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SUPPORT POINTS AND SUBORDINATION

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Abstract

Support points of the subordination family $s(F)$ are determined for an analytic function $F \in Hs(K)$. We show that $\operatorname{supp} s(F) = \{F \circ \phi : \phi \in \operatorname{supp} B_0\}$. As a corollary we obtain $\operatorname{supp} s(F)$ for $F \in H^{\infty}$ and prove the Abu-Muhanna conjecture.

1. Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. $A(\Delta)$ denotes the linear space of functions analytic in Δ with the topology of uniform convergence on compact sets. $A(\Delta)$ is locally convex. Let $A(\Delta)^*$ be the space of continuous linear functionals on $A(\Delta)$.

The Krein-Milman theorem holds for every compact subset F of $A(\Delta)$. If HF denotes the closed convex hull of F and EHF denotes the set of its extreme points, then $HF = HEHF$. Furthermore, $F \supset EHF$ and for every functional $J \in A(\Delta)^*$,

$$\max_{f \in F} \operatorname{Re} J(f) = \max_{f \in EHF} \operatorname{Re} J(f).$$

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Let B_0 denote the class of functions $\phi \in A(\Delta)$ such that $|\phi(z)| < 1$, $z \in \Delta$, and $\phi(0) = 0$. Let $f, F \in A(\Delta)$. Then f is said to be *subordinate* to F if and only if there exists a function $\phi \in B_0$ such that $f = F \circ \phi$. The class of functions subordinate to F is denoted by $s(F)$.

A function f is called a *support point* of a compact subset F of $A(\Delta)$ if $f \in A(\Delta)$ and there exists a functional $J \in A(\Delta)^*$ such that $\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) : g \in F\}$ and $\operatorname{Re} J$ is non-constant on F. The set of support points of F is denoted by $\operatorname{supp} F$. Each $J \in A(\Delta)^*$ is uniquely represented by a sequence of complex numbers $\{b_n\}_{n=0}^{\infty}$ such that $\limsup_{n\to\infty} \sqrt[n]{|b_n|} < 1$ and $J(f) = \sum_{n=0}^{\infty} b_n a_n$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in Δ [7, p. 36].

The set $\operatorname{supp} B_0$ consists of all finite Blashke products in B_0 [2].

Abu-Muhanna proved in [1] that $\operatorname{supp} s(F) \subset \{F \circ \phi : \phi \in \operatorname{supp} B_0\}$ for any non-constant F, $F \in A(\Delta)$. There are a few known cases in which equality is attained. For example,

$$\operatorname{supp} s(F) = \{F \circ \phi : \phi \in \operatorname{supp} B_0\} \tag{1}$$

for $F \in K$, where K denotes the class of univalent convex mappings on Δ with $F(0) = F'(0) - 1 = 0$ [5]. The equality holds also for any non-constant function F analytic in the closed unit disc [1].

The class of bounded analytic functions is denoted by H^{∞} and

$$\| f \|_{\infty} = \lim_{r\to 1} \max_{0 \le \theta \le 2\pi} | f(re^{i\theta}) |.$$

Also in [1], Abu-Muhanna conjectured that (1) holds for any bounded analytic function. We prove this conjecture as a corollary to the main result in Section 2.

2. Support Points of the Subordination Families with Majorants Subordinate to Convex Functions

Let $s(K) = \{f \in A(\Delta) : \exists_{F \in K}\ f \in s(F)\}$. This is to say, each f in $s(K)$ is subordinate to a univalent convex mapping on Δ, with the standard normalization at 0.

The set of extreme points of the closed convex hull $EHs(K)$ of $s(K)$ consists of the functions $yz/(1 - xz)$, where $|x| = |y| = 1$ [4]. We use this fact to prove that (1) occurs whenever $F \in Hs(K)$.

Theorem 1. *If* $F \in Hs(K)$, *then* $\operatorname{supp} s(F) = \{F \circ \phi : \phi \in \operatorname{supp} B_0\}$.

Proof. We need only to prove that if $\phi \in \operatorname{supp} B_0$, then $F \circ \phi \in \operatorname{supp} s(F)$. Let $\bar{\phi}(z) = \overline{\phi(\bar{z})}$ and let $J \in [A(\Delta)]^*$ be given by coefficients of $\bar{\phi}$. Then

$$J(f) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\bar{\phi}\left(\frac{e^{-i\theta}}{r}\right)d\theta$$

for $r < 1$ sufficiently close to 1. It is easy to see that

$$J(\phi) = \frac{1}{2\pi}\int_0^{2\pi}\phi(e^{i\theta})\overline{\phi(e^{i\theta})}d\theta = 1.$$

Lemma 7.18 in [3] implies that $J(\phi^n) = 0$, for $n = 2, 3, \ldots$.

Hence $J(F \circ \phi) = J(\phi) = 1$. We also have

$$\max_{f \in s(F)} \operatorname{Re} J(f) \le \max_{f \in s(K)} \operatorname{Re} J(f) \le \max_{f \in Hs(K)} \operatorname{Re} J(f)$$

$$= \max_{f \in EHs(K)} \operatorname{Re} J(f) = \max_{x,\ y \in \partial\Delta} \operatorname{Re} J\left(\frac{yz}{1 - xz}\right).$$

Assume now that $\psi \in A(\bar{\Delta})$. Then, for $r < 1$ and sufficiently close to

1 and for $x \in \partial\Delta$,

$$\psi(x) = \frac{1}{2\pi i}\int_{|\xi|=1/r}\frac{\psi(\xi)}{\xi - x}\,d\xi$$

$$= \frac{1}{2\pi i}\int_{2\pi}^{0}\frac{\psi\left(\frac{e^{-i\theta}}{r}\right)}{\frac{e^{-i\theta}}{r} - x}\left(\frac{-ie^{-i\theta}}{r}\right)d\theta = \frac{1}{2\pi}\int_{0}^{2\pi}\frac{\psi\left(\frac{e^{-i\theta}}{r}\right)\frac{e^{-i\theta}}{r}}{\frac{e^{-i\theta}}{r} - x}\,d\theta$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi}\frac{\psi\left(\frac{e^{-i\theta}}{r}\right)}{1 - xre^{i\theta}}\,d\theta.$$

Let $\psi(z) = \frac{\bar{\phi}(z)}{z}$. Since $\bar{\phi}(0) = 0$ and $\bar{\phi} \in A(\bar{\Delta})$ also $\psi \in A(\bar{\Delta})$.

Hence,

$$J\left(\frac{yz}{1 - xz}\right) = \frac{1}{2\pi}\int_{0}^{2\pi}\frac{yre^{i\theta}}{1 - xre^{i\theta}}\,\bar{\phi}\left(\frac{e^{-i\theta}}{r}\right)d\theta = y\,\frac{1}{2\pi}\int_{0}^{2\pi}\frac{re^{i\theta}\bar{\phi}\left(\frac{e^{-i\theta}}{r}\right)}{1 - xre^{i\theta}}\,d\theta$$

$$= y\,\frac{1}{2\pi}\int_{0}^{2\pi}\frac{\varphi\left(\frac{e^{-i\theta}}{r}\right)}{1 - xre^{i\theta}}\,d\theta = y\psi(x) = y\,\frac{\bar{\phi}(x)}{x}.$$

It follows that $\max\limits_{x, y \in \partial\Delta} \operatorname{Re} J\left(\frac{yz}{1 - xz}\right) = \max\limits_{x, y \in \partial\Delta} \operatorname{Re}\left(y\,\frac{\bar{\phi}(x)}{x}\right) \le \max\limits_{x, y \in \partial\Delta}\left|\frac{y\bar{\phi}(x)}{x}\right| = 1.$

Therefore, $\max\limits_{f \in s(F)} \operatorname{Re} J(f) \le 1.$

Finally, since $F \circ \phi^2 \in s(F)$ and $J(F \circ \phi^2) = J(\phi^2) = 0$, $\operatorname{Re} J \ne \text{const}$ on $s(F)$. Hence $F \circ \phi \in \operatorname{supp} s(F)$.

Since $[f(z) - f(0)]/\| f \|_\infty \in s(z)$, for any $f \in H^\infty$, and $z \in K$, we have the following corollary.

Corollary. *If* $F \in H^\infty$, *then* $\operatorname{supp} s(F) = \{F \circ \phi : \phi \in \operatorname{supp} B_0\}$.

References

[1] Yusuf Abu-Muhanna, Variability regions and support points of subordinate families, J. London Math. Soc. (2) 29 (1984), 477-484.

[2] P. Cochrane and T. H. MacGregor, Frechet differentiable functionals and support points for families of analytic functions, Trans. Amer. Math. Soc. 236 (1978), 75-92.

[3] D. J. Hallenbeck and A. E. Livingston, Applications of extreme point theory to classes of multivalent functions, Trans. Amer. Math. Soc. 221 (1976), 339-359.

[4] D. J. Hallenbeck and T. H. MacGregor, Subordination and extreme point theory, Pacific Math. J. 50 (1974), 455-468.

[5] D. J. Hallenbeck and T. H. MacGregor, Support points of families of analytic functions described by subordination, Trans. Amer. Math. Soc. 278 (1983), 523-546.

[6] D. J. Hallenbeck and Katarzyna Tkaczyńska, Extreme points and support points of subordination families with p-valent majorants, Annales Polonici Mathematici L (1989), 93-115.

[7] G. Schober, Univalent functions – selected topics, Lecture Notes in Math. 478, Springer-Verlag, Berlin, Heidelberg, New York, 1975.

[8] Katarzyna Tkaczyńska, On extreme points of subordination families with a convex majorant, J. Math. Anal. Appl. 145(1) (1990), 216-231.

[9] Katarzyna Tkaczyńska, Extreme points of classes of analytic functions with positive real part and restricted second coefficient, Math. Japon. 36(2) (1991), 221-223.

[10] Katarzyna Tkaczyńska-Hallenbeck, On properties of extreme points of subordination families with a convex majorant, Math. Japon. 41(3) (1995), 537-543.

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