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SOLVING A SYSTEM OF THIRD-ORDER OBSTACLE BOUNDARY VALUE PROBLEMS WITH BERNSTEIN POLYNOMIAL

Feng Gao

Science School
Qingdao Technological University
Qingdao, 266033, P. R. China
e-mail: gaofeng99@sina.com

Abstract

In this paper, we use Bernstein polynomial to approximate the analytical solution of a system of third-order nonlinear boundary problems associated with obstacle, unilateral and contact problems. A new approach is illustrated to provide the solution in the form of Bernstein polynomial, which dramatically reduces the size of work. An example is given to illustrate the efficiency of the method.

1. Introduction

In recent years, much attention has been given to solving the system of high order boundary problems, see [1-10]. In this paper, we consider the following systems of third-order nonlinear boundary value problems:

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$$\begin{cases} u'''(x) = a_1 u(x) + r_1(x), & a < x < c, \\ u'''(x) = a_2 u(x) + r_2(x), & c < x < d, \\ u'''(x) = a_3 u(x) + r_3(x), & d < x < b \end{cases} \tag{1.1}$$

with boundary conditions:

$$u(a) = \alpha, \quad u'(a) = \beta, \quad u(b) = \gamma \tag{1.2}$$

or

$$u(a) = \alpha, \quad u(b) = \beta, \quad u'(b) = \gamma,$$

and continuity conditions of $u(x), u'(x), u''(x)$ at internal points c and d of the interval $[a, b]$.

Here α, β, γ are real constants, and $r_i(x), i = 1, 2, 3$ are continuous functions on corresponding intervals. Such type of problems arise in the study of obstacle, contact, unilateral and equilibrium problems in economics, transportation, nonlinear optimization, fluid flow through porous media and some other branches of pure and applied sciences. Some techniques have been used to solve this type of problem. Noor and Tirmizi [7] applied finite difference method for unilateral problems, Al-Said et al. [2] used finite-difference method for obstacle problems, Momani et al. [5] used decomposition method for solving a system of obstacle problems, and Gao and Chi [3] applied quadric B-spline method for third-order obstacle problems. Noor et al. [8] used modified variation of parameters for solving a system of third-order boundary problems. These authors mainly considered the case $r_i(x) = r_i = \text{const}$ and $r_1 = r_3$. Some of these methods are numerical and require huge computational work. Here, we consider Bernstein polynomial technique. In the literature, some scholars employ Bernstein polynomials to numerically solve ODE and PDE (see [11]). The advantage of Bernstein polynomials is that this is a kind of tool that is easy to manipulate especially when it comes to approximating piecewise smooth function (say the solution associated with obstacle problems). The main idea is to approximate the solution with Bernstein polynomials piece by piece and at the same time connect the piecewise Bernstein polynomials at the joint points with some degree of smoothness. An important advantage of Bernstein polynomial technique is that by using Bernstein polynomials, we can void discretization in computing. In this paper, we illustrate a Bernstein polynomial method to solve the system of third-order boundary problems (1.1) with boundary conditions (1.2). It proves that this technique makes the solution procedure simple while still maintaining the higher accuracy. Also, the suggested technique is applied without any discretization, perturbation, decomposition, without any need to memorize special polynomials (Adomian polynomials, Hermite polynomials, etc.). An example is given to illustrate the proposed technique. We arrange our results as below. In Section 2, we introduce some basics for Bernstein polynomial. In Section 3, we put forward our method based on Bernstein polynomial and carry out a numerical test. And finally, Section 4 is our conclusion.

2. Some Basics for Bernstein Polynomial

We list below some basics for Bernstein basis polynomials. The $n + 1$ Bernstein basis polynomials of degree n are defined as

$$B_{i,n}(x) = \binom{n}{i} x^i (1 - x)^{n-i}, \quad i = 0, 1, \ldots, n,$$

where $\binom{n}{i}$ is a binomial coefficient. The Bernstein basis polynomials of degree n form a basis for the vector space of polynomials of degree n. A linear combination of Bernstein basis polynomials

$$B(x) = \sum_{i=0}^{n} \beta_i B_{i,n}(x)$$

is called a *Bernstein polynomial* or *polynomial in Bernstein form of degree n*. The coefficients β_i are called *Bernstein coefficients*.

The Bernstein basis polynomials have the following properties:

$$B_{i,n}(x) \geq 0, \quad B_{i,n}(1 - x) = B_{n-i,n}(x) \text{ for } x \in [0, 1].$$

The derivative can be written as a combination of two polynomials of lower degree:

$$B'_{i,n}(x) = n(B_{i-1,n-1}(x) - B_{i,n-1}(x)). \tag{2.1}$$

For Bernstein polynomials $B(x) = \sum_{i=0}^{n} \beta_i B_{i,n}(x)$, based on (2.1), we can easily find that $B'(x) = n\sum_{i=0}^{n-1} (\beta_{i+1} - \beta_i) B_{i,n-1}(x)$. Therefore,

$$B'(0) = n(\beta_1 - \beta_0), \quad B''(0) = n(n-1)(\beta_2 - 2\beta_1 + \beta_0) \tag{2.2}$$

and

$$B'(1) = n(\beta_n - \beta_{n-1}), \quad B''(1) = n(n-1)(\beta_n - 2\beta_{n-1} + \beta_{n-2}), \tag{2.3}$$

which are actually also Bernstein polynomials of lower degree. This property is very useful when Bernstein polynomial is used to solve ODE because discretization of the corresponding interval can be voided by using this property. The Bernstein basis polynomials of degree n form a partition of unity:

$$\sum_{i=0}^{n} B_{i,n}(x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} = 1.$$

Bernstein basis polynomials are often used to approximate continuous functions: Let f be a continuous function on the interval $[0, 1]$. Consider the Bernstein polynomial

$$B_n(f)(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) B_{i,n}(x).$$

It can be shown that

$$B_n(f)(0) = f(0), \quad B_n(f)(1) = f(1)$$

and

$$\lim_{n\to\infty} B_n(f)(x) = f(x) \text{ and } \lim_{n\to\infty} \| f^{(k)} - B_n^{(k)}(f) \| \to 0, \; k = 0, 1, \ldots.$$

In many cases, Bernstein basis polynomials

$$B_{i,n}(x) = \frac{1}{(b-a)^n}\binom{n}{i}(x-a)^i(b-x)^{n-i} \tag{2.4}$$

defined on interval $[a, b]$ are preferred than that defined on $[0, 1]$. In these cases, the Bernstein basis polynomials have the properties mentioned above accordingly too. In many cases, Bernstein polynomials are good tools to approximate the solution of differential equations because Bernstein polynomials are easy to manipulate, in particular, when it comes to approximate piecewise function which has certain continuity at the joint points.

3. Bernstein Polynomial Technique for Solving System of Third-order Obstacle Boundary Value Problems

We use Bernstein polynomials to approximate the solution of (1.1) with boundary conditions (1.2) numerically. Let

$$u(x) = \begin{cases} u_1(x), & a \le x \le c, \\ u_2(x), & c \le x \le d, \\ u_3(x), & d \le x \le b \end{cases}$$

be the solution of (1.1) with boundary conditions (1.2), where

$$u_1(x) = \sum_{i=0}^{n} p_i B_{i,n}^1(x), \quad u_2(x) = \sum_{i=0}^{n} q_i B_{i,n}^2(x) \text{ and}$$

$$u_3(x) = \sum_{i=0}^{n} w_i B_{i,n}^3(x), \tag{3.1}$$

$B^1_{i,n}(x)$, $B^2_{i,n}(x)$ and $B^3_{i,n}(x)$, $i = 0, 1, ..., n$ are Bernstein basis polynomials defined on $[a, c]$, $[c, d]$, $[d, b]$, respectively, and p_i, q_i, w_i are coefficients which are to be defined to identify $u(x)$.

Substituting (3.1) into (1.1) and utilizing the boundary and continuity conditions, we can get a linear system of unknowns of p_i, q_i and w_i, $i = 0, 1, ..., n$. Solving this system, we can identify p_i, q_i and w_i, and thus get the approximation to the analytic solution of (1.1) in the form of Bernstein polynomials.

To illustrate the method clearly, we use 4 degree Bernstein polynomial to solve the below typical third-order obstacle boundary value problems:

$$u'''(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{4}, \\ u(x) - 1, & \frac{1}{4} \le x \le \frac{3}{4}, \\ 0, & \frac{3}{4} \le x \le 1. \end{cases}$$

We assume

$$u(x) = \begin{cases} u_1(x), & 0 \le x \le \frac{1}{4}, \\ u_2(x), & \frac{1}{4} \le x \le \frac{3}{4}, \\ u_3(x), & \frac{3}{4} \le x \le 1, \end{cases}$$

where

$$u_1(x) = \sum_{i=0}^{4} p_i B^1_{i,n}(x),\ u_2(x) = \sum_{i=0}^{4} q_i B^2_{i,n}(x) \text{ and } u_3(x) = \sum_{i=0}^{4} w_i B^3_{i,n}(x)$$

are Bernstein polynomials defined on $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{3}{4}\right]$ and $\left[\frac{3}{4}, 1\right]$, respectively. From (2.4), we can easily identify that

$$B_{i,n}^1(x) = \binom{4}{i}(4x)^i(1-4x)^{4-i}, \quad B_{i,n}^2(x) = \binom{4}{i}\left(2x - \frac{1}{2}\right)^i\left(\frac{3}{2} - 2x\right)^{4-i},$$

$$B_{i,n}^3(x) = \binom{4}{i}(4x-3)^i(4-4x)^{4-i}, \quad i = 0, 1, ..., n.$$

By the use of (2.1), (2.2) and (2.3), we differentiate $u_1(x)$ in interval $\left[0, \frac{1}{4}\right]$ for three times, and we get

$$u_1'''(x) = 4^3 \cdot 24(p_3 - 3p_2 + 3p_1 - p_0)(1-4x)$$
$$+ 4^3 \cdot 24(p_4 - 3p_3 + 3p_2 - p_1)(4x).$$

Actually, $u_1'''(x)$ is a Bernstein polynomial of degree 1, and we have

$$u_1'''(0) = 4^3 \cdot 24(p_3 - 3p_2 + 3p_1 - p_0) \text{ and}$$

$$u_1'''\left(\frac{1}{4}\right) = 4^3 \cdot 24(p_4 - 3p_3 + 3p_2 - p_1). \tag{3.2}$$

Similarly, we have

$$u_2'''(x) = 2^3 \cdot 24(q_3 - 3q_2 + 3q_1 - q_0)\left(\frac{3}{2} - 2x\right)$$
$$+ 2^3 \cdot 24(q_4 - 3q_3 + 3q_2 - q_1)\left(2x - \frac{1}{2}\right),$$

$$u_2'''\left(\frac{1}{4}\right) = 2^3 \cdot 24(q_3 - 3q_2 + 3q_1 - q_0),$$

$$u_2'''\left(\frac{3}{4}\right) = 2^3 \cdot 24(q_4 - 3q_3 + 3q_2 - q_1) \tag{3.3}$$

and

$$u_3'''(x) = 4^3 \cdot 24(w_3 - 3w_2 + 3w_1 - w_0)(4-4x)$$
$$+ 4^3 \cdot 24(w_4 - 3w_3 + 3w_2 - w_1)(4x-3),$$

$$u_3'''\left(\frac{3}{4}\right) = 4^3 \cdot 24(w_3 - 3w_2 + 3w_1 - w_0),$$

$$u_3'''(1) = 4^3 \cdot 24(w_4 - 3w_3 + 3w_2 - w_1). \tag{3.4}$$

From the boundary conditions $u(0) = 0$, $u'(0) = 0$, $u'(1) = 0$, we get $u_1(0) = 0$, $u_1'(0) = 0$, $u_3'(1) = 0$. By using the differentiation approach in (2.1) and (2.2), we have

$$\begin{cases} p_0 = 0, \\ p_0 - p_1 = 0, \\ -w_3 + w_4 = 0. \end{cases} \tag{3.5}$$

Because $u''' = 0$, we have $u_1''' = 0$ in the interval $\left[0, \frac{1}{4}\right]$, in particular, $u_1'''(0) = 0$, $u_1'''\left(\frac{1}{4}\right) = 0$. Substituting into (3.2), we get

$$\begin{cases} -p_0 + 3p_1 - 3p_2 + p_3 = 0, \\ -p_1 + 3p_2 - 3p_3 + p_4 = 0. \end{cases} \tag{3.6}$$

At the point $x = \frac{1}{4}$, we consider the continuity condition:

$$\begin{cases} u_1\left(\frac{1}{4}\right) = u_2\left(\frac{1}{4}\right), \\ u_1'\left(\frac{1}{4}\right) = u_2'\left(\frac{1}{4}\right), \\ u_1''\left(\frac{1}{4}\right) = u_2''\left(\frac{1}{4}\right). \end{cases}$$

By using the differentiation approach in (2.1) and (2.2), we get

$$\begin{cases} p_4 = q_{0,} \\ 2(p_4 - p_3) = q_1 - q_0, \\ 4(p_4 - 2p_3 + p_2) = q_2 - 2q_1 + q_0. \end{cases} \tag{3.7}$$

In the interval $\left[\frac{1}{4}, \frac{3}{4}\right]$, we have $u_2''' = u_2$, in particular, $u_2'''\left(\frac{1}{4}\right) = u_2\left(\frac{1}{4}\right)$, $u_2'''\left(\frac{3}{4}\right) = u_2\left(\frac{3}{4}\right)$, that leads to

$$\begin{cases} 2^3 \cdot 24(q_3 - 3q_2 + 3q_1 - q_0) = q_0 - 1, \\ 2^3 \cdot 24(q_4 - 3q_3 + 3q_2 - q_1) = q_4 - 1. \end{cases} \tag{3.8}$$

At the point $x = \frac{3}{4}$, we consider the continuity condition:

$$\begin{cases} u_2\left(\frac{3}{4}\right) = u_3\left(\frac{3}{4}\right), \\ u_2'\left(\frac{3}{4}\right) = u_3'\left(\frac{3}{4}\right), \\ u_2''\left(\frac{3}{4}\right) = u_3''\left(\frac{3}{4}\right) \end{cases}$$

and get

$$\begin{cases} q_4 - w_0 = 0, \\ -q_3 + q_4 + 2w_0 - 2w_1 = 0, \\ q_2 - 2q_3 + q_4 - 4w_0 + 8w_1 - 4w_2 = 0. \end{cases} \tag{3.9}$$

From the conditions $u_3'''\left(\frac{1}{4}\right) = 0$, $u_3'''(1) = 0$ and (3.4), we have

$$\begin{cases} w_3 - 3w_2 + 3w_1 - w_0 = 0, \\ w_4 - 3w_3 + 3w_2 - w_1 = 0. \end{cases} \tag{3.10}$$

We are now to determine $p_i, q_i, w_i, i = 0, 1, ..., n.$ It can be easily found $p_0 = p_1 = 0$ and for the rest unknowns, we put (3.5)-(3.10) together and get the linear system $AX = b,$ where

$$X = (p_2, p_3, p_4, q_0, q_1, q_2, q_3, q_4, w_0, w_1, w_2, w_3, w_4)^T,$$

$$b = (0, 0, 0, 0, 0, -1, -1, 0, 0, 0, 0, 0, 0)^T$$

and

$$A=\begin{pmatrix}
-3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -2 & 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
4 & -8 & 4 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -193 & 576 & -576 & 192 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -192 & 576 & -576 & 191 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 2 & -2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & -4 & 8 & -4 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -3 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -3 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{pmatrix}.$$

Solving this linear system, we get

$$(p_0,\ p_1,\ p_2,\ p_3,\ p_4) = \left(0,\ 0,\ \frac{94}{74005},\ \frac{289}{75842},\ \frac{289}{37921}\right),$$

$$(q_0,\ q_1,\ q_2,\ q_3,\ q_4) = \left(\frac{289}{37921},\ \frac{275}{18042},\ \frac{1055}{37754},\ \frac{157}{3871},\ \frac{131}{2722}\right),$$

$$(w_0,\ w_1,\ w_2,\ w_3,\ w_4) = \left(\frac{131}{2722},\ \frac{72}{1387},\ \frac{256}{4703},\ \frac{289}{5189},\ \frac{289}{5189}\right),$$

and thus the approximation solution $u(x)$ to (1.1) with boundary conditions (1.2) is identified. The analytic solution to (1.1) with boundary conditions (1.2) is

$$u(x) = \begin{cases} \frac{1}{2}a_1x^2, \quad 0 \le x \le \frac{1}{4}, \\ 1 + a_2e^x + e^{-\frac{x}{2}}\left(a_3\cos\frac{\sqrt{3}}{2}x + a_4\sin\frac{\sqrt{3}}{2}x\right), \quad \frac{1}{4} \le x \le \frac{3}{4}, \\ a_5x\left(\frac{1}{2}x - 1\right) + a_6, \quad \frac{3}{4} \le x \le 1, \end{cases}$$

where

$$a_1 = 0.14520742362098, \quad a_2 = -0.21130240827197,$$

$$a_3 = -0.78610085318732, \quad a_4 = -0.24585768969643,$$

$$a_5 = 0.05860440434801, \quad a_6 = 0.04768241777632.$$

It can be easily found out that the observed error of our Bernstein polynomial method of degree 4 is 10^{-3}. Although this accuracy is not as good as the result obtained by finite difference method, we can get much better result if higher degree Bernstein polynomials are employed.

4. Conclusion

In this paper, we illustrate a Bernstein polynomial technique to solve a typical system of third-order nonlinear boundary value problems. It should be noted that the technique used, in this paper, is a simple technique and can be applied to much more general complicated physical problems in the form of (1.1). Generally speaking, more accuracy can be achieved if more higher degree Bernstein polynomials are employed. This method provides a good solution and is capable of reducing the volume of the computational work and at the same time, discretization of interval can be avoided.

References

[1] E. A. Al-Said, Numerical solutions for system of third-order boundary value problems, Int. J. Comput. Math. 78 (2001), 111-122.

[2] E. A. Al-Said, M. A. Noor, D. Kaya and K. Al-Khalid, Finite-difference method for solving fourth-order obstacle problems, Int. J. Comput. Math. 81 (2004), 741-748.

[3] F. Gao and C. M. Chi, Solving third-order obstacle problems with quartic B-splines, Appl. Math. Comput. 180 (2006), 270-274.

[4] H. Lewy and G. Stampacchia, On the regularity of the solutions of a variational inequality, Comm. Pure Appl. Math. 22 (1969), 153-188.

[5] S. Momani, K. Moadi and M. A. Noor, Decomposition method for solving a system of fourth-order obstacle boundary value problems, Appl. Math. Comput. 175 (2006), 923-931.

[6] M. A. Noor, Some developments in general variational inequalities, Appl. Math. Comput. 152 (2004), 199-277.

[7] M. A. Noor and S. I. A. Tirmizi, Numerical methods for unilateral problems, J. Comput. Appl. Math. 16 (1986), 387-395.

[8] M. A. Noor, K. I. Noor, A. Waheed and Eisa A. Al-Said, An efficient method for solving system of third-order nonlinear boundary value problems, J. Math. Probl. Eng. 2011, Art. ID 250184, 14 pp.

[9] M. A. Noor and Eisa A. Al-Said, Quartic splines solutions of third order obstacle problems, Appl. Math. Comput. 153 (2004), 307-316.

[10] M. A. Noor and A. K. Khalifa, A numerical approach for odd-order obstacle problems, Int. J. Comput. Math. 54 (1994), 109-116.

[11] S. A. Yousefi, Z. Barikbin and Mehdi Dehghan, Bernstein Ritz-Galerkin method for solving an initial-boundary value problem that combines Neumann and integral condition for the wave equation, Numer. Methods Partial Differential Equations 26 (2010), 1236-1246.