2; \ldots; x_{n+1}).$$

This implies

$$\int_1^\infty \omega(x_2; \ldots; x_{n+1}) \sim \cdots \sim \int_0^{x_{n+1}} \omega(x_2; \ldots; x_{n+1}).$$

On the other hand,

$$\int_1^\infty \omega(x_2; \ldots; x_{n+1}) = B(1-\mu_1; 1-\mu_{n+2}) F(a; b_2; \ldots; b_{n+1}; c; x_2; \ldots; x_{n+1}),$$

and for all $k \in \{2; \ldots; n+1\}$,

$$\int_0^{x_k} \omega(x_2; \ldots; x_{n+1}) \sim B(1-\mu_k; 1-\mu_0)$$

$$\times F\Bigg(1-b_k; b_2; \ldots; a+1-c; \ldots; b_{n+1};$$

$$2 + \sum_{i=2, i\neq k}^{n+1} b_i - c; \frac{x_k}{x_k - x_2}; \ldots; \frac{x_k}{x_k - 1}; \ldots; \frac{x_k}{x_k - x_{n+1}}\Bigg),$$

where the k-th parameter is given by $a+1-c$ and the $(k-1)$-th variable is given by $\frac{x_k}{x_k - 1}$. Now, it is well known that

$$B(1-p; 1-q) \sim \frac{\pi}{B(p; q)} \text{ for } p, q \notin \mathbb{Z}$$

which proves the first part of Lemma 5. For the remainder of that lemma, we use the following expressions. For $(x_2; \ldots; x_{n+1}) \in \overline{\mathbb{Q}}^n \cap \mathcal{Q}$, not in Z', we have

$$\int_1^{\infty} u^l \omega_{N-1}(x_2; \ldots; x_{n+1})$$

$$= B(\mu_{n+2} + n - l - 1; \mu_1) F(n - l - a; 1 - b_2; \ldots; 1 - b_{n+1}; n + 1 - l - c; x_2; \ldots; x_{n+1})$$

$$\sim B(\mu_1; \mu_{n+2}) F(n - l - a; 1 - b_2; \ldots; 1 - b_{n+1}; n + 1 - l - c; x_2; \ldots; x_{n+1})$$

and

$$\int_0^{x_k} u^l \omega_{N-1}(x_2; \ldots; x_{n+1})$$

$$= x_k^{\mu_0 + \mu_k + l - 1}(x_k - 1)^{\mu_1 - 1} \prod_{i=2, i \neq k}^{n+1} (x_k - x_i)^{\mu_i - 1} B(\mu_k; \mu_0 + l)$$

$$\times F\Bigg(b_k; 1 - b_2; \ldots; c - a; \ldots; 1 - b_{n+1}; c + l - \sum_{i=2, i \neq k}^{n+1} b_i; \frac{x_k}{x_k - x_2}; \ldots; \frac{x_k}{x_k - 1}; \ldots; \frac{x_k}{x_k - x_{n+1}}\Bigg)$$

$$\sim B(\mu_0; \mu_k) F\Bigg(b_k; 1 - b_2; \ldots; c - a; \ldots; 1 - b_{n+1}; c + l - \sum_{i=2, i \neq k}^{n+1} b_i; \frac{x_k}{x_k - x_2}; \ldots; \frac{x_k}{x_k - 1}; \ldots; \frac{x_k}{x_k - x_{n+1}}\Bigg),$$

where the k-th parameter is given by $c - a$ and the $(k-1)$-th variable is given by $\frac{x_k}{x_k - 1}$. This completes the proof of Lemma 5.

Now, when hypotheses of Theorem 1 are true, $T(x_2; \ldots; x_{n+1})$ is CM, because it is isogenous to the CM variety T_0. Then, the condition $F(a; b_2; \ldots; b_{n+1}; c; x_2; \ldots; x_{n+1}) \in \overline{\mathbb{Q}}^*$ implies

$$F\Bigg(1 - b_k; b_2; \ldots; a + 1 - c; \ldots; b_{n+1}; 2 + \sum_{i=2, i \neq k}^{n+1} b_i - c; \frac{x_k}{x_k - x_2}; \ldots; \frac{x_k}{x_k - 1}; \ldots; \frac{x_k}{x_k - x_{n+1}}\Bigg)$$

$$\sim \frac{B(1 - \mu_k; 1 - \mu_0)}{B(1 - \mu_1; 1 - \mu_{n+2})} \sim \frac{B(\mu_k; \mu_0)}{B(\mu_1; \mu_{n+2})} = \delta_{CM}^{k}{}^{-1},$$

where the k-th parameter is given by $a + 1 - c$ and the $(k - 1)$-th variable is given by $\frac{x_k}{x_k - 1}$.

A similar expression can be found for the $2n$ numbers studied in this theorem. Finally, one proves that the numbers δ_{CM}^k are transcendental. The following lemma gives this proof.

Lemma 6 [29]. *A non-zero period of the first kind and a non-zero period of the second kind on the same abelian variety defined over* $\overline{\mathbb{Q}}$ *are* $\overline{\mathbb{Q}}$ *linearly independent.*

In the present case, the periods $B(\mu_1; \mu_{n+2})$ and $B(\mu_k; \mu_0)$ are respectively of the first and the second kind, because by hypothesis $1 - c > 0$, that is, $\mu_1 + \mu_{n+2} > 1$ and so $\mu_k + \mu_0 < 1$ for all $k \in \{2; ...; n + 1\}$. As they are non-zero periods of the same abelian variety $T(x_2; ...; x_{n+1})$ defined on $\overline{\mathbb{Q}}$ because $(x_2; ...; x_{n+1}) \in \overline{\mathbb{Q}}^n \cap \mathcal{Q}$, they are linearly independent on $\overline{\mathbb{Q}}$. Thus the δ_{CM}^k are transcendental numbers.

Remark. In the case of one variable, applying this to the classical Gauss hypergeometric function leads to the following result (see [13, p. 29]):

For all $a, b, c \in \mathbb{Q} \cap]0; 1[,\ a, b < c$. For all $x \in \overline{\mathbb{Q}}$, when $F(a; b; c; x)$ is an algebraic number, then $F(b + 1 - c; a + 1 - c; 2 - c; x)$ is a transcendental number.

In particular, using the results of Beukers and Wolfart [7] and Archinard [4], this gives explicit points at which the value of the Gauss hypergeometric function is transcendental.

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