Far East Journal of Mathematical Sciences (FJMS)
Volume 65, Number 1, 2012, Pages 11-27
Published Online: May 2012
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF PERTURBED NONLINEAR SYSTEMS

S. A. A. El-Marouf[1,2] **and A. N. Al-Rehaili**[1]

[1]Department of Mathematics
Faculty of Science
Taibah University
Madinahmonwarah, Kingdom of Saudi Arabia

[2]Department of Mathematics
Faculty of Science
Menoufia University
Shebin El-Koom, Egypt

Abstract

Boundedness and different notions of stability of solutions of the perturbed nonlinear system of the type $y' = f(t, y) + g(t, y, Ty)$ are discussed. Some new sufficient conditions are given. Examples on our results are introduced. The obtained results improve and generalize some of those given in the literature.

1. Introduction

In this paper, we study the asymptotic behaviour of solutions of the functional differential equation of the form

$$y' = f(t, y) + g(t, y, Ty), \tag{1.1}$$

2010 Mathematics Subject Classification: 45D05, 34K20.
Keywords and phrases: perturbed nonlinear systems, integro-differential equation, boundedness, stability.
Received November 29, 2011

knowing some asymptotic properties about the solution of the ordinary differential equation

$$x' = f(t, x). \tag{1.2}$$

Let $t \in J = [0, \infty)$, $x \in R^n$, $f \in C(J \times R^n, R^n)$, $f(t, 0) \equiv 0$, and the derivative $f_x \in C(J \times R^n, R^n)$. The functional perturbation $g = g(t, y, z)$ $: J \times R^n \times R^n \to R^n$ is a continuous function and T is a continuous operator mapping $C(J, R^n)$ into $C(J, R^n)$. In this way, equation (1.1) may represent several interesting cases, namely, integro-differential equations (see [8] and [18]) as

$$y' = f(t, y) + g\left(t, y, \int_{t_0}^{t} k(t, s, y(s))ds\right), \tag{1.3}$$

functional (delay) differential equations as

$$y' = f(t, y) + g(t, y, y(t - \tau)),$$

etc., taking

$$Ty(t) = \int_{t_0}^{t} k(t, s, y(s))ds$$

and

$$Ty(t) = y(t - \tau),$$

respectively. For detailed meanings of the various functions arising in (1.3), see [1] and also [2-7], for more results, see [9, 10, 12, 14-17] and the references therein. This paper is organized as follows: in Section 2, we discuss the asymptotic behaviour of the solutions of some functional differential equations which include these classes of equations. Moreover, we determine the range of validity of the results. Thus, for example, we make precise the initial conditions (the radius of attraction) for which the solution tends to zero as $t \to \infty$. Furthermore, we obtain nice estimates for the solutions of (1.1) depending on the integral-norm $(L_1\text{-}norm)$ of the variable coefficients of g. All that yields a more natural approach to the nonlinear situation than the approach of Pachpatte [11]. Section 3 concerned with the stability, strong stability and asymptotic stability of solutions of (1.1). Finally, in Section 4, we give several examples to illustrate our obtained results.

2. Asymptotic Behaviour and Boundedness of Solutions

In this section, we discuss the asymptotic behaviour and boundedness of solutions of (1.3). Throughout this discussion, we consider $\Phi(t)$ to be the fundamental matrix of solutions of the nonlinear system (1.2), with the initial value $\Phi(t_0) = I$, where I is the identity matrix and $\|\cdot\|$ denotes the Euclidean norm which is defined by $\| A(t) \| = \sum_{i,\, j=1}^{n} | a_{i,\, j} |$. We give the following result which partially generalizes those of Pachpatte [11].

Theorem 2.1. *Let all the solutions of* (1.2) *be bounded. Suppose that the following assumptions are satisfied*:

(i)

$$\| g(t,\ y,\ z) \| \leq \lambda_1(t) \| y(t) \| + \lambda_2(t) \| z(t) \|, \tag{2.1}$$

(ii)

$$\lim_{t \to \infty} \int_{t_0}^{t} trA(s) ds > -\infty, \tag{2.2}$$

(iii)

$$\int_{t_0}^{\infty} \lambda_2(s) \| Ty(s) \| ds < \infty \quad \textit{and} \quad \int_{t_0}^{\infty} \lambda_1(s) ds < \infty. \tag{2.3}$$

Then the solutions of (1.1) *are also bounded on J*.

Proof. Let $x(t)$ and $y(t)$ be the solutions of (1.2) and (1.1), respectively, with the initial data $x(t_0) = y(t_0) = x_0$. Using the linear variation of constants formula (see [13]), we have

$$y(t) = x(t) + \int_{t_0}^{\infty} (\Phi(t)\Phi^{-1}(s) g(t, s, y(s)))ds$$

$$= \Phi(t) x_0 + \int_{t_0}^{\infty} (\Phi(t)\Phi^{-1}(s) g(t, s, y(s)))ds.$$

From equation (2.1), we obtain

$$\| y(t) \| \leq \| \Phi(t) \| \| x_0 \|$$

$$+ \int_{t_0}^{t} \| \Phi(t) \| \| \Phi^{-1}(s) \| [\lambda_1(s) \| y(s) \| + \lambda_2(s) \| z(s) \|] ds,$$

where

$$\Phi^{-1}(t) = \frac{adj\Phi(t)}{\det \Phi(t)} = \frac{adj\Phi(t)}{\exp \int_{t_0}^{t} trA(s)ds}. \tag{2.4}$$

Since $\| \Phi(t) \|$ is bounded, and by the assumption (2.2), it follows that $\| \Phi^{-1}(t) \|$ is also bounded. Now, let

$$c = \max(\sup_{t \geq t_0} \| \Phi(t) \|, \sup_{t \geq t_0} \| \Phi^{-1}(t) \|), \tag{2.5}$$

thus

$$\| y(t) \| \leq \| \Phi(t) \| \| x_0 \|$$

$$+ \int_{t_0}^{t} \| \Phi(t) \| \| \Phi^{-1}(t) \| [\lambda_1(s) \| y(s) \| + \lambda_2(s) \| z(s) \|] ds.$$

Therefore, from (2.4) and (2.5), we get

$$\| y(t) \| \leq c \| x_0 \| + \int_{t_0}^{t} \| \Phi(t) \| \| \Phi^{-1}(t) \| \lambda_1(s) \| y(s) \| ds$$

$$+ \int_{t_0}^{t} \| \Phi(t) \| \| \Phi^{-1}(t) \| \lambda_2(s) \| z(s) \| ds$$

or

$$\| y(t) \| \le c\| x_0 \| + c_1\int_{t_0}^{t} \lambda_1(s)\| y(s) \| ds + c_1\int_{t_0}^{t} \lambda_2(s)\| Ty(s) \| ds.$$

Setting $c_1 = c^2$, and $\int_{t_0}^{t} \lambda_2(s)\| Ty(s) \| ds = L$, we have

$$\| y(t) \| \le (c\| x_0 \| + L) + c_1 + \int_{t_0}^{t} \lambda_1(s)\| y(s) \| ds, \quad t \ge t_0 \ge 0,$$

for some positive constant L. Applying Gronwall's inequality (see [13]), we get

$$\| y(t) \| \le (c\| x_0 \| + L)\exp\left(c_1\int_{t_0}^{t} \lambda_1(s)ds\right).$$

It follows from the assumption (2.3) that

$$\| y(t) \| \le (c\| x_0 \| + L)N, \quad t \ge t_0 \ge 0,$$

for some positive constant N, which completes the proof.

Theorem 2.2. *Suppose that the conditions* (2.1), (2.2) *and* (2.3) *of Theorem* 2.1 *hold*. *If all the solutions of* (1.2) *approach zero as* $t \to \infty$, *then so do the solution of* (1.1).

Proof. Going through as in the proof of Theorem 2.1, we get

$$y(t) = \Phi(t)x_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)g(t, y(s), Ty(s))ds.$$

Since all the solutions of (1.2) approach zero as $t \to \infty$, $\| \Phi(t) \| \to 0$ as $t \to \infty$. In view of the assumption (2.3) and the fact that $\| \Phi(t) \| \to 0$ as $t \to \infty$, we have $\| \Phi^{-1}(t) \| \to 0$ as $t \to \infty$. But since both $\| \Phi(t) \|$ and $\| \Phi^{-1}(t) \|$ are constant and approach zero as $t \to \infty$, it follows that they are bounded for all $t \ge t_0 \ge 0$. Now let $c = \max(\sup_{tt_0}\| \Phi(t) \|, \sup_{tt_0}\| \Phi^{-1}(t) \|)$. Therefore, by the assumption (2.1), we have

$$\| y(t) \| \le \| \Phi(t) \| \| x_0 \| + \int_{t_0}^{t} \| \Phi(t) \| \| \Phi^{-1}(s) \| \| g(t, \; y(s), \; Ty(s)) \| ds,$$

It follows from (2.1) that

$$\| y(t) \| \le \| \Phi(t) \| \| x_0 \|$$

$$+ \int_{t_0}^{t} \| \Phi(t) \| \| \Phi^{-1}(s) \| [\lambda_1(s) \| y(s) \| + \lambda_2(s) \| Ty(s) \|] ds,$$

thus

$$\| y(t) \| \le \| \Phi(t) \| \| x_0 \| + c \int_{t_0}^{t} \| \Phi(t) \| [\lambda_1(s) \| y(s) \| + \lambda_2(s) \| Ty(s) \|] ds$$

$$\le \frac{\| y(t) \|}{\| \Phi(t) \|} \| x_0 \| + c \int_{t_0}^{t} \lambda_1(s) \| y(s) \| + \lambda_2(s) \| Ty(s) \| ds.$$

Taking

$$c = \frac{1}{\| \Phi(t) \|},$$

we get

$$\| y(t) \| \le \frac{\| y(t) \|}{\| \Phi(t) \|} \| x_0 \| + c^2 \int_{t_0}^{t} \frac{\| y(s) \|}{\| \Phi(t) \|} \lambda_1(s) ds + c \int_{t_0}^{t} \lambda_2(s) \| Ty(s) \| ds.$$

Setting $c \int_{t_0}^{t} \lambda_2(s) \| Ty(s) \| ds = L,$ by using (2.3), we have

$$\| y(t) \| \le \frac{\| y(t) \|}{\| \Phi(t) \|} (\| x_0 \| + L) + c^2 \int_{t_0}^{t} \frac{\| y(s) \|}{\| \Phi(t) \|} \lambda_1(s) ds.$$

Applying Gronwall's inequality (see [8]), we get

$$\| y(t) \| \le \frac{\| y(t) \|}{\| \Phi(t) \|} (\| x_0 \| + L) \exp\left(c^2 \int_{t_0}^{t} \frac{\| y(s) \|}{\| \Phi(t) \|} \lambda_1(s) ds \right), \quad t \ge t_0 \ge 0,$$

where

$$\exp\left(c^2 \int_{t_0}^{t} \frac{\| y(t) \|}{\| \Phi(t) \|} \lambda_1(s) ds\right) = N,$$

thus

$$\| y(t) \| \leq \frac{\| y(t) \|}{\| \Phi(t) \|} (\| x_0 \| + L) N$$

$$\leq \| y(t) \| (\| x_0 \| + L) N \| \Phi(t) \|, \quad t \geq t_0 \geq 0,$$

for some constant N, where $N = \exp\left(c^2 \int_{t_0}^{t} \lambda_1(s) ds\right)$. Therefore, all the solutions of (1.1) approach zero as $t \to \infty$.

Theorem 2.3. *Assume that the fundamental matrix of* (1.2) *satisfies the condition*

$$\| \Phi(t) \| \leq M \quad \text{and} \quad \| \Phi(t) \Phi^{-1}(s) \| \leq N, \tag{2.6}$$

for some positive constants M *and* N, $t \geq t_0 \geq 0$. *Suppose that the assumptions* (2.1), (2.2) *and* (2.3) *of Theorem* 2.1 *are satisfied*. *Then the relation*

$$\| y(t) \| \leq (\| x_0 \| + k) L, \quad t \geq t_0 \geq 0, \tag{2.7}$$

holds for some constants $k > 0$ *and* $L > 0$. *Moreover, if* $x(t) \to 0$ *as* $t \to \infty$, *then* $y(t) \to 0$ *as* $t \to \infty$.

Proof. As in the proof of Theorem 2.1, we get

$$y(t) = \Phi(t) x_0 + \int_{t_0}^{t} \Phi(t) \Phi^{-1}(s) g(t, y(s), Ty(s)) ds.$$

Using the assumptions (2.1), (2.2) and (2.6), we have:

$$\| y(t) \| = \| \Phi(t) \| \| x_0 \| + \int_{t_0}^{t} \| \Phi(t) \Phi^{-1}(s) \| \| g(s, y(s), Ty(s)) \| ds$$

$$\leq \| \Phi(t) \| \| x_0 \| + \int_{t_0}^{t} \| \Phi(t) \Phi^{-1}(s) \| [\lambda_1 \| y(s) \| + \lambda_2 \| Ty(s) \|] ds$$

$$\leq M\| x_0 \| + N\int_{t_0}^{t} [\lambda_1\| y(s)\| + \lambda_2\| Ty(s)\|]ds$$

$$\leq M\| x_0 \| + N\left[\int_{t_0}^{t} \lambda_1\| y(s)\| ds + \int_{t_0}^{t} \lambda_2\| Ty(s)\| ds\right].$$

Let $N\int_{t_0}^{t} \lambda_2\| Ty(s)\| ds = K.$ Then

$$\| y(t)\| \leq (M\| x_0 \| + K) + N\int_{t_0}^{t} \lambda_1\| y(s)\| ds.$$

Applying Gronwall's inequality (see [13]), we obtain

$$y(t) \leq (M\| x_0 \| + K)\exp N\int_{t_0}^{t} \lambda_1(s)ds, \quad t \geq t_0 \geq 0$$

$$\leq (M\| x_0 \| + K)L, \quad t \geq t_0 \geq 0,$$

for some positive constant $L = \exp N\int_{t_0}^{t} \lambda_1(s)ds.$ This proves (2.7). Now let $\lim_{t\to\infty} x(t) = 0.$ Then given any $\varepsilon > 0,$ there exists a constant $T = T(\varepsilon) > 0$ such that $\| x(t)\| < \varepsilon$ for all $t \geq T.$ Thus, from (2.7), we have:

$$y(t) \leq (\varepsilon + K)L, \quad t \geq t_0 \geq 0$$

$$y(t) = \varepsilon L, \quad t \geq t_0 \geq 0,$$

for some positive constant L independent of ε and $T(\varepsilon).$ Hence $y(t) \to 0$ as $t \to \infty.$ This completes the proof.

3. Stability Properties

In this section, we discuss the stability properties of solutions of (1.1) using the those of (1.2) and we first introduce the following lemma:

Lemma 3.1. *Let the functions* $a(t), b_i(t)$ *and* $c_i(t), i = 1, 2, ..., n$ *be* *nonnegative continuous functions on* $J = [0, \infty)$, *and* u_0 *be a nonnegative constant such that*

$$u(t) \le u_0 + \int_{t_0}^{t} a(s)u(s)ds + \sum_{i=1}^{n} b_i(s)\left(\int_{t_0}^{s} c_i(\tau)d\tau\right)ds, \; i = 1, 2, ..., n. \quad (3.1)$$

Then

$$u(t) \le u_0 \exp\left[\int_{t_0}^{t}\left(a(s) + \sum_{i=1}^{n} b_i(s)\int_{t_0}^{s} c_i(\tau)d\tau\right)ds\right], \quad i = 1, 2, ..., n. \quad (3.2)$$

Proof. Let

$$v(t) = u_0 + \int_{t_0}^{t} a(s)u(s)ds + \sum_{i=1}^{n}\int_{t_0}^{t} b_i(s)\left(\int_{t_0}^{s} c_i(\tau)u(\tau)d\tau\right)ds.$$

Thus, by (3.1), we have $u(t) \le V(t)$. Moreover,

$$v'(t) \le a(t)v(t) + \sum_{i=1}^{n} b_i(t)\int_{t_0}^{t} c_i(s)v(s)ds,$$

since $v(t)$ is nondecreasing, the above inequality can be written as

$$v'(t) \le v(t)\left(a(t) + \sum_{i=1}^{n} b_i(t)\int_{t_0}^{t} c_i(s)ds\right).$$

Applying Gronwall's inequality, we get

$$v(t) \le u_0 \exp\left[\int_{t_0}^{t}\left(a(s) + \sum_{i=1}^{n} b_i(s)\int_{t_0}^{t} c_i(\tau)d\tau\right)ds\right].$$

But since $u(t) \le v(t)$, it follows that

$$u(t) \le u_0 \exp\left[\int_{t_0}^{t}\left(a(s) + \sum_{i=1}^{n} b_i(s)\int_{t_0}^{t} c_i(\tau)d\tau\right)ds\right],$$

which completes the proof.

Theorem 3.2. *Assume that all the solutions of* (1.2) *are uniformly stable*. *Suppose that the following conditions hold*:

(i)

$$\| g(t, y, Ty) \| \le a(t) \| y \| + b(t) \int_{t_1}^{t} c(s) \| y(s) \| ds, \tag{3.3}$$

where $a(t)$, $b(t)$ *and* $c(t)$ *are nonnegative continuous functions on* $J = [0, \infty)$.

(ii)

$$\int_{t_1}^{t} \left[a(s) + b(s) \int_{t_1}^{s} c(\tau) d\tau \right] ds < \infty, \quad t \ge t_1 \ge t_0 \ge 0. \tag{3.4}$$

Then the zero solution of (1.1) *is uniformly stably*.

Proof. Let $x(t)$ and $y(t)$ be the solutions of (1.2) and (1.1), respectively, with the initial condition $x(t_1) = y(t_1) = x_1$. By the nonlinear variation of constant formula, we obtain

$$\begin{aligned} y(t) &= x(t) + \int_{t_1}^{t} \Phi(t)\Phi^{-1}(s) g(s, y(s), Ty(s)) ds \\ &= x_1 + \int_{t_1}^{t} \Phi(t)\Phi^{-1}(s) g(s, y(s), Ty(s)) ds \\ &= \Phi(t)\Phi^{-1}(t_1) x_1 + \int_{t_1}^{t} \Phi(t)\Phi^{-1}(s) g(s, y(s), Ty(s)) ds. \end{aligned}$$

Since the solutions of (1.2) are uniformly stable, there exists a positive constant M such that

$$\| \Phi(t)\Phi^{-1}(s) \| \le M, \quad t \ge t_1 \ge t_0 \ge 0. \tag{3.5}$$

In view of the assumptions (3.3) and (3.5), we obtain

$$\begin{aligned} \| y(t) \| &\le \| \Phi(t)\Phi^{-1}(t_1) \| \| x_1 \| + \int_{t_1}^{t} \| \Phi(t)\Phi^{-1}(s) \| \| g(s, y(s), Ty(s)) \| ds \\ &\le M \| x_1 \| + M \int_{t_1}^{t} \left(a(s) \| y(s) \| + b(s) \int_{t_1}^{s} c(\tau) \| y(\tau) \| d\tau \right) ds. \end{aligned}$$

Now, applying Lemma 3.1, we have

$$\| y(t) \| \leq M \| x_1 \| \exp\left(M \int_{t_1}^{t} a(s) + b(s) \int_{t_1}^{s} c(\tau) d\tau \right) ds.$$

Consequently, by the assumption (3.4), it follows that

$$\| y(t) \| \leq ML \| x_1 \|, \quad t \geq t_1 \geq t_0 \geq 0,$$

for some positive constant L. This completes the proof.

Corollary 3.3. *Assume that the solutions of* (1.2) *are asymptotically stable. Suppose that the assumptions* (3.3) *and* (3.4) *of Theorem* 3.2 *are satisfied. Then the zero solution of* (1.1) *is asymptotically stable.*

Proof. Since the solutions of (1.2) are asymptotically stable, it follows that $\| \Phi(t) \| \to 0$ as $t \to \infty$ which implies $\| x(t) \| \to 0$ as $t \to \infty$. Let $y(t)$ be a solution of (1.1) with the initial condition: $x(t_0) = y(t_0) = x_0$. Then

$$y(t) = x(t) + \int_{t_0}^{t} \Phi(t) \Phi^{-1}(s) g(s, y(s), Ty(s)) ds.$$

Since $\| x(t) \| \to 0$ as $t \to \infty$, given any $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that $\| x(t) \| < \varepsilon$ for all $t \geq T(\varepsilon) > 0$. But since $\Phi(t)$ is continuous and $\| \Phi(t) \| \to 0$ as $t \to \infty$, it follows that $\| \Phi(t) \|$ and $\| \Phi^{-1}(t) \|$ are bounded. Thus from the assumptions (3.3) and (3.4), we obtain

$$\| y(t) \| \leq \| x(t) \| + \int_{t_0}^{t} \| \Phi(t) \| \| \Phi^{-1}(s) \| \| g(t, y(s), Ty(s)) \| ds.$$

Setting

$$\| \Phi(t) \| \| \Phi^{-1}(s) \| = M^2,$$

we get

$$\| y(t) \| \leq \varepsilon + M^2 \int_{t_0}^{t} \left(a(s) \| y(s) \| + b(s) \int_{t_0}^{s} c(\tau) \| y(\tau) \| d\tau \right) ds.$$

Now, applying Lemma 3.1, we have

$$\| y(t) \| \leq \varepsilon \exp\left(M^2 \int_{t_0}^{t} \left(a(s) + b(s) \int_{t_0}^{s} c(\tau) d\tau \right) ds \right) \leq \varepsilon L,$$

for some positive constant L, independent on ε, for all $t \geq t_0 \geq 0$. Thus the desired result follows.

Theorem 3.4. *Suppose that the assumptions* (3.3) *and* (3.4) *of Theorem* 3.2 *are satisfied*. *If the solutions of* (1.2) *are strongly stable*, *then so does the zero solution of* (1.1).

Proof. It follows from the strong stability of (1.2) that there exists a positive constant M such that

$$\| \Phi(t) \| \leq M \quad \text{and} \quad \| \Phi^{-1}(t) \| \leq M, \quad t \geq t_0 \geq 0. \tag{3.6}$$

Now, using the nonlinear variation of constant formula given in [8], we obtain

$$y(t) = x_1 + \int_{t_1}^{t} \Phi(t) \Phi^{-1}(s) g(s, y(s), Ty(s)) ds$$

$$= \Phi(t) \Phi^{-1}(t_1) x_1 + \int_{t_1}^{t} \Phi(t) \Phi^{-1}(s) g(s, y(s), Ty(s)) ds,$$

$$\| y(t) \| \leq \| \Phi(t) \| \| \Phi^{-1}(t_1) \| \| x_1 \|$$

$$+ \int_{t_1}^{t} \| \Phi(t) \| \| \Phi^{-1}(t_1) \| \| g(s, y(s), Ty(s)) \| ds.$$

Thus, in view of the assumptions (3.3) and (3.6), we get

$$\| y(t) \| \leq M^2 \| x_1 \| + M^2 \int_{t_1}^{t} \left(a(s) \| y(s) \| + b(s) \int_{t_1}^{s} c(\tau) \| y(\tau) \| d\tau \right) ds.$$

By Lemma 3.1, we obtain

$$\| y(t) \| \leq M^2 \| x_1 \| \exp\left(M^2 \int_{t_1}^{t} \left(a(s) + b(s) \int_{t_1}^{s} c(\tau) d\tau \right) ds \right).$$

Again, from the assumption (3.4), we have

$$\| y(t) \| \leq M^2 \| x_1 \| L$$

$$\leq \| x_1 \| L^*,$$

for some positive constant L^*, $t \geq t_1 \geq t_0 \geq 0$. This completes the proof.

Theorem 3.5. *Assume that the solutions of* (1.2) *are uniformly asymptotically stable*. *Suppose that the assumptions* (3.3) *and* (3.4) *of Theorem* 3.2 *are satisfied*. *Then the zero solution of* (1.1) *is uniformly asymptotically stable*.

Proof. Since the solutions of (1.2) are uniformly asymptotically stable then, there exist positive constants α and M such that

$$\| \Phi(t)\Phi^{-1}(s) \| \leq Me^{-\alpha(t-s)}, \quad t \geq s \geq t_0 \geq 0. \tag{3.7}$$

As in the proof of Theorem 3.2, we have, for any $t_1 \geq t_0$,

$$y(t) = \Phi(t)\Phi^{-1}(t_1)x_1 + \int_{t_1}^{t} \Phi(t)\Phi^{-1}(s) g(s, y(s), Ty(s))ds,$$

$$\| y(t) \| \leq \| \Phi(t)\Phi^{-1}(t_1) \| \| x_1 \| + \int_{t_1}^{t} \| \Phi(t)\Phi^{-1}(s) \| \| g(s, y(s), Ty(s)) \| ds.$$

In view of the assumptions (3.3) and (3.7), we obtain

$$\| y(t) \| \leq M \| x_1 \| e^{-\alpha(t-s)}$$

$$+ M \int_{t_1}^{t} e^{-\alpha(t-s)} a(s) \| y(t) \| + b(s) \left(\int_{t_1}^{s} c(\tau) \| y(\tau) \| e^{\alpha t} d\tau \right) ds$$

or

$$\| y(t) \| e^{\alpha t} \leq M \| x_1 \| e^{\alpha t_1}$$

$$+ M \int_{t_1}^{t} \left(a(s) \| y(s) \| e^{\alpha s} + b(s) \int_{t_1}^{s} c(\tau) \| y(\tau) \| e^{\alpha \tau} d\tau \right) ds.$$

It follows from Lemma 3.1 that

$$\| y(t) \| e^{\alpha t} \leq M \| x_1 \| e^{\alpha t_1} \left(\exp M \int_{t_1}^{t} \left(a(s) + b(s) \int_{t_1}^{s} c(\tau) d\tau \right) ds \right), \quad t \geq t_1 \geq t_0$$

$$\leq M \| x_1 \| e^{\alpha t_1} L$$

or

$$\| y(t) \| \leq ML \| x_1 \| e^{\alpha(t - t_1)}, \quad t \geq t_1 \geq t_0 \geq 0.$$

This completes the proof.

4. Examples

In this section, we give several examples to illustrate our obtained results.

Example 4.1. Consider the perturbed differential equation of the form

$$y' = (\sin 4t) y + g(t, y, Ty), \tag{4.1}$$

where $t \in R^+$, $y \in R$ and $g(t, y, Ty) = e^{-t} y + e^{-2t} \int_{t_0}^{t} h(s) y(s) ds$, $h(t) \in L_1[0, \infty)$. Consider the linear differential equation of the form

$$x' = (\sin 4t) x, \quad x(t_0) = x_0, \quad t \geq t_0 \geq 0. \tag{4.2}$$

It is clear that the general solution of (4.2) is given by

$$x(t, t_0, x_0) = x_0 \exp\left[\frac{-1}{4} (\cos 4t - \cos 4t_0) \right], \quad t \geq t_0 \geq 0.$$

Also, the fundamental matrix $\Phi(t)$ is given by

$$\Phi(t) = \Phi(t, t_0, x_0) = \exp\left[\frac{-1}{4} (\cos 4t - \cos 4t_0) \right], \quad t \geq t_0 \geq 0.$$

Hence

$$| \Phi(t, t_0, x_0) | = \left| \exp\left[\frac{-1}{4} (\cos 4t - \cos 4t_0) \right] \right| \leq e, \quad t \geq t_0 \geq 0.$$

Therefore, the solution of (4.2) are bounded for all $t \geq t_0 \geq 0.$ By returning of Theorem 2.1, we obtain $\lambda_1(t) = e^{-t}$ and $\lambda_2(t) = e^{-2t}$, and $Ty(t) = e^{-2t}\int_{t_0}^{t} h(s)y(s)ds.$ Clearly, the assumptions (2.1), (2.2) and (2.3) of Theorem 2.1 are satisfied. Then all the solutions of (4.1) are uniformly bounded for all $t \geq t_0 \geq 0.$

Example 4.2. Consider the perturbed differential equation of the form

$$y' = -t^2 y + g(t, y, Ty), \tag{4.3}$$

where $t \in R^+,$ $y \in R$ and $g(t, y, Ty) = e^{-t}y + e^{-2t}\int_{t_0}^{t} h(s)y(s)ds,$ $h(t) \in L_1[0, \infty).$ It is clear that the general solution of the linear differential equation

$$x' = -t^2 x, \quad x(t_0) = x_0, \quad t \geq t_0 \geq 0 \tag{4.4}$$

is given by

$$x(t, t_0, x_0) = x_0 \exp\left[\frac{-1}{3}(t^3 - t_0^3)\right].$$

Therefore,

$$\Phi(t) = \Phi(t, t_0, x_0) = \exp\left[\frac{-1}{3}(t^3 - t_0^3)\right], \quad t \geq t_0 \geq 0.$$

Hence, all the solutions of (4.4) approach zero as $t \to \infty.$ Then by Theorem 2.2, it follows that all the solutions of (4.3) approach zero as $t \to \infty.$

Remark 4.1. In the preceding example, we have

$$|\Phi(t, t_0, x_0)| = \left|\exp\left[\frac{-1}{3}(t^3 - t_0^3)\right]\right| \to 0 \quad \text{as} \quad t \to \infty.$$

Then the solutions of (4.4) are asymptotically stable. Moreover, by Corollary 3.3, it follows that, the zero solution of (4.3) is also asymptotically stable.

Example 4.3. Consider the perturbed differential equation of the type

$$y' = (\cos 3t - 3)y + g(t, y, Ty), \tag{4.5}$$

where $t \in R^+$, $y \in R$ and

$$g(t, y, Ty) = e^{-2t} + e^{-t}\int_{t_0}^{t} h(s)y(s)ds, \quad h(t) \in L_1[0, \infty).$$

The solution of the linear differential equation

$$x' = (\cos 3t - 3)x, \quad x(t_0) = x_0, \quad t \geq t_0 \geq 0 \tag{4.6}$$

is given by

$$x(t, t_0, x_0) = x_0 \exp\left[\frac{1}{3}(\sin 3t - \sin 3t_0)\right] \cdot \exp[-3(t - t_0)], \quad t \geq t_0 \geq 0.$$

Hence the fundamental matrix is of the form

$$\Phi(t) = \Phi(t, t_0, x_0) = \exp\left[\frac{-1}{3}(\sin 3t - \sin 3t_0)\right] \cdot \exp[-3(t - t_0)], \; t \geq t_0 \geq 0.$$

Thus the inverse matrix $\Phi^{-1}(t)$ is given by

$$\Phi^{-1}(t) = \frac{adj\Phi(t)}{\det \Phi(t)} = 1, \quad t \geq t_0 \geq 0.$$

Then equation (4.6) is uniformly asymptotically stable. Thus the assumptions of Theorem 3.5 are satisfied, and so the zero solution of (4.5) is also uniformly asymptotic stable.

References

[1] Zh. S. Athanassov, Perturbation theorems for nonlinear systems of ordinary differential equations, J. Math. Anal. Appl. 86 (1982), 194-207.

[2] T. A. Burton, Uniform stabilities of Volterra equations, J. Differential Equations 36 (1980), 40-53.

[3] F. Dannan and S. Elaydi, Lipschitz stability of nonlinear systems of differential equations, J. Math. Anal. Appl. 113 (1986), 562-577.

[4] F. M. Dannan and S. Elaydi, Lipschitz stability of nonlinear systems of differential equations II Liapunov function, J. Math. Anal. Appl. 143 (1989), 517-529.

[5] S. Elaydi and M. Rama Mohana Rao, Lipschitz stability for nonlinear Volterra integrodifferential systems, Appl. Math. Comput. 27 (1988), 191-199.

[6] S. Elaydi and S. Sivasundaram, A unified approach to stability in integrodifferential equations via Liapunov functions, J. Math. Anal. Appl. 144 (1989), 503-531.

[7] M. M. A. El-Sheikh and S. A. A. El-Mahrouf, On stability and bifurcation of periodic solutions of delay differential equations, J. Appl. Math. Comput. 19 (2005), 281-295.

[8] P. Gonzalez and M. Pinto, Stability properties of the solutions of the nonlinear functional differential systems, J. Math. Anal. Appl. 181 (1994), 562-573.

[9] S. Köksal, Boundedness properties and perturbing Lyapunov functions, JMAA 163(1) (1992), 73-78.

[10] Lianzhong Li, Fanwei Meng and Peijun Ju, Some new integral inequalities and their applications in studying the stability of nonlinear integro-differential equations with time delay, J. Math. Anal. Appl. 377 (2011), 853-862.

[11] B. G. Pachpatte, Stability and asymptotic behaviour of perturbed nonlinear systems, J. Differential Equations 16 (1974), 14-25.

[12] R. M. Rao and P. Srinivas, Asymptotic behavior of solutions of Volterra integro-differential equations, Proc. Amer. Math. Soc. 94 (1985), 55-60.

[13] M. Rama Mohana Rao, Ordinary Differential Equations: Theory and Applications, Edward Arnold, London, 1981.

[14] J. Sugie, D.-L. Chen and H. Matsunaga, On global asymptotic stability of systems of Liénard type, J. Math. Anal. Appl. 219 (1998), 140-164.

[15] Shi Gui Ruan, Stability of Volterra integrodifferential systems, J. Math. Anal. Appl. 137 (1989), 471-476.

[16] A. A. Soliman and M. H. Abdalla, Integral stability criteria of nonlinear differential systems, Math. Comput. Modelling 48 (2008), 258-267.

[17] T. Taniguchi, Stability theorems of perturbed linear ordinary differential equations, J. Math. Anal. Appl. 149 (1990), 583-598.

[18] Liguang Xu and Daoyi Xu, Exponential stability of nonlinear impulsive neutral integro-differential equations, Nonlinear Anal. 69 (2008), 2910-2923.