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# ON NEVANLINNA DIRECTION OF MEROMORPHIC FUNCTION DEALING WITH MULTIPLE VALUES

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## Abstract

In this paper, by using Ahlfors' theory of covering surface, the existence of Nevanlinna direction of meromorphic function dealing with multiple values is obtained. Results are obtained extending the previous results.

## 1. Introduction and Main Results

In this paper, meromorphic function always means a function meromorphic in the whole complex plane. Assume that the basic definitions, theorems and standard notations of the Nevanlinna theory for meromorphic function are known (see [2] or [12]). The singular direction of meromorphic function $f(z)$ is one of main objects of value distribution theory. Since Julia introduced the concept of Julia direction and showed its existence for a



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meromorphic function in 1919, several types of singular directions have been introduced and studied. In 1928, Valiron [9] introduced the concept of Borel direction and established its existence for meromorphic function. After that, there are many investigated results on the study of the singular direction of meromorphic function, we refer the reader to the book Chuang [1] and Yang [12]. In 1983, Lü and Zhang [4] introduced the concept of Nevanlinna direction. They firstly defined the deficiency and deficient value with respect to a direction and finally by a Nevanlinna direction meant a direction for which the total sum of deficiencies does not exceed 2. Later, Sun [5] and Zhang [14] gave some new definition of Nevanlinna direction and got some better results. However, it was not discussed whether there exists a Nevanlinna direction concerning multiple values. Recently, [11] considered the existence of Nevanlinna direction concerning multiple values with multiplicity no less than 3. In this paper, we continuously investigate this problem in general. Especially, we considered the existence of Nevanlinna direction concerning multiple values with multiplicity no less than $l(\geq 1)$.

Suppose that $E$ is a subset of $\mathbb{C}$, let

$$S(E, f) = \frac{1}{\pi}\int_E \left(\frac{|f'(z)|}{(1+|f(z)|^2)}\right)^2 rd\theta dr, \quad z = re^{i\theta}.$$

When $E = \{z \in \mathbb{C}, |z| < r\}$, we denote $S(E, f) = S(r, f)$ and

$$T(r, f) = \int_0^r \frac{S(t, f)}{t} dt,$$

where $T(r, f)$ is the *Ahlfors-Shimizu's characteristic function*. Denote the following angular domain by

$$\Omega(\theta, \varepsilon) = \{z \in \mathbb{C}, |\arg z - \theta| < \varepsilon\}.$$

When $E$ is a sector $\{z \in \mathbb{C}, |z| < r\} \cap \Omega(\theta, \varepsilon)$, we denote $S(E, f) = S(r, \Omega(\theta, \varepsilon), f)$ and

$$T(r, \Omega(\theta, \varepsilon), f) = \int_0^r \frac{S(t, \Omega(\theta, \varepsilon), f)}{t} dt.$$

For any $a \in \mathbb{C}_\infty$, let $n(r, \theta, \varepsilon, a)$ be the number of zeros, counted according to their multiplicities, of $f(z) - a$ in the sector $\{z \in \mathbb{C}, |z| < r\} \cap \Omega(\theta, \varepsilon)$, and $n^{l)}(r, \theta, \varepsilon, a)$ be the number of zeros with multiplicities $\leq l$, of $f(z) - a$ in the sector $\{z \in \mathbb{C}, |z| < r\} \cap \Omega(\theta, \varepsilon)$, where $l$ is any positive integer. Denote

$$N(r, \theta, \varepsilon, a) = \int_0^r \frac{n(t, \theta, \varepsilon, a) - n(0, \theta, \varepsilon, a)}{t} dt + n(0, \theta, \varepsilon, a) \log r;$$

$$N^{l)}(r, \theta, \varepsilon, a) = \int_0^r \frac{n^{l)}(t, \theta, \varepsilon, a) - n^{l)}(0, \theta, \varepsilon, a)}{t} dt + n^{l)}(0, \theta, \varepsilon, a) \log r.$$

In addition, we also need the notations (see [13])

$$L(r, \psi_1, \psi_2) = \int_{\psi_1}^{\psi_2} \frac{|f'(re^{i\psi})|}{(1 + |f(re^{i\psi})|^2)} r d\psi,$$

$$L(r, \psi) = \int_1^r \frac{|f'(te^{i\psi})|}{(1 + |f(te^{i\psi})|^2)} dt.$$

Follows Lü and Zhang’s definitions of Nevanlinna direction of $f(z)$, we give the following definitions.

**Definition 1.** For any $a \in \mathbb{C}_\infty$, set

$$\Theta(a, \theta) = 1 - \limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \frac{N(r, \theta, \varepsilon, a)}{T(r, \Omega(\theta, \varepsilon), f)};$$

and

$$\Theta_{l)}(a, \theta) = 1 - \limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \frac{N^{l)}(r, \theta, \varepsilon, a)}{T(r, \Omega(\theta, \varepsilon), f)}.$$

We call $\Theta(a, \theta)$(or $\Theta_{l)}(a, \theta)$) the *deficiency* (*precise deficiency*) of the value $a$ with respect to direction $L : \arg z = \theta$. We call the value $a$ the *deficient* (*precise deficient*) value of $f(z)$ with respect to direction $L : \arg z = \theta$ if $\Theta(a, \theta) > 0$ (or $\Theta_{l)}(a, \theta) > 0$).

**Definition 2.** We call $L : \arg z = \theta$ the *Nevanlinna direction of* $f(z)$ if

$$\sum_{a \in \mathbb{C}_\infty} \Theta(a, \theta) \leq 2$$

holds for any finitely many deficient value $a$; and call $L : \arg z = \theta$ the *Nevanlinna direction of* $f(z)$ *dealing with multiple values* if

$$\sum_{a \in \mathbb{C}_\infty} \Theta_{l)}(a, \theta) \leq \frac{2l+2}{l}$$

holds for any finitely many deficient value $a$.

In [4], Lü and Zhang proved the following theorem for the existence of Nevanlinna direction of $f(z)$.

**Theorem A.** *Let* $f(z)$ *be a meromorphic function and satisfy*

$$\limsup_{r \to \infty} \frac{T(r, f)}{\log^2 r} = \infty. \tag{1}$$

*Then* $f(z)$ *must have a Nevanlinna direction*.

In this paper, we shall study the existence of Nevanlinna direction dealing with multiple values and prove the following theorem.

**Theorem 1.** *Let* $f(z)$ *be a meromorphic function and satisfy* (1). *Then there at least exists a direction* $L : \arg z = \theta$ *which is a Nevanlinna direction of* $f(z)$ *dealing with multiple values*.

## 2. Some Lemmas

In order to prove Theorem 1, we need the following lemmas.

**Lemma 1** [10]**.** *Let* $f(z)$ *be meromorphic in the complex plane*. *If* $\{a_v\}$ *are* $q\left(> \left[\frac{2l+2}{l}\right]\right)$ *distinct points on* $\mathbb{C}_\infty$, *then we have*

$$\left(q-2-\frac{2}{l}\right)S(r,\ \Omega(\theta,\ \varphi),\ f)$$

$$\le \sum_{v=1}^{q} n^{l)}(r,\ \theta,\ \delta,\ a_v) + \frac{2\pi H^2}{\left(q-2-\frac{2}{l}\right)(\delta-\varphi)}\log r$$

$$+\left(q-2-\frac{2}{l}\right)S(1,\ \Omega(\theta,\ \varphi),\ f)$$

$$+ HL(1,\ \theta-\delta,\ \theta+\delta) + HL(r,\ \theta-\delta,\ \theta+\delta) \tag{2}$$

*and*

$$\left(q-2-\frac{2}{l}\right)T(r,\ \Omega(\theta,\ \varphi),\ f)$$

$$\le \sum_{v=1}^{q} N^{l)}(r,\ \theta,\ \delta,\ a_v) + \frac{2\pi H^2}{\left(q-2-\frac{2}{l}\right)(\delta-\varphi)}\log^2 r$$

$$+\left(q-2-\frac{2}{l}\right)T(1,\ \Omega(\theta,\ \varphi),\ f)$$

$$+\left(q-2-\frac{2}{l}\right)S(1,\ \Omega(\theta,\ \varphi),\ f)\log r$$

$$+ HL(1,\ \theta-\delta,\ \theta+\delta)\log r + \chi(r,\ \theta-\delta,\ \theta+\delta) \tag{3}$$

*for any* $\varphi,\ 0<\varphi<\delta,$ *where* $H$ *is a constant depending only on* $a_v,\ \ v = 1,\ 2,\ \ldots,\ q$ *and* $\chi(r,\ \theta-\delta,\ \theta+\delta) = H\int_1^r \frac{L(t,\ \theta-\delta,\ \theta+\delta)}{t}\,dt.$

**Lemma 2** (Zhang [13])**.** *Under the condition of Lemma* 1, *we have*

$$\chi(r,\ \theta-\delta,\ \theta+\delta) = H\int_1^r \frac{L(t,\ \theta-\delta,\ \theta+\delta)}{t}\,dt$$

$$\le H\sqrt{2\delta\pi S(r,\ \Omega(\theta,\ \delta),\ f)\log r}$$

*or*

$$\chi(r,\ \theta - \delta,\ \theta + \delta) \leq H\sqrt{2\delta\pi T(r,\ \Omega(\theta,\ \delta),\ f)}\log T(r,\ \Omega(\theta,\ \delta),\ f) \tag{4}$$

*with at most one exceptional set* $E_\delta$ *of* $r$*, where* $E_\delta$ *consists of a series of intervals and satisfies*

$$\int_{E_\delta} \frac{1}{r\log r}\,dr \leq \frac{1}{\log T(r,\ \Omega(\theta,\ \delta),\ f)} < \infty.$$

**Lemma 3** (Li and Gu [3])**.** *Suppose that* $\Psi(r)$ *is a nonnegative increasing function in* $(1,\ \infty)$ *and satisfies*

$$\limsup_{r\to\infty} \frac{\Psi(r)}{(\log r)^2} = \infty.$$

*Then for any set* $E \subset (1,\ \infty)$ *such that* $\int_E \frac{1}{r\log r}\,dr < \frac{1}{3}$*, we have*

$$\limsup_{r\to\infty;\, r\in(1,\infty)-E} \frac{\Psi(r)}{(\log r)^2} = \infty.$$

We are now in the position to prove Theorem 1.

To prove of Theorem 1 suppose that $\delta \in (0,\ 2\pi)$, using Lemma 3 and the same argument as Li and Gu [3], there exists a sequence of $\{r_n\}$ and some $\theta$ such that for any $0 < \varphi < \delta$, we have

$$\limsup_{n\to\infty} \frac{T(r_n,\ f)}{\log^2 r_n} = \infty, \tag{5}$$

and

$$\limsup_{r\to\infty} \frac{T(r_n,\ \Omega(\theta,\ \varphi),\ f)}{T(r_n,\ f)} > 0. \tag{6}$$

We are now in the position to prove that $L : \arg z = \theta$ is the Nevanlinna direction in Theorem 1.

Otherwise, for an arbitrary sufficiently small $\varepsilon > 0$, there exists $q\left(> \left[\frac{2l+2}{l}\right]\right)$ distinct complex number $\{a_i\}$, $i = 1, \ldots, q$ such that the following inequality holds:

$$\frac{l}{l+1}\sum_{i=1}^{q}\Theta_{l)}(a_i, \theta) > 2 + 2\varepsilon.$$

By Definition 1, we have

$$\frac{l}{l+1}\left(q - \sum_{i=1}^{q}\limsup_{\varphi\to 0}\limsup_{r\to\infty}\frac{N^{l)}(r, \theta, \varphi, a)}{T(r, \Omega(\theta, \varphi), f)}\right) > 2 + 2\varepsilon,$$

then there exists some $\varphi' : \varphi' \in (0, \delta)$, such that for any $\varphi \in (0, \varphi')$, the following inequality holds:

$$\limsup_{r\to\infty}\sum_{i=1}^{q}\frac{N^{l)}(r, \theta, \varphi, a)}{T(r, \Omega(\theta, \varphi), f)} \le q - \frac{2+2l}{l} - \frac{l+1}{l}\varepsilon. \tag{7}$$

For any $\varphi \in (0, \varphi')$, we define an increasing function as following:

$$T(\varphi) = \limsup_{n\to\infty}\frac{T(r_n, \Omega(\theta, \varphi), f)}{T(r_n, f)}.$$

From (6) we deduce $T(\varphi) \in (0, 1]$. By the increasing of $T(\varphi)$ in interval $[0, \varphi']$ and the continuous theorem for monotonous function, we see that all discontinuous points of $T(\varphi)$ constitute a countable set at most (see [11]). Thus, by Lemmas 1 and 2, we have

$$\begin{aligned}&\left(q - 2 - \frac{2}{l}\right)T(r_n, \Omega(\theta, \varphi), f)\\ &\le \sum_{v=1}^{q}N^{l)}(r_n, \theta, \delta', a_v) + O(\log^2 r_n)\\ &\quad + O(\sqrt{T(r_n, \Omega(\theta, \delta'), f)})\log T(r_n, \Omega(\theta, \delta'), f)\end{aligned} \tag{8}$$

holds for any $0 < \varphi < \delta' < \varphi' < \delta$. It follows from (5), (7), (8) and the definition of $T(\varphi)$ that

$$\left(q - 2 - \frac{2}{l}\right)T(\varphi) \leq \left(q - \frac{2 + 2l}{l} - \frac{l + 1}{l}\varepsilon\right)T(\delta'). \tag{9}$$

Note that $T(\varphi) \to T(\delta')$ as $\varphi \to \delta'$. Combining this result and (9), we have $T(\delta') = 0$. This contradicts with $T(\delta') \in (0, 1]$. Theorem 1 follows.

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