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SEIFERT SURFACES FOR GRAPHS, ASSOCIATED LINKS, AND PLANARITY

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Abstract

In this note, we establish necessary and sufficient conditions to determine whether a given embedding of a graph in S^3 is ambient isotopic to a planar embedding. Our methods are geometric and our results apply to any tame embedding of any finite, abstractly planar graph.

We also consider the problem of whether an associated link determines an embedded graph. For graphs of rank less than three, the associated link is trivial if and only if the graph is planar. For graphs of rank at least three, we prove the existence of nonplanar graphs with trivial associated links using an example of P. Zhao [Topology Appl. 57 (1994), 23-30].

A fundamental problem in low-dimensional topology is the classification of embeddings of graphs in S^3 up to ambient isotopy. In particular, determining whether an embedding of a graph in S^3 is ambient isotopic to a planar embedding has received considerable attention [1-8]. Some approaches are algebraic while others are geometric with applications of knot and link theory techniques to knots and links associated with the graph. In Theorem 1, we use the entire graph, its Seifert surface, and boundary, the associated link, to determine

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conditions for planarity. In Theorems 2 and 3, we classify graphs that are determined by their associated links.

To begin, all embeddings are tame, all graphs are finite, connected and abstractly planar, and all surfaces are orientable. Let G_{ab} denote a finite, connected, abstractly planar graph, and let G denote a tame embedding of G_{ab} in S^3. Consider a planar embedding G_{pl} of G_{ab}. Let Γ_{pl} denote an ε neighborhood of G_{pl} in S^2. The Seifert surface, Γ_G, of G is a tame embedding of Γ_{pl} into S^3 such that Γ_G deformation retracts to G, its spine, and $lk(\partial_i, \partial_j) = 0$, where ∂_i and ∂_j are any two distinct boundary components of Γ_G that share a band. The existence and, under certain conditions, uniqueness of Γ_G has already been established [1, 3]. The boundary of Γ_G, $\partial\Gamma_G$, is an embedded link in S^3 with $r+1$ components, where r = rank of $\pi_1(G, *)$. We call $\partial\Gamma_G$ the *associated link* of G.

We now consider Γ_G and the associated link to determine conditions for planarity of G. At first it might seem possible that G is planar if and only if $\partial\Gamma_G$ is a trivial link. However, P. Zhao provides an interesting counterexample to this statement in [8]. In this counterexample, G is a nonplanar embedding of K_4 (the complete graph on four vertices).

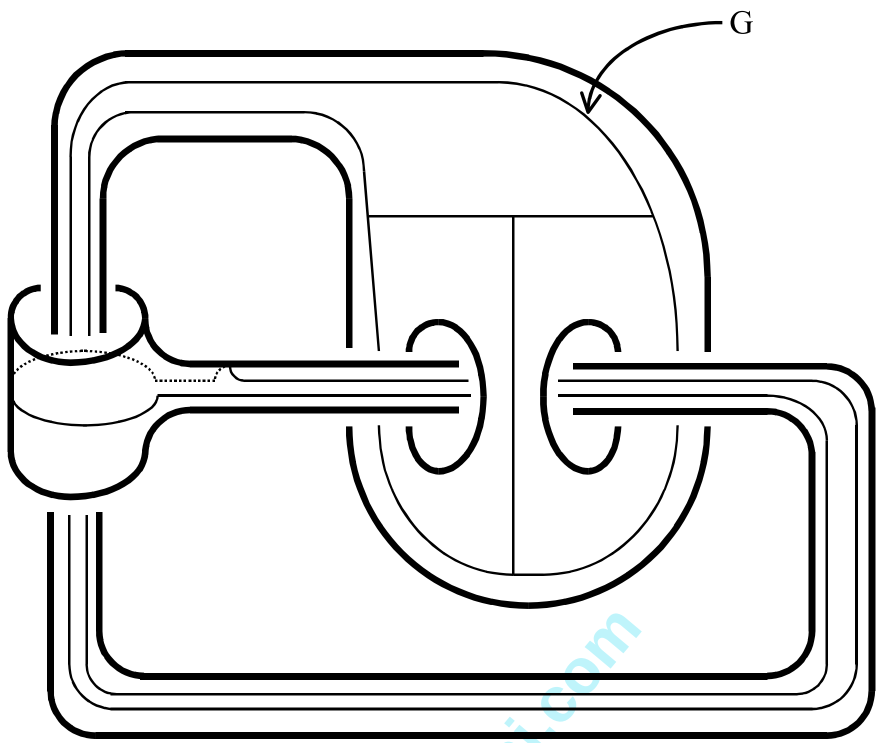

Figure 1

In fact, Zhao's example illustrates the essential point that the associated link components do not bound two-discs in the complement of Γ_G or G.

Theorem 1. *Let G be a tame embedding of a finite, abstractly planar graph G_{ab}, let Γ_G denote a tame embedding of Γ_{pl} into S^3 such that Γ_G deformation retracts to G, its spine, and let $\bigcup_{i=1}^{r+1} \partial_i$ denote the link components of $\partial\Gamma_G$. Then G is ambient isotopic to a planar embedding if and only if for some Γ_G, $\bigcup_{i=1}^{r+1} \partial_i$ bound embedded two-discs in $S^3 - G$.*

Proof. Suppose G is ambient isotopic to a planar embedding denoted by G_{pl}. Let $\phi_t : S^3 \times I \to S^3$ denote the isotopy, where ϕ_0 = identity and $\phi_1(G) = \phi_1|_G = G_{pl}$. Consider $\Gamma_{G(pl)}$, an orientable ε neighborhood of G_{pl} embedded in S^2. $\partial\Gamma_{G(pl)}$ is a trivial link all of whose components bound mutually disjoint embedded two-discs, $\bigcup_{i=1}^{r+1} D_i^2 = S^2 - \Gamma_{G(pl)}$. So $\Gamma_{G(pl)} \cup \bigcup_{i=1}^{r+1} D_i^2$ is S^2. Set $\Gamma_G = \phi_1^{-1}(\Gamma_{G(pl)})$. Then $\phi_1^{-1}\left(\bigcup_{i=1}^{r+1} D_i^2\right)$ is a collection of mutually disjoint embedded two-discs in $S^3 - G$ whose boundary coincides with $\partial\Gamma_G$.

Now, suppose for some Γ_G, $\partial\Gamma_G$ bounds $\bigcup_{i=1}^{r+1} D_i^2$, a collection of embedded two-discs in $S^3 - G$. If $\bigcup_{i=1}^{r+1} D_i^2 \cap \mathrm{Int}(\Gamma_G) \neq \varnothing$, then use general position and transversality to isotope any intersections off $\mathrm{Int}(\Gamma_G)$. So then $\left(\bigcup_{i=1}^{r+1} D_i^2 \cap (S^3 - G)\right) \subset (S^3 - \Gamma_G)$.

If $\bigcap_{i=1}^{r+1} D_i^2 \neq \varnothing$, then use general position and transversality to isotope $\bigcup_{i=1}^{r+1} D_i^2$ so that $\bigcap_{i=1}^{r+1} D_i^2 = \bigcup_{i=1}^{k} S_i^1$, a finite collection of intersection circles in the interior of the two-discs. Begin with an "innermost" intersection circle S_1^1, and isotope the two-discs that form S_1^1 to "cap off" and remove this intersection circle. Move to another "innermost" intersection circle S_2^1 and continue with this isotopy and "cap off" process to remove this intersection circle. Repeat this isotopy and "cap off" procedure to remove all remaining intersection circles. Then the resulting two-discs, $\bigcup_{i=1}^{r+1} \widetilde{D}_i^2$, are mutually disjoint embedded two-discs. Using attaching homeomorphisms h_i, $1 \leq i \leq r+1$, attach $\widetilde{D}_i^2$ to Γ_G along their respective boundaries. The result, $\Gamma_G \cup \bigcup_{i=1}^{r+1} \widetilde{D}_i^2$, is an embedded two-sphere in S^3. (Recall the Euler characteristic of Γ_G, $\chi(\Gamma_G) = \chi(G) = 1 - r$, where r = rank of $\pi_1(G, *)$. So $\Gamma_G \cup \bigcup_{i=1}^{r+1} \widetilde{D}_i^2$ is a closed, orientable surface with $\chi\left(\Gamma_G \cup \bigcup_{i=1}^{r+1} D_i^2\right) = 1 - r + (r+1) = 2.$)

By Alexander's Theorem, this embedded two – sphere bounds a three – disc, and is isotopic to the standard embedded two – sphere in S^3. Using this isotopy, G is ambient isotopic to a planar embedding.

Although Zhao's example demonstrated that, in general, a trivial associated link does not guarantee planarity, there are special graphs for which this is true.

Theorem 2. *Let G denote a tame embedding of a connected, abstractly planar rank r graph, $r = 1, 2$. Then G is ambient isotopic to a planar embedding if and only if $\partial\Gamma_G$ is a trivial link for some Γ_G.*

Proof. Suppose G is isotopic to a planar embedding. Let $\phi_t : S^3 \times I \to S^3$ denote the isotopy, where ϕ_0 = identity and $\phi_1(G) = \phi_1|_G = G_{pl}$. Consider $\Gamma_{G(pl)}$, an orientable ε neighborhood of G_{pl} embedded in S^2. $\partial\Gamma_{G(pl)}$ is a trivial link. Isotope $\Gamma_{G(pl)}$ using ϕ^{-1}, and set $\Gamma_G = \phi_1^{-1}(\Gamma_{G(pl)})$. $\partial\Gamma_G$ is a trivial link.

Now, suppose $\partial\Gamma_G$ is a trivial link for some Γ_G. If $r = 1$, then G is a knot. So Γ_G is homeomorphic to an orientable $S^1 \times [0, 1]$, where $(S^1 \times \{0\}) \cup (S^1 \times \{1\})$ is a trivial link. If G is not ambient isotopic to the unknot, then $(S^1 \times \{0\}) \cup (S^1 \times \{1\})$ is not ambient isotopic to a trivial link.

If $r = 2$, then G is either an embedded theta graph, handcuff graph, or figure eight graph (one point wedge of two circles). (Rank two graphs have exactly two "independent" cycles which are either disjoint, or not. If the two cycles are disjoint, then they are connected via an isthmus. If the two cycles intersect, then the intersection set is either a point, or an edge.)

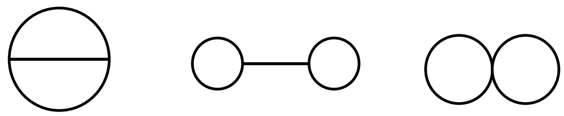

Figure 2

The case when G is an embedded theta graph has been proved in [1, 8].

Suppose G is either a handcuff or figure eight graph. For any rank two graph G, Γ_G is homeomorphic to $S^2 - \bigcup_{i=1}^{3} \overset{\circ}{D}{}_i^2$, a "pair of pants". If G is not ambient isotopic to a planar embedding, then there exists a Γ_G with spine G and $\partial\Gamma_G$ a trivial link. But then the theta graph may be drawn on this surface Γ_G (as any rank two graph may be drawn on Γ_G) contradicting the result in [1, 8].

For graphs of rank at least three, we present a collection of nonplanar graphs with trivial associated links by a simple modification of Zhao's example [8].

Theorem 3. *Let G_{ab} denote a connected, abstractly planar rank r graph, $r \geq 3$. Then there exists a nonplanar tame embedding G of G_{ab} and associated surface Γ_G such that $\partial\Gamma_G$ is a trivial link.*

Proof. We provide a construction of Γ_G for any abstractly planar graph G of rank r, $r \geq 3$. The construction is based on the example provided by P. Zhao in [8], and the observation that Γ_G is homeomorphic to $S^2 - \bigcup_{i=1}^{r+1} \overset{\circ}{D}{}_i^2$, for any graph G of rank r. From this observation, it follows that all graphs G of rank r may be drawn on the same $\Gamma_{G'}$, where rank $G' = r$.

Let Γ^* denote the surface provided by P. Zhao in [8] (See Figure 1). Γ^* is homeomorphic to $S^2 - \bigcup_{i=1}^{4} \overset{\circ}{D}{}_i^2$. To construct Γ_G for a graph G of rank r, remove $(r+1) - 4$ open two-discs from $\mathrm{Int}(\Gamma^*)$ (See Figure 3).

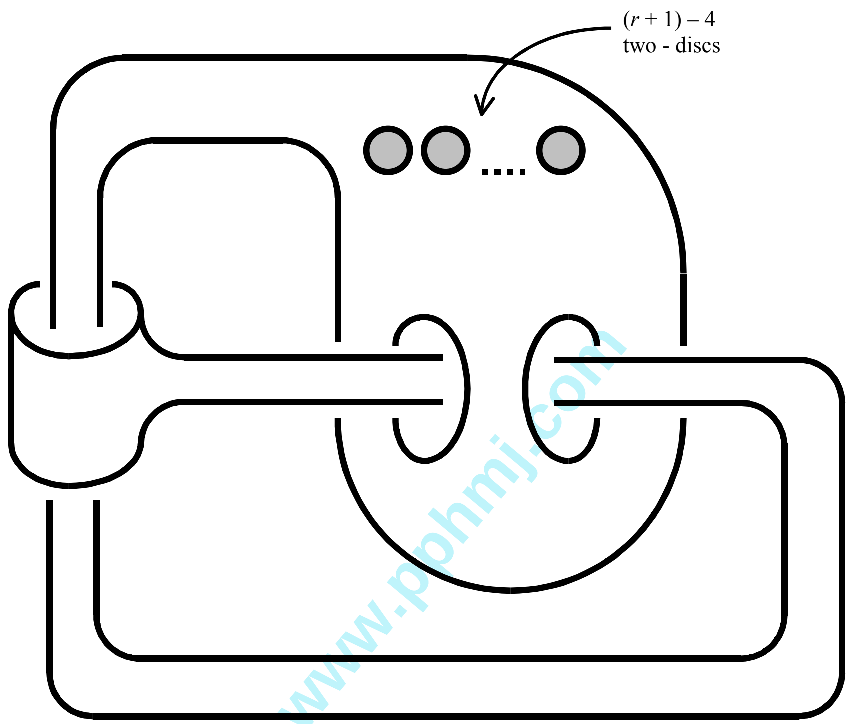

Figure 3

Then G may be drawn on Γ_G as its spine, $\partial\Gamma_G$ is a trivial link, but G is not ambient isotopic to a planar embedding as it contains a nonplanar embedded rank three subgraph.

For the infinite class of graphs with unique Seifert surfaces, an interesting area for future research is the development of algebraic invariants of graphs using their associated links.

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