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MODIFICATION OF CZ_3-FREE CONDITION TO HAVE A CYCLE CONTAINING SPECIFIED VERTICES

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Abstract

In [4], Faudree et al. showed that if a 2-connected graph contains no $K_{1,3}$ and Z_3 as an induced subgraph, then the graph is Hamiltonian (except for specified graphs). In this paper, we consider the extension of this result to cycles passing through specified vertices. We define the families of graphs which are extension of the forbidden pair $K_{1,3}$ and Z_6, and consider that the forbidden families imply the existence of cycles passing through specified vertices.

1. Introduction

In this paper, we only consider finite undirected graphs without loops or multiple edges. For standard graph-theoretic terminology not explained in this paper, we refer the reader to [3].

For a family $\mathcal{F} = \{H_1, H_2, ..., H_k\}$ of graphs, a graph G is called an $\mathcal{F}$*-free graph* if G contains no induced subgraph isomorphic to any H_i with

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$i = 1, 2, ..., k$. A path or a cycle that includes all vertices of the graph is called a *hamiltonian path* or a *hamiltonian cycle*, respectively. A hamiltonian graph is one that contains a hamiltonian cycle, and a traceable graph is one that contains a hamiltonian path.

There are a lot of results on the existence of a hamiltonian path or cycle in graphs. As the generalizations of such research, some studies on the existence of a path or a cycle passing through specified vertices have been done ([1], [2] and [5]). Though forbidden subgraphs are major tool to find a hamiltonian path or cycle, there are few results using the condition on forbidden subgraphs to find a cycle passing through specified vertices. (The result in [5] uses degree condition in addition to the condition on forbidden subgraphs.) Our main results shown later use only the condition on forbidden subgraphs.

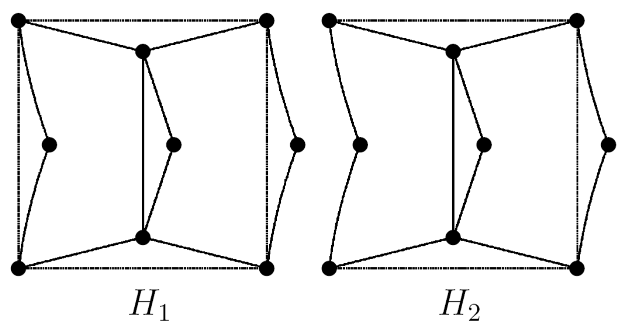

Figure 1. H_1 and H_2.

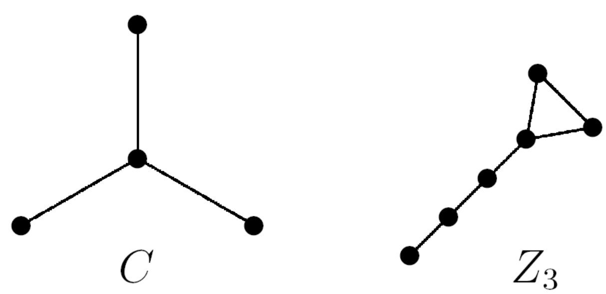

Figure 2. C and Z_3.

In 1995, Faudree et al. gave the following result. The graphs H_1 and H_2 are shown in Figure 1.

Theorem 1 (Faudree et al. [4]). *Let G be a* 2*-connected graph*. *If G is a* CZ_3*-free graph*, *then G is either hamiltonian or isomorphic to* H_1 *or* H_2.

In this section, we consider the generalization of Theorem 1, for the existence of cycles passing through specified vertices. We propose the following problem.

Problem 2. Let G be a 2-connected graph and $S \subseteq V(G)$. If G is a $CZ_3(S)$-free graph, then G contains a cycle D such that $S \subseteq V(D)$ (except for some graphs).

For the definition of $CZ_3(S)$-free graphs, we define families of graphs. Let G be a graph and $S \subseteq V(G)$. In [6], $C(S)$ is defined.

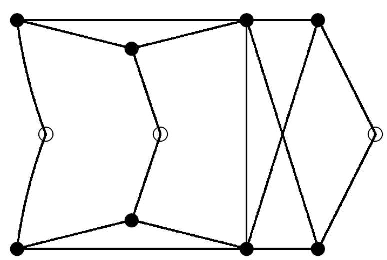

Figure 3. A graph G' and S'.

We define a family of graphs $Z_3(S)$ whose members are the graphs F' satisfying the following properties:

(1) F' contains a triangle T with $V(T) = \{x_1, x_2, x_3\}$ (we call x_i a *root* of F'),

(2) there exists a path P^1 such that x_1 is an end vertex of P^1, $|V(P^1)| \geq 4$ and $V(F') = V(P^1) \cup V(T)$,

(3) the end vertex of P^1 which is not x_1 is a vertex of S (we call such a vertex of P^1 a *leaf* of F'),

(4) internal vertices of P^1 (except for x_1^+, x_1^{+2}) are contained in $V(G)\backslash S$ and

(5) $E(F') = E(P^1) \cup E(T)$.

If there exists no induced subgraph which is a member of $C(S)$ and $Z_3(S)$ in G, then we call G a $CZ_3(S)$*-free graph*. It is clear that a $CZ_3(V(G))$-free graph is a CZ_3-free graph, and a CZ_3-free graph is a $CZ_3(S)$-free graph for every subset S of $V(G)$.

Theorem 1 implies that there exists an exception of Problem 2 for the case $S = V(G)$. Now we consider the exceptions of Problem 2. By the definition of $CZ_3(S)$-free, H_1 and H_2 are also exceptions of Problem 2 (if $S = V(H_1)$ or $V(H_2)$). But the graph G' with $S' \subseteq V(G')$ (see Figure 3) shows the existence of more exceptions.

We define several families of graphs which will be parts of the exceptions of Problem 2. First, we define a family $\mathcal{A}$ to be the set of graphs A satisfying the following properties:

(1) A contains a triangle T $(V(T) = \{a_1, a_2, a_3\})$,

(2) A does not contain triangles other than T, and

(3) A is 2-connected.

We define a family $\mathcal{A}'$ to be the set of graphs A' satisfying the following properties:

(1) A' contains a triangle T $(V(T) = \{a_1, a_2, a_3\})$,

(2) for any triangle T' in A', $\{a_1, a_2\} \in V(T')$ (we call a_1, a_2 *bases* of A'), and

(3) A' is 2-connected.

We define a family $\mathcal{B}$ to be the set of graphs B satisfying the following properties:

(1) B contains a $K_{1,2}R$ $(V(R) = \{b_1, b_2, b_3\}$ and $d_R(b_1) = d_R(b_2) = 1)$,

(2) B does not contain triangles, and

(3) $B + b_1b_2$ is 2-connected.

We define a family $\mathcal{B}'$ to be the set of graphs B' satisfying the following properties:

(1) B' contains a $K_{1,2}R(V(R) = \{b_1, b_2, b_3\}$ and $d_R(b_1) = d_R(b_2) = 1)$,

(2) B' does not contain triangles,

(3) $B' + b_1b_2$ is 2-connected, and

(4) every induced b_1-b_2 path has length at most 2.

We define a family $\mathcal{C}$ to be the set of graphs D satisfying the following properties:

(1) D contains an edge c_1c_2 (we call c_1, c_2 *bases* of D),

(2) D contains an independent set of vertices M $(|M| \geq 2)$,

(3) D contains a vertex c_3 (we call c_3 the *root* of D),

(4) for each $i \in \{1, 2, 3\}$, c_i is adjacent to all vertices of M,

(5) $D \backslash M$ is disconnected,

(6) D is 2-connected, and

(7) every triangle of D contains both c_1 and c_2.

Now we define six families of exceptions.

Let $\mathcal{G}_1$ be a family of pairs of graphs and their subsets of vertices (G, S_1) satisfying the following properties:

(1) G contains three subgraphs A_1, A_2, A_3 such that for each $i \in \{1, 2, 3\}$, A_i is isomorphic to a member of $\mathcal{A}'$ containing a unique triangle T_i with $V(T_i) = \{a_1^i, a_2^i, a_3^i\}$ (a_1^i and a_2^i are bases of A_i),

(2) G contains two subgraphs A_4, A_5 such that for each $i \in \{4, 5\}$, A_i is isomorphic to a member of $\mathcal{A}$ containing a unique triangle T_i with $V(T_i) = \{a_1^i, a_2^i, a_3^i\}$,

(3) A_1, A_2, A_3 are pairwise disjoint, and $A_4 \cap A_5 = \emptyset$,

(4) for each $j \in \{1, 2, 3\}$, $A_j \cap A_4 = \{a_1^j\} = \{a_j^4\}$, $A_j \cap A_5 = \{a_2^j\} = \{a_j^5\}$, and

(5) $S_1 \subseteq V(G)$ satisfies $\{a_3^1, a_3^2, a_3^3\} \subseteq S_1 \subseteq \{a_1^1, a_2^1, a_3^1, a_1^2, a_2^2, a_3^2, a_1^3, a_2^3, a_3^3\}$.

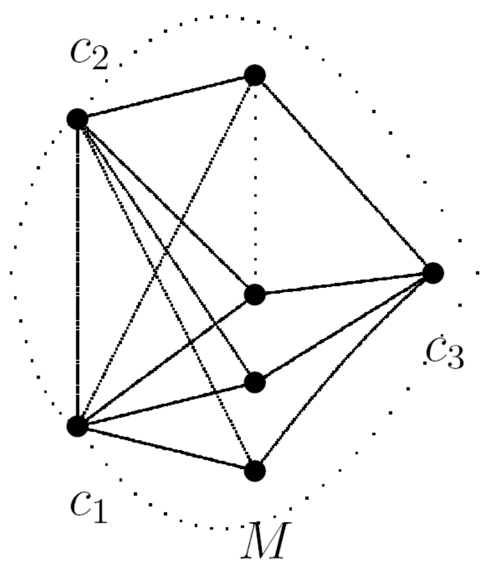

Figure 4. Example $D \in \mathcal{C}$.

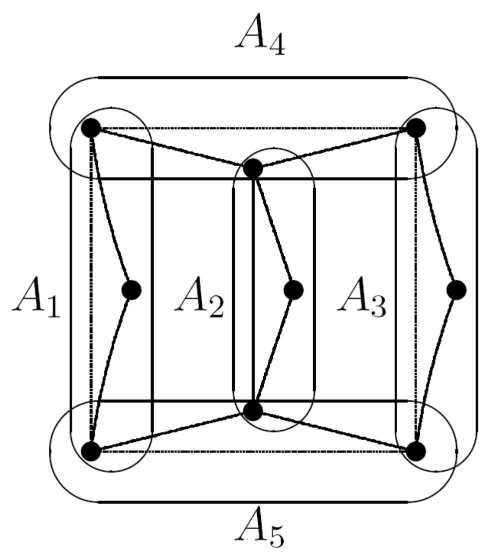

Figure 5. A graph $G \in \mathcal{G}_1$.

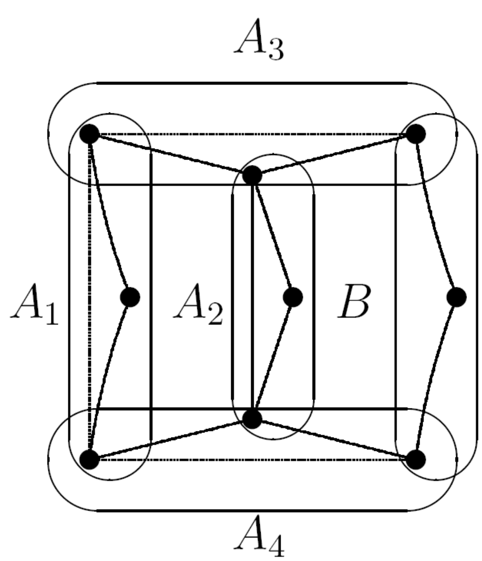

Figure 6. A graph $G \in \mathcal{G}_2$.

Let $\mathcal{G}_2$ be a family of pairs of graphs and their subsets of vertices (G, S_2) satisfying the following properties:

(1) G consists of five subgraphs A_1, A_2, A_4, A_5, B such that for each $i \in \{1, 2\}$, A_i is isomorphic to a member of $\mathcal{A}'$ containing a unique triangle T_i with $V(T_i) = \{a_1^i, a_2^i, a_3^i\}$ (a_1^i and a_2^i are bases of A_i), for each $i \in \{4, 5\}$, A_i is isomorphic to a member of $\mathcal{A}$ containing a unique triangle T_i with $V(T_i) = \{a_1^i, a_2^i, a_3^i\}$ and B is isomorphic to a member of $\mathcal{B}$ containing a unique path T_3 with $V(T_3) = \{a_1^3, a_2^3, a_3^3\}$, $E(T_j) = \{a_1^j a_3^j, a_2^j a_3^j\}$,

(2) A_1, A_2, B are pairwise disjoint, and $A_4 \cap A_5 = \emptyset$,

(3) for each $j \in \{1, 2\}$, $A_j \cap A_4 = \{a_1^j\} = \{a_j^4\}$, $A_j \cap A_5 = \{a_2^j\} = \{a_j^5\}$,

(4) $B \cap A_4 = \{a_1^3\} = \{a_3^4\}$, $B \cap A_5 = \{a_2^3\} = \{a_3^5\}$, and

(5) $S_2 \subseteq V(G)$ satisfies $\{a_3^1, a_3^2, a_3^3\} \subseteq S_2 \subseteq \{a_3^1, a_2^1, a_3^1, a_1^2, a_2^2, a_3^2, a_3^3\}$.

Let $\mathcal{G}_2'$ be a family of pairs of graphs and their subsets of vertices (G, S_2') satisfying the following properties:

(1) G consists of five subgraphs A_1, A_2, A_4, A_5, B such that for each $i \in \{1, 2\}$, A_i is isomorphic to a member of $\mathcal{A}'$ containing a unique triangle T_i with $V(T_i) = \{a_1^i, a_2^i, a_3^i\}$ (a_1^i and a_2^i are bases of A_i), for each $i \in \{4, 5\}$, A_i is isomorphic to a member of $\mathcal{A}$ containing a unique triangle T_i with $V(T_i) = \{a_1^i, a_2^i, a_3^i\}$ and B' is isomorphic to a member of $\mathcal{B}'$ containing a unique path T_3 with $V(T_3) = \{a_1^3, a_2^3, a_3^3\}$, $E(T_j) = \{a_1^j a_3^j, a_2^j a_3^j\}$,

(2) A_1, A_2, B are pairwise disjoint, and $A_4 \cap A_5 = \emptyset$,

(3) for each $j \in \{1, 2\}$, $A_j \cap A_4 = \{a_1^j\} = \{a_j^4\}$, $A_j \cap A_5 = \{a_2^j\} = \{a_j^5\}$,

(4) $B' \cap A_4 = \{a_1^3\} = \{a_3^4\}$, $B' \cap A_5 = \{a_2^3\} = \{a_3^5\}$, and

(5) $S_2' \subseteq V(G)$ satisfies $\{a_3^1, a_3^2, a_3^3\} \subseteq S_3' \subseteq \{a_3^1, a_2^1, a_3^1, a_1^2, a_2^2, a_3^2, a_1^3, a_2^3, a_3^3\}$.

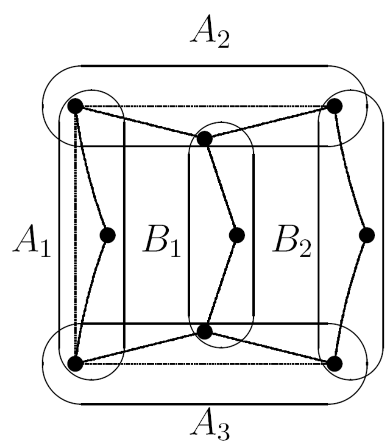

Figure 7. A graph $G \in \mathcal{G}_3$.

Let $\mathcal{G}_3$ be a family of pairs of graphs and their subsets of vertices (G, S_3) satisfying the following properties:

(1) G consists of five subgraphs A_1, A_4, A_5, B_2, B_3 such that A_1 is isomorphic to a member of $\mathcal{A}'$ containing a unique triangle T_i with $V(T_i) = \{a_1^1, a_2^1, a_3^1\}$ (a_1^1 and a_2^1 are bases of A_i), for each $i \in \{4, 5\}$, A_i is isomorphic to a member of $\mathcal{A}$ containing a unique triangle T_i with $V(T_i) = \{a_1^i, a_2^i, a_3^i\}$ and for each $j \in \{2, 3\}$, B_j is isomorphic to a member of $\mathcal{B}$ containing a unique path T_j with $V(T_j) = \{a_1^j, a_2^j, a_3^j\}$, $E(T_j) = \{a_1^j a_3^j, a_2^j a_3^j\}$,

(2) A_1, B_2, B_3 are pairwise disjoint, and $A_4 \cap A_5 = \varnothing$,

(3) $A_1 \cap A_4 = \{a_1^1\} = \{a_1^4\}$, $A_1 \cap A_5 = \{a_2^1\} = \{a_1^5\}$,

(4) for each $j \in \{2, 3\}$, $B_j \cap A_4 = \{a_1^j\} = \{a_j^4\}$, $B_j \cap A_5 = \{a_2^j\} = \{a_j^5\}$, and

(5) $S_3 \subseteq V(G)$ satisfies $\{a_3^1, a_3^2, a_3^3\} \subseteq S_3 \subseteq \{a_3^1, a_2^1, a_3^1, a_3^2, a_3^3\}$.

Let $\mathcal{G}_4$ be a family of pairs of graphs and their subsets of vertices (G, S_4) satisfying the following properties:

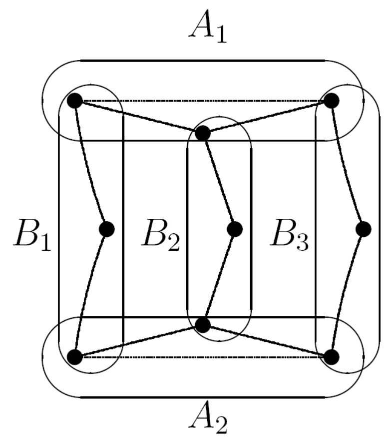

Figure 8. A graph $G \in \mathcal{G}_4$.

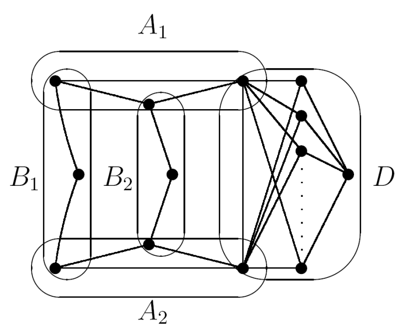

Figure 9. A graph $G \in \mathcal{G}_5$.

(1) G consists of five subgraphs B_1, B_2, B_3, A_4, A_5 such that for each $i \in \{4, 5\}$, A_i is isomorphic to a member of $\mathcal{A}$ containing a unique triangle T_i with $V(T_i) = \{a_1^i, a_2^i, a_3^i\}$ and for each $j \in \{1, 2, 3\}$, B_j

is isomorphic to a member of $\mathcal{B}$ containing a unique path T_j with $V(T_j) = \{a_1^j, a_2^j, a_3^j\}$, $E(T_j) = \{a_1^j a_3^j, a_2^j a_3^j\}$,

(2) B_1, B_2, B_3 are pairwise disjoint, and $A_4 \cap A_5 = \emptyset$,

(3) for each $i \in \{1, 2, 3\}$, $B_i \cap A_4 = \{a_1^i\} = \{a_i^4\}$, $B_i \cap A_5 = \{a_2^i\} = \{a_i^5\}$, and

(4) $S_4 \subseteq V(G)$ satisfies $S_4 = \{a_3^1, a_3^2, a_3^3\}$.

Let $\mathcal{G}_5$ be a family of pairs of graphs and their subsets of vertices (G, S_5) satisfying the following properties:

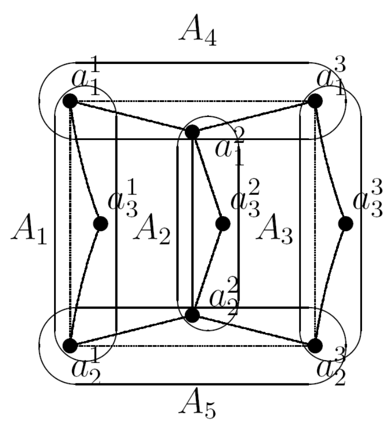

Figure 10. (G, S).

(1) G consists of five subgraphs B_1, B_2, A_4, A_5, D such that for each $i \in \{4, 5\}$, A_i is isomorphic to a member of $\mathcal{A}$ containing a unique triangle T_i with $V(T_i) = \{a_1^i, a_2^i, a_3^i\}$, for each $j \in \{1, 2\}$, B_j is isomorphic to a member of $\mathcal{B}$ containing a unique path T_j with $V(T_j) = \{a_1^j, a_2^j, a_3^j\}$, $E(T_j) = \{a_1^j a_3^j, a_2^j a_3^j\}$, and D is isomorphic to a member of $\mathcal{C}$ containing a graph X $(V(X) = \{x_1, x_2, x_3, x_4, x_5\}$, $E(X) = \{x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_5, x_4x_5\})$,

(2) B_1, B_2, D are pairwise disjoint, and $A_4 \cap A_5 = \emptyset$,

(3) for each $i \in \{1, 2\}$, $B_i \cap A_4 = \{a_1^i\} = \{a_i^4\}$, $B_i \cap A_5 = \{a_2^i\} = \{a_i^5\}$,

(4) $D \cap A_4 = \{x_1\} = \{a_3^4\}$, $D \cap A_5 = \{x_2\} = \{a_3^5\}$, and

(5) $S_5 \subseteq V(G)$ satisfies $S_5 = \{a_3^1, a_3^2, x_3\}, \{a_3^1, a_3^2, x_4\}, \{a_3^1, a_3^2, x_5\}$, $\{a_3^1, a_3^2, x_3, x_5\}$ or $\{a_3^1, a_3^2, x_4, x_5\}$.

Let $\mathcal{G}_S$ be a set of pairs of a graph G and a subset of $V(G)$ such that

$$\mathcal{G}_S = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_2' \cup \mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5.$$

Proposition 3. *If* $(G, S) \in \mathcal{G}_S$, *then G is a* 2-*conncected* $CZ_3(S)$*-free graph.*

Proof. We suppose $(G, S) \in \mathcal{G}_1$. We assume that (G, S) satisfies the following properties:

(1) G contains three subgraphs A_1, A_2, A_3 such that for each $i \in \{1, 2, 3\}$, A_i is isomorphic to a member of $\mathcal{A}'$ containing a unique triangle T_i with $V(T_i) = \{a_1^i, a_2^i, a_3^i\}$ (a_1^i and a_2^i are bases of A_i),

(2) G contains two subgraphs A_4, A_5 such that for each $i \in \{4, 5\}$, A_i is isomorphic to a member of $\mathcal{A}$ containing a unique triangle T_i with $V(T_i) = \{a_1^i, a_2^i, a_3^i\}$,

(3) A_1, A_2, A_3 are pairwise disjoint, and $A_4 \cap A_5 = \emptyset$,

(4) for each $j \in \{1, 2, 3\}$, $A_j \cap A_4 = \{a_1^j\} = \{a_j^4\}$, $A_j \cap A_5 = \{a_2^j\} = \{a_j^5\}$, and

(5) $\{a_3^1, a_3^2, a_3^3\} \subseteq S \subseteq \{a_1^1, a_2^1, a_3^1, a_1^2, a_2^2, a_3^2, a_1^3, a_2^3, a_3^3\}$.

First, we assume $S = \{a_1^1, a_2^1, a_3^1, a_1^2, a_2^2, a_3^2, a_1^3, a_2^3, a_3^3\}$. Assume to the contrary, suppose that G contains an induced subgraph F which is a member of $C(S)$ or $Z_3(S)$.

Case 1. F is a member of $C(S)$.

By the definition of $C(S)$, F contains three independent vertices in S such that these vertices are leaves of F.

Subclaim 1.1. $\{a_3^1, a_3^2, a_3^3\} \subseteq V(F)$ and these vertices are leaves of F.

Since F is a connected graph and $\{a_1^i, a_2^i\}$ $(i \in \{1, 2, 3\})$ is a cut set in G, F contains three vertices a_h^1, a_j^2 and a_k^3 $(h, j, k = 1 \text{ or } 2)$, If $h, j, k = 1$, then $\langle a_1^1, a_1^2, a_1^3 \rangle$ is isomorphic to K_3. But F contains no K_3, a contradiction. By the same argument, if $h, j, k = 2$, then we can lead a contradiction. If $h, j = 1$ and $k = 2$, then F does not contain a_1^3. On the other hand, by the definition of $\mathcal{G}_1$, for any a_3^2-a_2^3 path P, P contains a_2^1, a_2^2 or a_1^3. Hence F contains a_2^1 or a_2^2. Since F is an induced subgraph in G, we obtain a $d_F(a_3^1) \geq 2$ or $d_F(a_3^2) \geq 2$. But a_3^1 and a_3^2 are leaves of F, a contradiction. Similarly, we can prove the proposition for the other cases.

Subclaim 1.2. $\{a_3^i, a_3^j, a_2^k\}$ or $\{a_3^i, a_3^j, a_1^k\} \subseteq V(F)$ $(\{i, j, k\} = \{1, 2, 3\})$ and these vertices are leaves of F.

We suppose $\{a_3^1, a_3^2, a_2^3\} \subseteq V(F)$ and these three vertices are leaves of F. By the same argument in Case 1, F contains three vertices a_h^1, a_j^2 and a_k^3 $(h, j, k = 1 \text{ or } 2)$. If $h = j = k$, then we can find a triangle in F, a contradiction. By $a_2^3 \in V(F)$, a_2^1 or a_2^2 are not contained in $V(F)$. Without loss of generality, we may assume $a_2^2 \notin V(F)$. Hence $V(F)$ contains a vertex a_1^2. For any a_1^2-a_2^3 path P which does not contain a_2^2, P contains $\{a_1^3\}$ or $\{a_1^1, a_2^1\}$. If $\{a_1^1, a_2^1\} \subseteq V(F)$, then $d_F(a_3^1) \geq 2$, a contradiction. Hence $a_1^3 \in V(F)$. Since F contains no triangle, $a_1^1 \notin V(F)$, otherwise we can find a triangle in F. We obtain $a_2^1 \in V(F)$. But since F is an induced subgraph in G and $a_2^1a_2^3, a_1^3a_2^3 \in E(G)$, we obtain $d_F(a_2^3) \geq 2$, a contradiction. By a similar argument, we can prove the proposition for the other cases.

Subclaim 1.3. $\{a_1^i, a_2^j, a_3^k\} \subseteq V(F)$ $(\{i, j, k\} = \{1, 2, 3\})$ and these vertices are leaves of F.

We suppose $\{a_1^1, a_2^2, a_3^3\} \subseteq V(F)$ and these vertices are leaves of F. Since $\{a_1^3, a_2^3\}$ is a two cut in G, F contains a_1^3 or a_2^3. Without loss of generality, we assume $a_1^3 \in V(F)$. Since a_1^1 is a leaf of F, F is an induced subgraph in G and $a_1^1a_1^3, a_1^1a_2^1 \in E(G)$, we obtain $a_2^1 \notin V(F)$. Similarly, we obtain $a_2^3 \notin V(F)$. Since $\{a_2^1, a_1^2, a_2^3\}$ is a minimal three cut in G, F contains a vertex a_1^2. But $\langle\{a_1^1, a_1^2, a_1^3\}\rangle$ is a triangle. Hence we can find a triangle in F, a contradiction. By a similar argument, we can prove the proposition for the other cases.

Case 2. F is a member of $Z_3(S)$.

By the definition of $Z_3(S)$, F contains a triangle T and a vertex $x \in S$. By the definition of $\mathcal{G}_1$, G contains only five triangles $\langle\{a_1^1, a_2^1, a_3^1\}\rangle$, $\langle\{a_1^2, a_2^2, a_3^2\}\rangle$, $\langle\{a_1^3, a_2^3, a_3^3\}\rangle$, $\langle\{a_1^1, a_1^2, a_1^3\}\rangle$ and $\langle\{a_2^1, a_2^2, a_2^3\}\rangle$.

Subclaim 2.1. F contains $\langle\{a_1^1, a_2^1, a_3^1\}\rangle$, $\langle\{a_1^2, a_2^2, a_3^2\}\rangle$ or $\langle\{a_1^3, a_2^3, a_3^3\}\rangle$.

We assume that F contains $\langle\{a_1^1, a_2^1, a_3^1\}\rangle$. By $a_1^1a_1^2, a_1^1a_1^3, a_2^1a_2^2, a_2^1a_2^3 \in E(G)$, $x \notin \{a_1^2, a_2^2, a_1^3, a_2^3\}$. Hence $x \in \{a_3^2, a_3^3\}$. We suppose $x = a_3^2$. Since $\{a_1^2, a_2^2\}$ is a cut set in G, F contains a_1^2 or a_2^2. But $a_1^1a_1^2, a_2^1a_2^2 \in E(G)$. Hence, a v-$V(T)$ path P in F has length 2. But it contradicts the definition of $Z_3(S)$. By a similar argument, we can prove the proposition for the other cases.

Subclaim 2.2. F contains $\langle\{a_1^1, a_1^2, a_1^3\}\rangle$ or $\langle\{a_2^1, a_2^2, a_2^3\}\rangle$.

We assume that F contains $\langle\{a_1^1, a_1^2, a_1^3\}\rangle$. By the definition of $Z_3(S)$, $x \in S \backslash V(T)$. By $a_1^1a_2^1, a_1^1a_3^1, a_1^2a_2^2, a_1^2a_3^2, a_1^3a_2^3, a_1^3a_3^3 \in E(G)$, $E(V(T), \{x\}) \neq \emptyset$, a contradiction. By a similar argument, we can prove the proposition for the other cases.

By a similar argument, we can prove the proposition for

$$S \subsetneq \{a_1^1, a_2^1, a_3^1, a_1^2, a_2^2, a_3^2, a_1^3, a_2^3, a_3^3\}.$$

In the case $(G, S) \in \mathcal{G}_S \backslash \mathcal{G}_1$, by a similar argument, we can show that G is a 2-connetcted $CZ_3(S)$-free graph. □

Easily, we can prove the following proposition.

Proposition 4. *Let* $(G, S) \in \mathcal{G}_S$. *Then G contains no cycle D such that* $S \subseteq V(D)$.

By Propositions 3 and 4, every member of $\mathcal{G}_S$ is an exception of Problem 2. The following proposition says that the class $\mathcal{G}_S$ is a maximal in a sense as an exceptional class of Problem 2.

Proposition 5. *Let G be a graph and* $S \subseteq V(G)$ *such that* $(G, S) \in \mathcal{G}_S$. *Let* $H = G \cup P$ *be a graph obtained from G by adding a path P so that P is a* $V(G)$-$V(G)$ *path in H. Then H satisfies one of the following statements*:

(1) *H is not* $CZ_3(S)$*-free*,

(2) *H contains a cycle passing through all vertices of S*, *or*

(3) $(H, S) \in \mathcal{G}_S$.

Proof. We suppose $(G, S) \in \mathcal{G}_1$. We assume (G, S) satisfies the following properties:

(1) G contains three subgraphs A_1, A_2, A_3 such that for each $i \in \{1, 2, 3\}$, A_i is isomorphic to a member of $\mathcal{A}'$ containing a unique triangle T_i with $V(T_i) = \{a_1^i, a_2^i, a_3^i\}$ (a_1^i and a_2^i are bases of A_i),

(2) G contains two subgraphs A_4, A_5 such that for each $i \in \{4, 5\}$, A_i is isomorphic to a member of $\mathcal{A}$ containing a unique triangle T_i with $V(T_i) = \{a_1^i, a_2^i, a_3^i\}$,

(3) A_1, A_2, A_3 are pairwise disjoint, and $A_4 \cap A_5 = \emptyset$,

(4) for each $j \in \{1, 2, 3\}$, $A_j \cap A_4 = \{a_1^j\} = \{a_j^4\}$, $A_j \cap A_5 = \{a_2^j\} = \{a_j^5\}$ and

(5) $\{a_3^1, a_3^2, a_3^3\} \subseteq S \subseteq \{a_1^1, a_2^1, a_3^1, a_1^2, a_2^2, a_3^2, a_1^3, a_2^3, a_3^3\}$.

First, we assume $S = \{a_1^1, a_2^1, a_3^1, a_1^2, a_2^2, a_3^2, a_1^3, a_2^3, a_3^3\}$.

Let $G' = \langle\{a_1^1, a_2^1, a_3^1, a_1^2, a_2^2, a_3^2, a_1^3, a_2^3, a_3^3\}\rangle$. If for any $a_{i'}^i, a_{j'}^j$ such that $a_{i'}^i a_{j'}^j \notin E(G)$ $(i, j \in \{1, 2, 3, 4, 5\}, i', j' \in \{1, 2, 3\})$, $\langle\{H \setminus G'\} \cup \{a_{i'}^i, a_{j'}^j\}\rangle$ contains an $a_{i'}^i$-$a_{j'}^j$ path, then we can find a cycle D such that $S \subseteq V(D)$ in H. Hence there exists i such that P is an A_i-A_i path $(i \in \{1, 2, 3, 4, 5\})$.

Case 1. $i \in \{4, 5\}$.

We suppose $i = 4$. If for any triangle T in A_4, $E(P) \cap E(T) = \emptyset$, then $A_4 \in \mathcal{A}$. By the definition of $\mathcal{A}$, $(H, S) \in \mathcal{G}_1$. Hence there exists a triangle T such that $E(P) \cap E(T) \neq \emptyset$. Let P' be a $(V(T) \setminus \{a_1^1, a_1^2, a_1^3\})$-$\{a_1^1, a_1^2, a_1^3\}$ path such that $|V(P')|$ is as small as possible. Without loss of generality, we may assume $\{a_1^1\} \subseteq \{a_1^1, a_1^2, a_1^3\} \cap V(P')$. Let $a_1^{1^-} = N_{P'}(a_1^1)$. If $|N_H(a_1^{1^-}) \cap \{a_1^1, a_1^2, a_1^3\}| \geq 2$, then there exists $k \in \{2, 3\}$ such that $a_1^k \in$

$N_H(a_1^{1^-}) \cap \{a_1^1, a_1^2, a_1^3\}$. In this case, $\langle\{a_1^{1^-}, a_1^1, a_1^k, a_2^1, a_2^h, a_3^h\}\rangle$ is a member of $Z_3(S)$ $(h \notin \{1, k\})$. Hence H is not $CZ_3(S)$-free. By the above argument, for any $x \in V(A_4)\backslash\{a_1^1, a_1^2, a_1^3\}$, we obtain $|N_H(x) \cap \{a_1^1, a_1^2, a_1^3\}| \leq 1$. Hence $|V(T) \cap \{a_1^1, a_1^2, a_1^3\}| \leq 1$. We obtain $N_H(a_1^{1^-}) \cap \{a_1^1, a_1^2, a_1^3\} = \{a_1^1\}$. If there exists a vertex $a' \in V(T)\backslash\{a_1^{1^-}\}$ such that $a'a_1^1 \in E(G)$, then $\langle\{a', a_1^{1^-}, a_1^1, a_2^1, a_2^2, a_3^2\}\rangle$ is a member of $Z_3(S)$. By the above argument, there exists $k \in \{2, 3\}$ such that $\langle V(T) \cup V(P') \cup \{a_2^1, a_2^k, a_3^k\}\rangle$ contains a member of $Z_3(S)$. We obtain H is not $CZ_3(S)$-free.

By a similar argument, we can prove the proposition for $i = 5$.

Case 2. $i \in \{1, 2, 3\}$.

We suppose $i = 1$. If for any triangle T in A_1, $E(P) \cap E(T) = \emptyset$, then $A_1 \in \mathcal{A}'$. Hence by the definition of $\mathcal{A}'$, $(H, S) \in \mathcal{G}_1$. There exists a triangle T such that $E(P) \cap E(T) \neq \emptyset$. Let P' be a $(V(T)\backslash\{a_1^1, a_2^1, a_3^1\})$-$\{a_1^1, a_2^1, a_3^1\}$ path such that $|V(P')|$ is as small as possible. We suppose $\{a_3^1\} \subseteq \{a_1^1, a_2^1, a_3^1\} \cap V(P')$. Let $a_3^{1^-} = N_{P'}(a_3^1)$. If $|N_H(a_3^{1^-}) \cap \{a_1^1, a_2^1, a_3^1\}| \geq 2$, then without loss of generality, we may assume $\{a_1^1, a_3^1\} \subseteq N_H(a_3^{1^-}) \cap \{a_1^1, a_2^1, a_3^1\}$. $\langle\{a_3^{1^-}, a_1^1, a_3^1, a_1^2, a_2^2, a_2^3\}\rangle$ is a member of $Z_3(S)$. Hence $\{a_3^1\} = N_H(a_3^{1^-}) \cap \{a_1^1, a_2^1, a_3^1\}$. If $N_G(V(T)\backslash\{a_3^{1^-}\}) \cap \{a_1^1, a_2^1, a_3^1\} = \emptyset$, $\langle V(T) \cup V(P') \cup \{a_2^1, a_2^2, a_3^2\}\rangle$ contains a member of $Z_3(S)$. If there exists $i \in \{1, 2\}$ such that $|E(V(T), \{a_i^1\})| = 2$, then $\langle V(F) \cup \{a_i^1, a_i^2, a_j^2, a_j^3\}\rangle$ contains a member of $Z_3(S)$ $(j \in \{1, 2\}\backslash\{i\})$. If for any $i \in \{1, 2\}$, $|E(V(T), \{a_i^1\})| = 0$, then there exists a vertex $a' \in V(T)\backslash\{a_3^{1^-}\}$ such that $a'a_3^1 \in E(G)$. By the choice of P', $a_3^{1^-} \in V(T)$. $\langle\{a', a_3^{1^-}, a_3^1, a_1^1, a_1^2, a_3^2\}\rangle$ contains a member of $Z_3(S)$. Hence for any $i \in \{1, 2\}$, $|E(V(T), \{a_i^1\})| \le 1$ and there exists $i \in \{1, 2\}$ such that $|E(V(T), \{a_i^1\})| = 1$. But $\langle V(T) \cup \{a_i^1, a_i^2, a_j^2, a_j^3\}\rangle$ contains a member of $Z_3(S)$ $(j \in \{1, 2\}\backslash\{i\})$. Therefore, we obtain $a_3^1 \notin \{a_1^1, a_2^1, a_3^1\} \cap V(P')$. Without loss of generality, we can assume $a_1^1 \in \{a_1^1, a_2^1, a_3^1\} \cap V(P')$. If $E(V(T), \{a_1^1, a_2^1, a_3^1\}) = \emptyset$, then $\langle V(T) \cup V(P') \cup \{a_1^2, a_2^2, a_2^3\}\rangle$ contains a member of $Z_3(S)$. Hence $E(V(T), \{a_1^1, a_2^1, a_3^1\}) \neq \emptyset$. By the choice of P', $E(V(T), \{a_1^1\}) \neq \emptyset$. If $|E(V(T)\backslash\{a_2^1\}, \{a_1^1\})| \ge 2$, let $a', a'' \in V(T)\backslash\{a_2^1\}$ such that $a'a_1^1, a''a_1^1 \in E(G)$. But $\langle\{a', a'', a_1^1, a_1^2, a_2^2, a_2^3, a_3^3\}\rangle$ contains a member of $Z_3(S)$. Hence $|E(V(T)\backslash\{a_2^1\}, \{a_1^1\})| = 1$. Let $a' \in V(T)\backslash\{a_2^1\}$ such that $a'a_1^1 \in E(G)$. If $V(T) = \{a', a_1^1, a_2^1\}$, then (H, S) is also a member of $\mathcal{G}_1$. Suppose $V(T) \neq \{a', a_1^1, a_3^1\}$. If $V(T) = \{a', a_1^1, a_3^1\}$, then $\langle\{a', a_1^1, a_3^1, a_1^2, a_2^2, a_2^3\}\rangle$ contains a member of $Z_3(S)$. Other cases, $\langle V(T) \cup V(P') \cup \{a_1^1, a_1^2, a_3^2, a_2^3\}\rangle$ contain a member of $Z_3(S)$. By a similar argument, we can prove the proposition for $i \in \{2, 3\}$.

Similarly, we can prove the proposition for

$$S \subsetneq \{a_1^1, a_2^1, a_3^1, a_1^2, a_2^2, a_3^2, a_1^3, a_2^3, a_3^3\}.$$

By a similar argument, we can also prove the proposition for the case $(G, S) \in \mathcal{G}_S \backslash \mathcal{G}_1$. $\square$

Finally in this section, we propose the following conjecture.

Conjecture 6. Let G be a connected graph and $S \subseteq V(G)$. If G is a $CZ_3(S)$-free graph, then G contains a cycle D such that $S \subseteq V(D)$ or $(G, S) \in \mathcal{G}_S$.

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