JP Jour. Algebra, Number Theory & Appl. **5**(3) (2005), 421-427

ON THE COMMUTATIVE-TRANSITIVE KERNEL OF CERTAIN INFINITE GROUPS

COSTANTINO DELIZIA
Dipartimento di Matematica e Informatica
Università di Salerno
Via Ponte don Melillo
84084 *Fisciano* (*SA*), *Italy*
e-mail: cdelizia@unisa.it

CHIARA NICOTERA
Dipartimento di Matematica e Informatica
Università di Salerno
Via Ponte don Melillo
84084 *Fisciano* (*SA*), *Italy*
e-mail: cnicotera@unisa.it

Abstract

We study the commutative-transitive kernel of some infinite groups. In particular, we prove that supersoluble groups are 1-(commutative-transitive).

We say that a group G is *commutative-transitive* (briefly: a CT-group) if $[a, b] = 1$ and $[b, c] = 1$ imply that $[a, c] = 1$ for all $a, b, c \in G \setminus \{1\}$. Then CT-groups are precisely those groups in which the relation of commutativity is transitive on the set of non-trivial elements.

2000 Mathematics Subject Classification: 20E15, 20F22, 20F16.

Key words and phrases: commutative transitivity, commutative-transitive kernel, infinite groups, supersoluble groups.

Communicated by Martyn Dixon

Received February 21, 2005

It is obvious that abelian groups are *CT*-groups. Moreover, a group having non-trivial center is commutative-transitive if and only if it is abelian. Thus a non-abelian nilpotent group cannot be a *CT*-group.

Of course, a group G is commutative-transitive if and only if the centralizer $C_G(x)$ is abelian, for all $x \in G \setminus \{1\}$. If G is a free group, then the centralizer of every non-identity element of G is cyclic. It follows that free groups are commutative-transitive, and so the class of *CT*-groups is not closed under taking homomorphic images. On the contrary, this class is obviously closed under taking subgroups.

The classification of all locally finite *CT*-groups has been given by Wu (see [4, Theorems 10 and 11]).

In [2], the authors introduced an ascending series

$$\{1\} = T_0(G) \subseteq T_1(G) \subseteq \cdots \subseteq T_n(G) \subseteq \cdots$$

of characteristic subgroups of a group G contained in the derived subgroup G' of G. By definition $T_1(G)$ is the subgroup of G' generated by those commutators $[a, c]$ such that $a, c \in G \setminus \{1\}$ and there exist a positive integer t and elements $x_1, \ldots, x_t \in G \setminus \{1\}$ with $[a, x_1] = [x_1, x_2] = \cdots = [x_t, c] = 1$; if $n > 1$, then $T_n(G)$ is defined by $T_n(G)/T_{n-1}(G) = T_1(G/T_{n-1}(G))$. The commutative-transitive kernel of G is the subgroup $T(G) = \bigcup_n T_n(G)$ of G'.

Obviously a group G is commutative-transitive if and only if $T(G) = \{1\}$, and for every group G the quotient $G/T(G)$ is commutative-transitive. We say that a group G is n-(*commutative-transitive*) if $T(G) = T_n(G)$ for some nonnegative integer n. So G is 0-(*commutative-transitive*) if and only if it is a *CT*-group. Of course, if a group is n-(commutative-transitive), then it is also m-(commutative-transitive), for all $m \geq n$.

Every group G having non-trivial center is 1-(commutative-transitive). More precisely, in this case $T(G) = T_1(G) = G'$.

In [1] we proved that locally finite groups are 1-(commutative-transitive). In this paper our purpose is to state similar results for other classes of infinite groups.

As pointed out in Theorem 12 of [4], the class of locally finite CT-groups is quotient closed. This fact plays a crucial role in our proof in [1]. Unfortunately, this closure property not always holds. For instance, the class of soluble CT-groups is not quotient closed, since all free soluble groups are commutative-transitive (see [4, Corollary 20]). The class of supersoluble CT-groups is not quotient closed either. For, the infinite dihedral group is a supersoluble CT-group (see [4, Lemma 7]) having some non-abelian nilpotent quotients.

On the other hand, it is easy to prove that a residually finite group whose finite quotients are CT-groups is itself a CT-group.

Proposition 1. *Let G be a residually finite group, and suppose that all finite quotients of G are CT-groups. Then G is a CT-group.*

Proof. We may assume that G is infinite. Suppose there exist elements $a, b, c \in G \setminus \{1\}$ such that $[a, b] = 1 = [b, c]$ and $[a, c] \neq 1$. Since G is residually finite, there exists a normal subgroup N of G such that G/N is finite and $a, b, c, [a, c] \notin N$. By hypothesis G/N is a CT-group. Moreover $aN, bN, cN \in G/N \setminus \{1\}$. Then $[aN, bN] = 1 = [bN, cN]$ implies that $[aN, cN] = 1$. It follows that $[a, c] \in N$, a contradiction.

In particular, Proposition 1 implies that if every finite factor of a polycyclic group G is commutative-transitive, then G is commutative-transitive.

Lemma 2. *Let G be an abelian-by-cyclic group. Then either $T(G) = \{1\}$ or $T(G) = T_1(G) = G'$.*

Proof. Let A be an abelian normal subgroup of G such that G/A is cyclic. If $[g, a] \neq 1$ for all $g \in G \setminus A$ and for all $a \in A \setminus \{1\}$, then G is a CT-group (see [4, Lemma 7]), so $T(G) = \{1\}$.

Otherwise there exist elements $g \in G \setminus A$ and $a_0 \in A \setminus \{1\}$ such that $[g, a_0] = 1$. Put $G/A = \langle xA \rangle$. Then $g = x^k b$, where $x^k \notin A$ and $b \in A$.

Hence $1 = [g, a_0] = [x^k b, a_0] = [x^k, a_0]^b = [x^k, a_0]$. From $[x^n, x^k] = 1 = [x^k, a_0] = [a_0, a]$ it follows that $[x^n, a] \in T_1(G)$ for all integers n and for all $a \in A$. Now for all elements $g_1 = x^{n_1}a_1$ and $g_2 = x^{n_2}a_2$ of G we get

$$[g_1, g_2] = [x^{n_1}a_1, x^{n_2}a_2] = [x^{n_1}a_1, a_2][x^{n_1}a_1, x^{n_2}]^{a_2} = [x^{n_1}, a_2][a_1, x^{n_2}].$$

Then $[g_1, g_2] \in T_1(G)$. Therefore $T_1(G) = G'$.

Theorem 3. *Let G be a group having a non-trivial normal subgroup N such that $AutN$ is cyclic. Then $T(G) = T_2(G)$.*

Proof. Let $C = C_G(N)$ be the centralizer of N in G. Hence G/C is cyclic. If $C = \{1\}$, then G is cyclic, thus $T(G) = T_0(G)$. Otherwise, for all $y \in N\setminus\{1\}$ and for all $c_1, c_2 \in C$, we get $[c_1, y] = 1 = [y, c_2]$. This means that $C' = T_1(C) \leq T_1(G) \leq G' \leq C$, so $C/T_1(G)$ is abelian. Therefore $G/T_1(G)$ is abelian-by-cyclic. By Lemma 2, either $G/T_1(G)$ is a CT-group, and therefore $T(G) = T_1(G)$, or $G'/T_1(G) = T_1(G/T_1(G)) = T_2(G)/T_1(G)$, and therefore $G' = T_2(G)$. In any case the result follows.

It is well known that every infinite supersoluble group has a cyclic normal subgroup of odd prime-power or infinite order (see, for instance, 5.4.8 in [3]). Hence from Theorem 3 and from the results in [1] it follows that every supersoluble group is 2-(commutative-transitive). In what follows we shall improve this result by proving that supersoluble groups are 1-(commutative-transitive).

Recall that if G is a supersoluble group, then the elements of odd order form a finite characteristic subgroup D of G. Moreover, the Fitting subgroup F of G is nilpotent, and G/F is a finite abelian group (see, for instance, 5.4.9 and 5.4.10 in [3]).

Proposition 4. *Let G be an infinite supersoluble group which is abelian-by-(finite abelian). Then either $T(G) = 1$ or $T_1(G) = G' = T(G)$.*

Proof. Let A be maximal in the set of all abelian subgroups of finite index of G containing the derived subgroup G'. Thus $G/A = \langle x_1A\rangle \times \langle x_2A\rangle \times \cdots \times \langle x_nA\rangle$, where $|x_iA| = p_i^{\alpha_i}$ for suitable primes p_i (not all necessarily distinct) and positive integers α_i. If $n = 1$, then G is abelian-by-cyclic, and the result follows from Lemma 2. So we may assume $n > 1$. We shall prove that $G' \leq T_1(G)$. It is easy to see that $G' = \langle [x_i, x_j], [A, x_i] \mid 1 \leq i, j \leq n \rangle^G$. So our purpose is to show that $[x_i, x_j] \in T_1(G)$ and $[x_i, a] \in T_1(G)$ for all $i, j \in \{1, 2, \ldots, n\}$ and for all $a \in A$.

If for every $i \in \{1, 2, \ldots, n\}$ there exists a non-trivial power of x_i, say $x_i^{r_i}$, which commutes with a non-trivial element a_i of A, then the result is true. For, in that case we get $[x_i, x_i^{r_i}] = 1 = [x_i^{r_i}, a_i] = [a_i, a_j] = [a_j, x_j^{r_j}] = [x_j^{r_j}, x_j]$, so $[x_i, x_j] \in T_1(G)$. Moreover, for all $a \in A$, we have $[x_i, x_i^{r_i}] = 1 = [x_i^{r_i}, a_i] = [a_i, a]$, so $[x_i, a] \in T_1(G)$.

Let $x \in G$. If x has infinite order, then there exists a non-trivial power of x which belongs to A, since G/A is finite. If x has odd order, then $x \in D$, where D is the subgroup consisting of all elements of G having odd order. Since $G/C_G(D)$ is finite and A is infinite, the centralizer $C_A(x)$ is not trivial. If x has order $2^\beta s$, where $s \neq 1$ is odd, then x^{2^β} has odd order. In each of the previous cases, it is evident that there exists a non-trivial power of x which commutes with a non-trivial element of A.

Therefore we may assume that for some $x \in \{x_1, x_2, \ldots, x_n\}$ the order of x is 2^t. Put $y = x^{2^{t-1}}$. So either y acts fixed-point-freely on A, or there exists a non-trivial power of x commuting with a non-trivial element of A.

Notice that if c and d are elements of G having order 2 modulo A, and both c and d act fixed-point-freely on A, then c and d cannot be linearly independent modulo A. In fact, for all $a \in A$, we get $(aa^c)^c = a^c a = aa^c$, so $a^c = a^{-1}$. In the same way $a^d = a^{-1}$ and $a^{cd} = a$ for all $a \in A$. It follows that $cd \in C_G(A) = A$. Therefore c and d are not linearly independent modulo A.

Therefore, without loss of generality, we may assume that for every $i \in \{2, 3, ..., n\}$ there exists a non-trivial power $x_i^{r_i}$ of x_i which commutes with a non-trivial element a_i of A. Moreover, we may assume that the order of x_1 is 2^t, and that $x_1^{2^{t-1}}$ acts fixed-point-freely on A.

Since A is infinite abelian and G is supersoluble, there exists a non-trivial normal subgroup $\langle b \rangle$ of G such that $\langle b \rangle$ is torsion-free and $\langle b \rangle \leq A$. Hence $b^{x_1} = b^{\pm 1}$. But $x_1^{2^{t-1}}$ acts fixed-point-freely on A, so $b^{x_1} = b^{-1}$. Therefore $t = 1$, and x_1 has order 2.

For every $i \in \{2, 3, ..., n\}$ let us consider the element $x_1 x_i$. Then there exists a non-trivial power $(x_1 x_i)^{u_i}$ of $x_1 x_i$ which commutes with a non-trivial element h_i of A. Otherwise we have $x_1 x_i$ of order 2, which acts fixed-point-freely on A, and that yields $x_1 A = x_1 x_i A$, a contradiction. Therefore, for all $a \in A$, we get $[x_1 x_i, (x_1 x_i)^{u_i}] = 1 = [(x_1 x_i)^{u_i}, h_i] = [h_i, a]$, so $[x_1 x_i, a] \in T_1(G)$. Since $[x_i, a] \in T_1(G)$, it easily follows that $[x_1, a] \in T_1(G)$. Finally, since $[x_1 x_i, (x_1 x_i)^{u_i}] = 1 = [(x_1 x_i)^{u_i}, h_i] = [h_i, a_i] = [a_i, x_i^{r_i}] = [x_i^{r_i}, x_i]$, we get $[x_1 x_i, x_i] \in T_1(G)$, and also $[x_1, x_i] \in T_1(G)$, as required.

Theorem 5. *Let G be an infinite supersoluble group. Then either $T(G) = 1$ or $T_1(G) = G' = T(G)$.*

Proof. Let F be the Fitting subgroup of G. If F is abelian, then G is abelian-by-(finite abelian), and the result follows from Proposition 4. So we may assume that F is not abelian.

We claim that for all $x \in G$ there exists a non-trivial power of x which commutes with a non-trivial element of F. If the order of x is not a power of 2, we can argue analogously as in Proposition 4, since F is infinite and its center is not trivial. Otherwise, if the order of x is 2^t, put $y = x^{2^{t-1}}$. Since F is not abelian, and y has order 2, the element y cannot act fixed-point-freely on F. So again there exists a non-trivial power of x which commutes with a non-trivial element of F.

Therefore for all $x_1, x_2 \in G$ there exist a non-trivial power $x_1^{n_1}$ of x_1 which commutes with a non-trivial element a_1 of F, and a non-trivial power $x_2^{n_2}$ of x_2 which commutes with a non-trivial element a_2 of F. Let z be a non-trivial element of the center of F. Then we get $[x_1, x_1^{n_1}] = 1 = [x_1^{n_1}, a_1] = [a_1, z] = [z, a_2] = [a_2, x_2^{n_2}] = [x_2^{n_2}, x_2]$. Thus $[x_1, x_2] \in T_1(G)$, and the result follows.

References

[1] C. Delizia and C. Nicotera, On the commutative-transitive kernel of locally finite groups, Algebra Colloq. 10(4) (2003), 567-570.

[2] B. Fine, A. Gaglione, G. Rosenberger and D. Spellman, The commutative transitive kernel, Algebra Colloq. 4(2) (1997), 141-152.

[3] D. J. S. Robinson, A Course in the Theory of Groups, 2nd ed., Springer-Verlag, New York, 1996.

[4] Y. F. Wu, Groups in which commutativity is a transitive relation, J. Algebra 207 (1998), 165-181.

■