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ON TOWERS OF SURVIVAL PAIRS OF COMMUTATIVE RINGS

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Abstract

Let $R \subseteq S \subseteq T$ be (commutative unital) rings. If *S* is integral over *R*, then (R, T) is a survival pair if and only if (S, T) is a survival pair. An example is given to show that if (R, S) is a survival pair and (S, T) is a survival pair, then (R, T) need not be a survival pair; in this example, *R* is a field, *T* is integral over *S*, and *S* and *T* are integral domains. The survival pair concept is used to determine which rings *S* are integral over *R*.

1. Introduction

All rings and algebras considered below are commutative with identity; all inclusions of rings and ring homomorphisms are unital. If *R* is a ring, then $\mathrm{Spec}(R)$ denotes the set of prime ideals of *R*. If $R \subseteq T$ are rings, then $[R, T]$ denotes the set of rings *A* such that $R \subseteq A \subseteq T$, that is, the set of all

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R-subalgebras of T. If $\mathcal{P}$ is a property of (some) ring extensions and $R \subseteq T$ are rings, then (R, T) is called a *P-pair* if $A \subseteq B$ satisfies $\mathcal{P}$ for all rings $R \subseteq A \subseteq B \subseteq T$, that is, for all $A \subseteq B$ in $[R, T]$. As in [5, p. 28], we let GU and LO denote the going-up and lying-over properties, respectively, of ring extensions. It is well known that GU $\Rightarrow$ LO and that the converse is false (cf. [5, Theorem 42; Exercise 3, p. 41]). However, the concepts of GU-pair and LO-pair are logically equivalent [3, Corollary 3.2]. Our focus in this note is on another equivalent concept, namely, that of survival pairs. Proposition 2.1 generalizes to extensions of arbitrary rings a characterization [2, Corollary 2.19] of survival pairs that Coykendall and Dutta recently gave for the context of certain extensions of integral domains. In fact, we proved Proposition 2.1 in its full generality more than 30 years ago, in [3, Lemma 2.11(b)], by reasoning with the lying-over property. In Proposition 2.1, we give a slightly different proof that focuses on the "survival" property, whose definition is recalled next.

Slightly modifying the usage in [5, p. 35], we say that a ring extension $R \subseteq T$ is a *survival extension* if $IT \neq T$ for each proper ideal I of R. (Since our rings are unital, one may restrict attention to $I \in \mathrm{Spec}(R)$.) It is clear that if a ring extension satisfies LO, then it is a survival extension, but the converse is false [3, Examples 2.6]. Nevertheless, the concepts of LO-pairs and of survival pairs are logically equivalent [1, Theorem 2.2]. According to the above terminology, (R, T) is a *survival pair* if and only if $A \subseteq B$ is a survival extension for all $A \subseteq B$ in $[R, T]$; equivalently, if and only if $IT \neq T$ whenever $I \in \mathrm{Spec}(A)$ for some $A \in [R, T]$. The above-mentioned result of Coykendall and Dutta [2, Corollary 2.19] states that if $R \subseteq T$ are integral domains such that (the quotient field of) T is algebraic over (the quotient field of) R and if R^* denotes the integral closure of R in T, then (R, T) is a survival pair if and only if (R^*, T) is a survival pair. Proposition 2.1 generalizes this assertion in several ways: R and T need not be domains; T need not be algebraic over R; and R^* may be replaced by any of its R-subalgebras.

The notion of survival pair (equivalently, LO-pair) has figured in some characterizations of integrality, beginning with the so-called Folklore Theorem [3, p. 454] (cf. also [3, Corollaries 3.5 and 3.6], [4, Theorem 2.2 and Proposition 2.3]). Corollary 2.2 shows how Proposition 2.1 leads easily to a contribution in the same vein. Example 2.4 then shows that one cannot, in general, interchange the ordering of the integral extension and the survival pair in the hypothesis of Proposition 2.1. This has the important consequence that if both (A, B) and (B, C) are survival pairs, then (A, C) need not be a survival pair. The reasoning in Example 2.4 leads naturally to Theorem 2.5, which uses the survival pair concept to determine which algebras are integral. The note concludes with Remark 2.7, which collects several relevant observations.

As usual, $\subset$ denotes proper inclusion and dim denotes Krull dimension. Any unexplained material is standard, as in [5].

2. Results

We move at once to the generalization of [2, Corollary 2.19] that was promised above.

Proposition 2.1 ([3, Lemma 2.11(b)])**.** *Let $R \subseteq T$ be rings and let $S \in [R, T]$ be such that S is integral over R. Then (R, T) is a survival pair if and only if (S, T) is a survival pair.*

Proof. Since $[S, T] \subseteq [R, T]$, the "only if" assertion is trivial. Conversely, suppose that (S, T) is a survival pair. By the above comments, it suffices to prove that if $A \in [R, T]$ and $I \in \operatorname{Spec}(A)$, then $IT \neq T$.

Consider $B := SA$, the subring of T generated by $S \cup A$. Observe that $A \subseteq B$ inherits integrality from $R \subseteq S$. Hence, by the classical Lying-over Theorem (cf. [5, Theorem 44]), $A \subseteq B$ satisfies LO. In particular, there exists $J \in \operatorname{Spec}(B)$ such that $J \cap A = I$. Since $B \in [S, T]$ and (S, T) is a survival pair, $JT \subset T$. As $IT \subseteq JT \subset T$, we have $IT \neq T$, as desired. $\square$

Consider any ring extension $A \subseteq B$. According to the Folklore Theorem [3, p. 454], B is integral over A if and only if (A, B) is both an LO-pair and an INC-pair. (As in [5, p. 28], INC denotes the incomparable property of ring extensions.) This result was sharpened in [4, Proposition 2.3] to the statement that B is integral over A if and only if (A, B) is an LO-pair such that $A \subseteq B$ satisfies INC. In conjunction with Proposition 2.1, this leads easily to the following characterization of integrality.

Corollary 2.2. *Let* $R \subseteq S \subseteq T$ *be rings. Then the following conditions are equivalent*:

(1) *S is integral over R,* (S, T) *is a survival pair, and* $R \subseteq T$ *satisfies INC*;

(2) *S is integral over R and T is integral over S*;

(3) *T is integral over R.*

Proof. Note that (3) $\Rightarrow$ (2) trivially; and it is well known that (2) $\Rightarrow$ (3) (cf. [5, Theorem 40]). Also, the implication (3) $\Rightarrow$ (1) follows from the fact that all integral extensions are survival extensions and satisfy INC (cf. [5, Theorem 44]). Thus, it remains only to show that (1) $\Rightarrow$ (3). If (1) holds, then Proposition 2.1 ensures that (R, T) is a survival pair, and so (3) then follows from an above-mentioned result, [4, Proposition 2.3] (and the fact that every survival pair is an LO-pair [1, Theorem 2.2]). $\square$

It seems natural to ask if one may interchange the ordering of the integral extension and the survival pair in the hypothesis of Proposition 2.1. In other words, given rings $R \subseteq S \subseteq T$ such that (R, S) is a survival pair and T is integral over S, must it be the case that (R, T) is a survival pair? One possible reason to expect an affirmative answer comes from the following example. We note that the data in Example 2.3 were first examined in [3, Remark 4.2(c)] for other purposes.

Example 2.3. Take R to be the field $\mathbb{R}$ of real numbers, S to be $\mathbb{R}[X^2]$, where X is an indeterminate over $\mathbb{R}$, and T to be $\mathbb{C}[X^2, X^3]$, where $\mathbb{C}$

denotes the field of complex numbers. Then (R, S) is a survival pair, T is integral over S, and (R, T) is a survival pair.

Proof. (R, S) is a survival pair since $(F, F[x])$ is an LO-pair whenever x is an indeterminate over a field F [3, Proposition 2.9]; T is integral over S, by applying [5, Theorem 40] to the tower

$$S \subseteq A := S[\sqrt{-1}] = \mathbb{C}[X^2] \subseteq A[X^3] = T;$$

and $(R, \mathbb{C}[X])$ is a survival pair (so that, *a fortiori*, (R, T) is also a survival pair), by applying Proposition 2.1 to the tower $R \subseteq \mathbb{C} \subseteq \mathbb{C}[X]$. □

Despite the affirmative answer in Example 2.3, caution may be appropriate for the general case, as a similar attempt to interchange the ordering of conditions involving the LO and INC properties in [4, Proposition 2.3] led to a negative answer to the analogous question in [4, Theorem 2.4(b)]. Caution is indeed appropriate for our general question, as Example 2.4 next answers it in the negative.

Example 2.4. If (R, S) is a survival pair and (S, T) is a survival pair, then (R, T) need not be a survival pair, even if R is a field, T is integral over S, and S and T are integral domains. For an example, it suffices to let R be any field F, let $S := F[X]$, where X is an indeterminate over F, and let $T := S[y] = F[X, y]$, where y is an element of an algebraic closure of $F(X)$ such that $y^2 + Xy - 1 = 0$.

Proof. Note that $T = S[y]$ is integral over S since y is integral over $F[X] = S$. In particular, (S, T) is a survival pair. Of course, (R, S) is a survival pair by [3, Proposition 2.9]. As T is an integral domain, it remains only to show that (R, T) is not a survival pair. To that end, note first that $y \neq 0$ (since $1 \neq 0$ in the given algebraic closure). Since $(1 - y^2)/y = X$ is not algebraic over F, it follows that y is not algebraic over F, that is, y is transcendental over F. Consider the polynomial ring $A := F[y] \in [R, T]$.

Then $I := yA$ is a proper (in fact, maximal) ideal of A. We claim that $IT = T$. To see this, note that $y^{-1} = y + X \in F[X, y] = T$, so that $1 = yy^{-1} \in IT$, which proves the claim. Accordingly, $A \subseteq T$ is not a survival extension, and so (R, T) is not a survival pair. □

By revisiting the data in Example 2.4, we next obtain a characterization of the algebras that are integral.

Theorem 2.5. *Let* $R \subseteq S$ *be rings. Then the following conditions are equivalent*:

(1) (R, S) *is a survival pair and, for all non-maximal* $Q \in \mathrm{Spec}(S)$ *such that* $M := Q \cap R$ *is a maximal ideal of R and for all integral ring extensions* $S/Q \subseteq T$, *one has that* $(R/M, T)$ *is a survival pair*;

(2) *S is an integral extension of R.*

Proof. (2) $\Rightarrow$ (1) Suppose (2). Since all integral extensions are survival extensions, it suffices to note that the extension $R/M \subseteq S/Q$ inherits integrality from $R \subseteq S$ and then observe, via [5, Theorem 40], that T is integral over R/M.

(1) $\Rightarrow$ (2) Suppose the assertion fails. As (R, S) is a survival pair such that S is not integral over R, we can apply [3, Proposition 4.5], and thus obtain a non-maximal $Q \in \mathrm{Spec}(S)$ such that $M := Q \cap R$ is a maximal ideal of R and an element $X \in S/Q$ such that X is transcendental over $F := R/M$ and the extension $F[X] \subseteq S/Q$ is integral. Motivated by the reasoning in the proof of Example 2.4, we work inside an algebraic closure of the quotient field of S/Q, where we find an element y such that $y^2 + Xy - 1 = 0$. As above, $y \neq 0$ since $1 \neq 0$. Note that $T := (S/Q)[y]$ is an integral extension of S/Q and so, by (1), (F, T) is a survival pair. In particular, $F[y] \subseteq T$ is a survival extension. Since $y^{-1} = y + X \in F[X, y] \subseteq T$, we see as above that

$I := yF[y]$ satisfies $IT = T$, and so $I = F[y]$. However, since $X = y^{-1} - y$ is not algebraic over F, we see that y is transcendental over F, whence I is a prime (hence, proper) ideal of $F[y]$, the desired contradiction. $\square$

Corollary 2.6. *Let F be a field and S be an F-algebra; as usual, view $F \subseteq S$. Then the following conditions are equivalent*:

(1) (F, S) *is a survival pair and, for all non-maximal* $Q \in \mathrm{Spec}(S)$ *and for all integral ring extensions* $S/Q \subseteq T$, *one has that* (F, T) *is a survival pair*;

(2) *S is an integral* (*that is, algebraic*) *extension of F.*

If, in addition, S is an integral domain, then the above equivalent conditions are also equivalent to

(3) *S is an algebraic field extension of F.*

Proof. The equivalence of (1) and (2) is a special case of Theorem 2.5. Finally, to see that (2) and (3) are equivalent when S is an integral domain, it suffices to recall that whenever an integral domain D is integral over a field F, D must be a field (cf. [5, Theorem 48]). $\square$

Remark 2.7. (a) We wish to state more clearly what Coykendall and Dutta established in [2, Corollary 2.19]. By way of proof for this result, Coykendall and Dutta have written only, "The following corollary is immediate, but worth noting," and so it seems fair to suppose that the rings R, T in [2, Corollary 2.19] admit the hypotheses of the result that immediately precedes [2, Corollary 2.19], namely, [2, Theorem 2.18]. That result uses some terminology, "strongly 1-almost integral" that was introduced earlier in that paper. When the definition of that terminology and the ambient assumptions are made explicit, we obtain the following complete statement of [2, Corollary 2.19]. Let $R \subseteq T$ be integral domains with corresponding quotient fields $K \subseteq F$ such that F is algebraic over K, and let R^* denote the integral closure of R in T. Then (R, T) is a survival pair $\Leftrightarrow (R^*, T)$ is a survival pair. If one includes the statement of [2, Theorem

2.18], then the above equivalent conditions are, under the above hypotheses, also equivalent to the following condition: for all $t \in T$, for all $A \in [R^*, T]$ such that A is integrally closed in T, and for all nonzero elements $c \in \{a \in A \mid at \in A\}$, we have $ct^n \in A$ for all positive integers n.

Note that [2, Corollary 2.19], as stated above, has the unnecessary restrictions that were mentioned in the introduction. In addition, as reconstituted above, [2, Corollary 2.19] cannot be used to study the survival pair that we treated in Example 2.3, the data discussed in Example 2.4, or the example discussed below in part (d); nor can [2, Corollary 2.19] give the generality of Proposition 2.1 or Corollary 2.2. For these reasons, we conclude that the final sentence in the abstract of [1] is somewhat misleading (or, at best, premature). However, in the spirit of that sentence, we would ask whether the concepts that were introduced in [1] can, with additional reasoning, be used to characterize LO-pairs of arbitrary rings.

(b) If one wishes only to give an example with the behavior noted in Example 2.3, then the following is a simpler way to proceed. Let F be a field, X be an indeterminate over F, and y be an element of the algebraic closure of $F(X)$ such that $y^2 = X$. As $F(X) \subseteq F(y)$, y is transcendental over F. Then, for the tower $R := F \subseteq S := F[X] \subseteq T := S[y] = F[y]$, we have, via [3, Proposition 2.9], that (R, S) and (R, T) are survival pairs; and, of course, $T = S[y]$ is integral over S.

(c) One can perhaps appreciate Example 2.4 more fully in view of the following example, which is in the spirit of Example 2.3 and part (b). If F is a field and (F, B) is any survival pair such that B is not integral (algebraic) over F, then there is a naturally associated tower $F \subseteq S \subseteq T$ such that (F, S) and (F, T) are survival pairs and $S \subseteq T$ is an integral extension. Indeed, by [3, Proposition 4.5], there exist a non-maximal $Q \in \mathrm{Spec}(B)$ and an element $X \in B/Q$ such that X is transcendental over F and the extension $F[X] \subseteq B/Q$ is integral. Then the assertion holds by taking $S := F[X]$ and

$T := B/Q$. For a proof, it suffices to use [3, Lemma 3.1(b)] to show that (F, T) inherits the survival pair property from (F, B).

(d) Recall that the data that were used in Example 2.4 had first appeared in [3, Remark 4.2(c)]. Given that the other data in Example 2.3 had (incorrectly) suggested a positive answer to the general question, one may wonder whether the data that appeared in [3, Remark 4.2(b)] also could have been used for the purposes of Example 2.4. The data in question consist of a field F, an indeterminate X over F, and the ring $B := F[X, (X-1)^{-1}]$. It was shown in [3, Remark 4.2(b)] that (F, B) is not a survival pair. Moreover, by [3, Proposition 2.9], $(F, F[X])$ is a survival pair. Nevertheless, the tower $F \subseteq F[X] \subseteq B$ cannot play the role of $R \subseteq S \subseteq T$ in Example 2.4, since the extension $F[X] \subseteq B$ is not integral. Indeed, if we let $Y := X - 1$, then $B = F[Y, Y^{-1}] = F[X]_Y$ is a proper flat overring of $F[X]$ and hence cannot be integral over $F[X]$ (cf. [5, Exercise 10, p. 24]).

(e) In condition (1) in the statement of Corollary 2.6, one cannot delete consideration of the rings T or even reduce consideration to the case $T = S/Q$. Indeed, since [3, Lemma 3.1(b)] ensures that $(F, S/Q)$ inherits the survival pair property from (F, S), the modified version of (1) would be equivalent to (F, S) being a survival pair, and this condition is not equivalent to S being integral over F, in view of the example, where $S = F[X]$ with X transcendental over F.

(f) In comparing the proof given above for Proposition 2.1 with the original argument given in [3, Remark 2.11(b)], we see that the former used both the survival property and the lying-over property, while the latter only used the lying-over property (via the classical Lying-over Theorem). It would be interesting to know if Proposition 2.1 can be proved without any implicit appeal to the lying-over property. We suspect a negative answer to this question, since prime ideals are germane, inasmuch as our proof used the fact that any proper ideal is a subset of some prime (in fact, maximal) ideal.

In any event, since survival pairs are logically equivalent to GU-pairs, we wish to close by asking if Proposition 2.1 can be proved by focusing on the going-up property.

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